The Goldstone model static solutions on $S^1$

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Abstract

We study in a systematic way all static solutions of the Goldstone model in 1+1 dimension with a periodicity condition imposed on the spatial coordinate. The solutions are presented in terms of the standard trigonometric functions and of Jacobi elliptic functions. Their stability analysis is carried out, and the complete list of classically stable quasi-topological solitons is given.
1 Introduction

The Goldstone model, the O(2) invariant Higgs model with Mexican hat potential, is well known as a theoretical laboratory for the study of spontaneous symmetry breaking in relativistic field theory. Recently, it was pointed out that the structure of the classical solutions of its 1+1 dimensional version on a spatial circle is strongly reminiscent of that of the localized solutions of the two-Higgs-doublet extension of the standard model (2HSM) of electroweak interactions [1]. Several branches of its solutions and their analogs in the 2HSM were explicitly constructed. Furthermore, the Goldstone model on $S^1$ is the simplest paradigm of field theories [1], [2], [3] whose topological properties are too trivial to lead to absolutely stable solitons of any kind, and which nevertheless support the existence of classically stable quasi-topological solutions for some range of their parameters, here the Higgs masses or equivalently the radius $L$ of spatial $S^1$. The purpose of this little note is to present the complete list of static classical solutions of the Goldstone model on $S^1$, the corresponding bifurcation tree and their stability properties. This analysis, apart from its own mathematical interest, may provide insight useful in our search for stable solitons in the 2HSM or in the special case of the minimal supersymmetric standard model.

2 The classical solutions

In this section we present all static classical solutions of the Goldstone model on spatial $S^1$. The model is defined with two real Higgs fields $\phi_1$ and $\phi_2$, whose dynamics is described by the action

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - V(\phi_1, \phi_2) \quad \mu = 0, 1$$

(1)

$$V(\phi_1, \phi_2) = \frac{1}{4}(\phi_1^2 + \phi_2^2 - 1)^2$$

(2)
and with periodic boundary condition on the space coordinate \( x \in [0, 2\pi L] \). The energy functional for static configurations is given by

\[
E = \int_0^{2\pi L} dx \left[ \frac{1}{2} \left( \frac{d\phi_1}{dx} \right)^2 + \frac{1}{2} \left( \frac{d\phi_2}{dx} \right)^2 + V(\phi_1, \phi_2) \right].
\]  

(3)

As usual, the static solutions of this model may be thought of as classical periodic motions of period \( 2\pi L \) of a particle in two dimensions under the influence of the inverted potential \( -V(\phi_1, \phi_2) \). Apart from the vacuum solutions, which have \( E_{\text{vac}} = 0 \), there is a trivial solution

\[
\phi_1 = \phi_2 = 0, \quad E_0 = \frac{L\pi}{2}
\]  

(4)

which exists for all values of \( L \). The corresponding small oscillation eigenmodes, labelled by \( j \), have

\[
\omega^2(j) = \frac{1}{L^2}(j^2 - L^2) \quad , \quad j = 0, 1, 2, \ldots .
\]  

(5)

They are four times degenerate, apart from the \( j = 0 \) one which is doubly degenerate. This solution is always unstable since \( \omega^2(0) = -1/2 \).

Many additional solutions, some of which were discussed in [1], [2] bifurcate from the solution \( \phi_1 = \phi_2 = 0 \) at critical values of \( L \). The generic solution of the model (1) can be expressed in terms of Jacobi elliptic functions; it has the form

\[
\phi_1 = \sqrt{R} \cos \Omega, \quad \phi_2 = \sqrt{R} \sin \Omega
\]  

(6)

\[
R(x) = a_1 + a_2 sn^2 \left( \sqrt{2} \lambda (x - x_0), k \right)
\]

\[
\Omega(x) = C \int_{\xi}^x \frac{1}{R(y)} \, dy
\]  

(7)

where \( a_1, a_2, \lambda, \xi, k, C, x_0 \) are constants, while \( sn \) denotes the Jacobi elliptic function; \( sn(z, k), sn(z, k)^2 \) are periodic functions on the real axis with periods \( 4K(k) \) and \( 2K(k) \), respectively. \( K(k) \) is the complete elliptic integral of the first kind. Inserting (6) and (7) into the equations corresponding to (1) leads to several conditions on the parameters and correspondingly to the following three types of non-trivial periodic solutions:
2.1 Type-I solution

