FERMI SPECTRA AND THEIR GAUGE INVARIANCE IN HOT AND DENSE ABELIAN AND NON-ABELIAN THEORIES

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Abstract

The one-loop Fermi spectra (one-particle and collective ones) are found for all momenta in the $T^2$-approximation and their gauge invariance in hot and dense Abelian and non-Abelian theories is studied. It is shown that the one-particle spectrum, if the calculation accuracy is kept strictly, is gauge invariant for all momenta and has two branches as the bare one. The collective spectrum always has four branches which are gauge dependent including also their $|q| = 0$ limit. The exception is the case $m, \mu = 0$ for which this spectrum is gauge invariant for all momenta as well.

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1 Introduction

The problem to obtain the gauge invariant results from gauge theories (especially from non-Abelian ones) is not new but is very actual for present-day physics. This problem is very many-sided and one easily calls many tasks which require the first rate solution. In particular, many questions arise when the effective electron (or quark) mass is calculated within hot and dense QED or QCD, including even the pure QFT (quantum field theory). Of course, the latter case is more studied but some problems (in the first turn, the proof that the fermion mass is infrared-finite) were solved only recently making calculation of the perturbative mass in QFT selfconsistent and proving its gauge invariance [1]. The completely different situation takes place when T (temperature) is nonzero. In this case the nonzero thermal mass (or thermal gap) is always generated [2,3] and all Bose and Fermi spectra (if we speak about so called collective ones) are split demonstrating themselves more pronounced than in QFT [2-9]. The more complicated scenario arises when the bare fermion mass m and temperature are nonzero simultaneously [10,11]. In this case the spectrum has the additional splitting and changes its long wavelength asymptotical behaviour: it becomes $q^2$ instead of $|q|$ when $m = 0$. If along with $m, T \neq 0$ the chemical potential $\mu$ is also nonzero the scenario becomes very cumbersome [12,13] and only the separate spectrum limits can be found analytically. To obtain the spectrum curves for all $|q|$ the numerical calculations are necessary that, unfortunately, hides many details from the further analysis. However, if scenarios with $\mu, T \neq 0$ and with $m, T \neq 0$ are considered separately the spectrum curves for all $|q|$ can be found analytically in any case within $T^2$-approximation [14,15]. These analytical expressions easily reproduce all known limits and are more convenient to investigate the spectrum gauge invariance and many other properties. For the general case when $\mu, m$ along with $T$ are nonzero only the effective thermal mass can be found analytically [15,16] and being calculated within different gauges at once demonstrates its gauge dependence. However the situation drastically changes if the one-particle spectrum [16] is considered instead of the collective one. This spectrum is analogous to the bare one but its properties are modified due to interaction with medium and completely different from the collective ones. Unlike the latter scenario the one-particle effective mass is gauge invariant in the leading $e^2$-order (and, possibly, in the higher orders as well) and has a real physical sense. The nonperturbative fermion mass (the $|q| = 0$-limit of the collective spectrum) is always gauge dependent as well as the full collective spectrum, and the additional resummation seems to be necessary.

Briefly speaking the question of the gauge invariance for many results found in statistical QCD is insufficiently studied and opens for discussion. Even in statistical QED the exact gauge invariance is only known for the photon thermal mass, but the same as in QCD, no exact results exist for the thermal fermion masses and each times their gauge invariance is necessary to investigate independently.

The goal of the present paper is to show that not only the perturbative mass but also the full one-particle Fermi spectrum for all $q^2$ is gauge invariant if the calculation accuracy is kept strictly. This invariance takes place for any $m, \mu$-parameters, although the case $\mu = 0$ is separated. In the case $\mu = 0$ the full one-particle Fermi spectrum is gauge invariant even beyond the perturbative accuracy in any case within the simplest summation of one-loop calculations. In the section 4 the collective spectrum is calculated for the case $m, \mu \neq 0$ in the Coulomb gauge to compare it with one found in the Feynman gauge. It is shown that this spectrum is always gauge dependent but, the Coulomb gauge seems to be more physical one.
for any applications. The exception is the case \( m, \mu = 0 [3,5] \) for which this spectrum is gauge invariant for all momenta including their effective mass. This scenario, probably, is valid in all perturbative orders since in this case within \( T^{2} \)-approximation no dimensional parameters (except \( T \)) are present and due to general theorems [17,18], such spectrum should be gauge invariant exactly (including all higher order corrections) even within hot QCD. Of course, the Abelian QED and non-Abelian QCD should generate the different gauge dependence of any spectra, but in the leading \( e^{2} \) (or \( g^{2} \)) order QED and QCD Fermi spectra are the same (if everywhere the numerical factor is changed due to the prescription \( e^{2} \to g^{2}(N^{2} - 1)/2N \)). Today there is a problem to find (or, at least, understand) what a difference arises within the spectrum found in QED when the higher order corrections are taken into account to distinguish QED from QCD. The strong infrared divergencies of hot QCD (see, e.g. [19] and other references within it) should display themselves via the higher order corrections and it is not excluded that effective thermal mass in QCD becomes the infrared-unstable. To start we choose hot and dense QED (not QCD) although all results are the same in the approximation considered. We also propose that everywhere the damping is small and can be neglected considering that this question should be investigated separately [20,21]. Our attention will be focused to solve the fermion dispersion relation in a more