For $a_2 = 0$ the function $R(x)$ is a constant and the solution reduces to the form

$$
\phi_1 = \sqrt{1 - \frac{N^2}{L^2}} \cos \left( \frac{Nx}{L} + \theta \right), \quad \phi_2 = \sqrt{1 - \frac{N^2}{L^2} \sin \left( \frac{Nx}{L} + \theta \right)}
$$

(8)

where $N$ is an integer and $\theta$ an arbitrary constant; it was first obtained in [2]. This solution bifurcates from (4) at $L = N$ i.e. when one of the $\tilde{\omega}$ of Eq. (5) crosses zero. The Higgs field winds $N$ times around the top of the Mexican hat. Its energy is given by

$$
E_I(L, N) = \frac{\pi N^2}{2L} (2 - \frac{N^2}{L^2})
$$

(9)

2.2 Type-II solution

For $C = 0$ the function $\Omega(x)$ vanishes, the parameter $a_1$ vanishes as well and the solution takes the form

$$
\phi_1 = 2k \Lambda \text{sn}(\sqrt{2}\Lambda x, k), \quad \phi_2 = 0, \quad \Lambda^2 = \frac{1}{2(1 + k^2)}
$$

(10)

The Higgs field oscillates in the $\phi_2 = 0$ plane about the origin $\phi_1 = 0 = \phi_2$. Evidently, this solution is the Manton-Samols sphaleron [4] embedded into the Goldstone model. The argument $k$ of the Jacobi elliptic function sn has to be chosen such that the periodicity condition is fulfilled, i.e.

$$
L = \frac{2K(k)m}{\pi} \sqrt{1 + k^2},
$$

(11)

for some integer $m$. When $k \to 0$ (i.e. $L \to m$) the solution (10) approaches (4). Note that $x$ can be translated by a constant and that the field $\phi_1 + i\phi_2$ can be rotated by a constant phase. The energy $E_{II}$ is identical to the one of the one-Higgs model [4]. The relevant integral reads

$$
E_{II}(L, m) = \frac{8m}{\sqrt{2\Lambda(1 + k^2)^2}} \int_0^{K(k)} dy \left[ (k^2 \text{sn}^2(y, k) - \frac{1}{2})^2 + \frac{2k^2 - 1 - k^4}{8} \right]
$$

(12)
and can be evaluated in terms of the elliptic functions $K(k)$, $E(k)$ by means of the integrals

\begin{align*}
\int_0^1 dy \, \text{sn}^2(y, k) &= \frac{K - E}{k^2} \\
\int_0^1 dy \, \text{sn}^4(y, k) &= \frac{(2 + k^2)K - 2(1 + k^2)E}{3k^4}
\end{align*}

In particular

\begin{align*}
\text{for } k = 0 & \quad E_{II}(L = m, m) = \frac{m\pi}{2} \\
\text{for } k = 1 & \quad E_{II}(L = \infty, m) = \frac{8m}{3\sqrt{2}}
\end{align*}

2.3 Type-III solution

When all the parameters entering in (6), (7) are non-zero the solutions are more involved. They were mentioned without details in [1] and we construct them here, as explicitly as possible. The equations imply the conditions

\begin{align*}
a_1 &= \frac{2}{3}(1 - 2\Lambda^2(1 + k^2)) \quad a_2 = 4k^2\Lambda^2 \\
C^2 &= \frac{4}{27}(1 + (4k^2 - 2)\Lambda^2)(1 + (4 - 2k^2)\Lambda^2)(1 - (2k^2 + 2)\Lambda^2)
\end{align*}

leaving a family of solutions depending on the four parameters $k, \Lambda, \xi, x_0$. The condition that $R$ and $C^2$ should be positive implies

\begin{equation}
\Lambda^2 \leq \frac{1}{2(1 + k^2)}
\end{equation}

When the equality holds one is led to the type-II solutions discussed above.

In order for $\Omega$ and $R$ to be periodic on $[0, 2\pi L]$ the following conditions must be satisfied

\begin{align*}
C \int_0^{2\pi L} \frac{1}{R(y)} dy &= 2\pi n \\
L &= \frac{mK(k)}{\sqrt{2\pi}\Lambda}
\end{align*}

for some positive integers $m, n$. These conditions fix $\Lambda$ and $k$ as functions of $L, m, n$. Thus, for a given $L$, the generic solution depends on the parameters $\xi, x_0, m$ and
n. The first two correspond to the arbitrary global phase and position of the configuration. The integer parameters \( m \) and \( n \) determine respectively the number of oscillations of the modulus of the Higgs field and the number of the Higgs field windings around the origin \( \phi_1 = 0 = \phi_2 \) in a period \( 2\pi L \).

Solving (18), (19) in the case \( k = 0 \) (with \( K(0) = \pi/2 \)) we find easily the critical values of \( L \) where the type-III solutions start to exist:

\[
\Lambda^2 = \frac{m^2}{4(6n^2 - m^2)} \Rightarrow L^2 = \frac{1}{2}(6n^2 - m^2)
\]

(20)

The expression for \( \Lambda^2 \) above combined with (17) lead to the condition \( 2n > m \) on the integers \( m \) and \( n \). The positivity of \( L^2 \) in (20) then follows automatically. This result also demonstrates that for \( n \) fixed there are \( 2n - 1 \) possible values of \( m \).

Due to the absence of a closed form for the integral

\[
\int_0^{K(k)} \frac{1}{c + \text{sn}^2(y, k)} dy
\]

(21)

for generic values of \( k \), the condition (18) is impossible to handle algebraically. Even so, one can make some progress with the analysis by means of a \( k^2 \) expansion.

For any \( n \) and \( m \neq 2n \) the coefficient of \( \text{sn}^2 \) in the integral giving \( \Omega(x) \) in (7) is proportional to \( k^2 \) and one may easily expand the solution in powers of \( k^2 \). We find

\[
\Lambda^2 = \frac{m^2}{4(6n^2 - m^2)} (1 + \frac{k^2}{2}) + O(k^4)
\]

\[
L^2 = \frac{6n^2 - m^2}{2} + O(k^4)
\]

\[
C^2 = \frac{2n^2(m^2 - 4n^2)^2}{(6n^2 - m^2)^3} + O(k^4)
\]

(22)

Inspection of the limit \( k = 0 \) shows that the \( 2n - 1 \) solutions of type-III bifurcates from the type-I solution with \( N = n \) at \( L^2 = (6n^2 - m^2)/2 \), for \( m = 1, 2, \ldots, 2n - 1 \).

The energy can also be expanded in powers of \( k^2 \) leading to

\[
E_{III}(k, m, n) = \frac{\pi \sqrt{2} n^2}{(6n^2 - m^2)^{3/2}} \left( 5n^2 - m^2 - k^4 \frac{3m^2(12n^4 - m^2 n^2 - m^4)}{64(6n^2 - m^2)(4n^2 - m^2)} + O(k^6) \right)
\]

(23)
The dependence on $L$ is recovered by (22). Correspondingly, the $k^2$-expansion of the energy of the Type-I solution about the point $L^2 = (6n^2 - m^2)/2$ leads to

$$E_I(k, N = n) = \frac{\pi \sqrt{2}n^2}{(6n^2 - m^2)^{3/2}} \left(5n^2 - m^2 - k^4m^2(4n^2 + m^2)(3n^2 - m^2) + O(k^6)\right)$$

(24)

The case $m = n = 1$ corresponds to the branch labelled $\tilde{W}_1$ in [1]. In this case the energies $E_I$ and $E_{III}$ deviate only from the $k^8$ term on:

$$E_I(k, 1) = \frac{8}{25} \sqrt{\frac{2}{2}} \left[1 - \frac{1}{128}(k^4 + k^6) - \frac{2105}{294912} k^8 + O(k^{10})\right]$$

(25)

$$E_{III}(k, 1, 1) = \frac{8}{25} \sqrt{\frac{2}{2}} \left[1 - \frac{1}{128}(k^4 + k^6) - \frac{2045}{294912} k^8 + O(k^{10})\right]$$

(26)

Thus $E_I(k, 1)$ is lower than $E_{III}(k, 1, 1)$ but only very slightly, as pointed out in [1] on the basis of a numerical study of these solutions.

We have studied Eqs. (18), (19) numerically, solving for $\Lambda^2$ as a function of $k^2$ for different values of $n/m$. For fixed values of $k, n/m$ we find a single solution $\Lambda^2(k, n/m)$ obeying the following property

$$\Lambda^2(k = 0, n/m) = \frac{m^2}{4(6n^2 - m^2)} \quad \Lambda^2(k = 1, n/m) = \frac{1}{4}$$

(27)

This is illustrated in Fig.1 for $n/m = 1$ and $n/m = 4/7$ by the solid lines, the dashed line representing the limit (17). The second limit (27) and the form of the solutions suggest that in the limit $k \to 1$, which corresponds to $L \to \infty$, solution III approaches solution II. To test this statement, it is interesting to compare their energies. The energy of the type-III solution is given by the integral

$$E_{III}(k, m, n) = \frac{m}{\sqrt{2\Lambda}} \int_0^{K(k)} dy \left[(a_1 - 1 + a_2 \sin^2(y, k))^2 + \frac{1}{6} \left(1 + 16\Lambda^4(k^2 - 1 - k^4)\right)\right]$$

(28)

For $k = 1$ one obtains

$$E_{III}(1, m, n) = \frac{4m}{3\sqrt{2}} = \frac{1}{2} E_{II}(1, m)$$

(29)

This verifies our expectation that solution III approaches solution II in the limit $k \to 1$. The occurrence of the factor $1/2$ is due to the fact that the solution III
depends on $sn^2$ which has a period $2K(k)$ while the solution II depends on $sn$ whose period is $4K(k)$.

The energies of the solutions of a few low lying branches are plotted in Fig. [2] as functions of $L$. For $L > \sqrt{5/2}$ all four types of solutions coexist and satisfy

$$E_I(L, N = 1) < E_{III}(L, m = 1, n = 1) < E_{II}(L, m = 1) < E_0(L) \quad (30)$$

The circle shows the bifurcation value $L^2 = 5/2$ of the $n = m = 1$ type-III solution from the $N = 1$ type-I solution. The stars indicate the three bifurcating values ($L^2 = 15/2, 10, 23/2$) of the $n = 2, m = 1, 2, 3$ type-III solution from the $N = 2$ type-I solution (energies of the $n = 2$ type-III solutions are not plotted). The numbers in parentheses in the figure represent the number of negative modes and the number of zero modes respectively of the corresponding solution. They follow from the stability analysis which is the content of the next section.

### 3 Stability

#### 3.1 Type-I solutions

We start with the stability analysis of the type-I solutions (8). This set contains the lowest energy non-trivial solution, the branch $W_1$ of [1], which was shown in [2] to be for $L > \sqrt{5/2}$ classically stable and therefore to be a soliton of the model. Do there exist more solitons in this class of solutions?

To analyse the stability of these solutions we will adopt the point of view of hidden algebra and of differential operators preserving finite dimensional spaces of polynomials [5], which in the case at hand is a rather straightforward application of Fourier analysis.

Perturbing as usual the fields $\phi_a$, $a = 1, 2$ around the classical solution (8), denoted here by $\phi_a^c$, $\phi_a(x) = \phi_a^c(x) + \eta_a(x) \exp(-i\omega t) \quad (31)$
leads to the following equation for the normal modes:

\[
A(N, L) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \omega^2 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}
\]

(32)

\[
A(N, L) \equiv -\frac{d^2}{dx^2} + 2\left(1 - \frac{N^2}{L^2}\right) \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix} - \frac{N^2}{L^2}
\]

(33)

where \( \omega^2 \) is the eigenvalue and, to simplify notation, we posed \( c = \cos(Nx/L) \) and \( s = \sin(Nx/L) \).

The complete list of eigenvalues of the operator \( A(N, L) \) for \( N \geq 1 \) can be obtained by classifying its invariant subspaces. This can be done by using the Fourier decomposition. After some algebra, one can check that for an integer \( n \geq N \) the following finite dimensional vector spaces are preserved by \( A(N, L) \)

\[
V_n = \text{Span} \{ \alpha_p \cos \frac{px}{L} + \beta_p \sin \frac{px}{L}, \ p - n = 0 \text{ (mod } 2N) \}, \quad |p| \leq n
\]

\[
\tilde{V}_n = \text{Span} \{ \alpha_p \sin \frac{px}{L} - \beta_p \cos \frac{px}{L}, \ p - n = 0 \text{ (mod } 2N) \}, \quad |p| \leq n
\]

provided

\[
\alpha_p = \beta_p \quad \text{if} \quad p - n + 2N > 0
\]

(35)

while the other constants \( \alpha_p, \beta_p \) are arbitrary. The operator \( A(N, L) \) can then be diagonalized on each of the finite dimensional vector space above, leading to a set of algebraic equations. One finds finally that the eigenvalues of \( A(N, L) \) on \( V_n \) read

\[
\omega^2(N, k, \pm 1) = \frac{1}{L^2}((L^2 - N^2) + k^2 \pm \sqrt{(L^2 - N^2)^2 + 4N^2k^2})
\]

(36)

for \( k = 0, 1, 2, \ldots \) and similarly on \( \tilde{V}_n \) for \( k = 1, 2, 3, \ldots \). The spectrum given by (36) fits exactly with (5) in the limit \( L = N \) for all \( N \) i.e. when \( L \) approaches the points of bifurcation of the type-I solutions from (4).

We can now discuss the stability of the solutions of type-I. First remark that \( \omega^2(N, 0, -1) = 0 \), it corresponds to the zero mode related to the invariance of the equations under translations of \( x \). For \( N \) fixed and \( L \) slightly greater than \( N \), the solution (8) possess \( 4N - 2 \) negative modes corresponding to \( \epsilon = -1 \) and \( k = 1, 2, 3, \ldots, 2N - 1 \)
in (36), remembering that they are twice degenerate. When $L$ increases, more and more of these eigenmodes become positive, crossing zero at the values

$$L^2 = \frac{1}{2}(6N^2 - m^2) \quad , \quad m = 1, 2, \ldots , 2N - 1$$

(37)
i.e. (cf. (20)) at those values of $L$ where the solutions of type-III with $n = N$ bifurcate from the solution of type-I. For

$$L^2 \geq L^2_\alpha(N) \equiv \frac{1}{2}(6N^2 - 1)$$

(38)
all the modes are positive and (8) are classically stable solitons. These results are illustrated on Fig.[2] for $N = 1$ and $N = 2$. The numbers in parenthesis represent the number of negative and of zero modes of the corresponding branch.

3.2 Alternative approach for Type-I

The stability analysis could also be performed by considering the polar decomposition of the Higgs fields [2]

$$\phi_1 = F \cos \Theta \quad , \quad \phi_2 = F \sin \Theta$$

(39)
The quadratic operator $Q$ associated with this parametrisation of the fields admits the following invariant subspaces

$$U_n = \text{span}\{ \frac{1}{\sqrt{\pi L}}(\cos \frac{n\pi}{L}, 0), \frac{1}{\sqrt{\pi L}}(0, \sin \frac{n\pi}{L}) \}$$

(40)
$$\hat{U}_n = \text{span}\{ \frac{1}{\sqrt{\pi L}}(\sin \frac{n\pi}{L}, 0), \frac{1}{\sqrt{\pi L}}(0, \cos \frac{n\pi}{L} x) \}$$

(41)
which, by Fourier theory, cover the whole relevant Hilbert space of periodic functions.

In the normalized basis (40), the eigenvalues of $Q$ on the subspace $U_n$ read

$$\omega^2(N, n, \pm 1) = \frac{1}{2L^2} \left\{ n^2 L^2 + \Delta^2(n^2 + 2L^2) \right\}$$

$$\pm \sqrt{\Delta^4(n^4 - 2n^2 L^2 + 4L^4) + 2n^2 L^2 \Delta^2(10L^2 - n^2) + n^4 L^4}$$

(42)
where we defined $\Delta^2 = L^2 - N^2$. The eigenvalues corresponding to $\hat{U}_n$ are identical to (42).
These values fail to approach the $\omega^2$ of Eq.(5) in the limit $L \to N$. This is due to the fact that the parametrization (39) is singular about the solution $\phi_1 = \phi_2 = 0$. Remarkably though, the values (36) and (42) have zero crossing at exactly the same values of $L^2$ and the conclusions about the stability of the solutions are identical in the two approaches.

3.3 Type-II and III solutions

The stability analysis about the solution of type-II can immediately be carried out. The equations for the fluctuations about the solution (10) decouple to take the form of the Lamé equations:

\begin{align*}
\{ -\frac{d^2}{dy^2} + 6k^2\text{sn}^2(y, k) \}\eta_1 &= \Omega^2_1\eta_1 \\
\{ -\frac{d^2}{dy^2} + 2k^2\text{sn}^2(y, k) \}\eta_2 &= \Omega^2_2\eta_2
\end{align*}

where we posed

$$y = \sqrt{1 + k^2}x, \quad \Omega^2_a \equiv (\omega^2_a + 1)(k^2 + 1), \quad a = 1, 2$$

$\omega^2_a$ being the effective eigenvalue of the relevant operator. Equations (43) and (44) admit five and three algebraic modes respectively, with corresponding eigenvalues

\begin{align*}
\Omega^2_1 & : \quad 4 + k^2, 1 + 4k^2, 1 + k^2, 2(1 + k^2) \pm 2\sqrt{1 - k^2 + k^4} \\
\Omega^2_2 & : \quad 1 + k^2, 1, k^2
\end{align*}

The corresponding values of $\omega^2$ follow immediately from (45); they have signature $(+, +, 0, +, -)$ and $(0, -, -)$ respectively for (46) and (47). Each of the equations (43),(44) therefore leads to a zero mode of the solution (10), their origin was discussed in Sect.2. It is a property of the Lamé equation that the solutions determined algebraically correspond to the solutions of lowest eigenvalues. The remaining part of the spectrum therefore consists of positive eigenmodes. The spectrum of Eq.(43) was studied perturbatively in [4], while the relation between the Lamé equation and
the Manton-Samols sphalerons was first pointed out in [6]. The presence of negative modes in the small oscillation spectrum of all type-II solutions means that no classically stable soliton exists among them.

The stability equation associated with the type-III solutions seems to be more difficult, mainly due to the lack of analytical expression for $\Omega(x)$ in Eq. (6). We have not obtained convincing algebraic expressions for their normal modes but a few of their properties can be pointed out. First, these solutions should possess a double zero mode due to the invariance of the solution (6), (15) under translations of $x$ and internal $O(2)$ rotation. Likely the eigenvalues relative to these solutions connect to the values (36) in the limit $L^2 \to (6n^2 - m^2)/2$, $N = n$. We conjecture that the solutions of type-III always possess at least one negative mode. In the case $n = m = 1$, the negative mode should be unique and meet the two zero modes at the point $L = \sqrt{5}/2$, $\omega^2 = 0$, indicated by the star of Fig. [3].

4 Conclusion

The model (1) on $S^1$ admits a rich set of solutions which bifurcate from each other at critical values of the spatial radius $L$. A detailed analytical study of these solutions was presented. Their stability analysis was carried out, all classically stable solitons were identified, together with the range of the parameter $L$ for which they are stable. The bifurcation pattern contains as a subset the branches corresponding to the solutions found in [4], and it is found to be richer than that of the gauged Higgs model [5], the gauged version of (1).

The spectrum of modes of small fluctuations around a classical solution is shown to depend on the parametrization employed for the complex scalar. In particular, we explicitly demonstrated this fact for the normal modes around the type-I solutions. As one varies the parameter $L$ to approach the bifurcation point $L = N$ along the type-I solution with winding $N$, the correct spectrum is the one which coincides with that of the main branch (4).
In this work we concentrated on static solutions. The study of time dependent ones, to search for the analogs of the breather solutions of the sine-Gordon equation, is particularly interesting. They may well be the prototypes of stable "breathing membranes" in realistic particle physics models.
References


Figure Captions

- **Figure 1.** The solutions of eqs. (18), (19) are plotted as functions of $k^2$ for two values of $n/m$. The dashed line indicates the limit (17).

- **Figure 2.** The energies of some of the solutions are plotted as functions of the parameter $L$. The circle indicates the bifurcation point ($L^2 = 5/2$) on the $N = 1$ type-I solution. The stars indicate the three bifurcation points ($L^2 = 15/2, 10, 23/2$) on the $N = 2$ type-I solution. The two numbers in parentheses refer to the number of negative modes and of zero modes of the corresponding solution.

- **Figure 3.** The values (5) are plotted as functions of $L$ for $j = 0, 1, 2$ (solid lines), together with the values $\omega^2(1, 1, -1), \omega^2(1, 0, -1), \omega^2(1, 0, 1), \omega^2(1, 2, -1), \omega^2(1, 1, 1), \omega^2(1, 3, -1)$ of (36) (dotted lines). The numbers indicate the multiplicity of the eigenvalues.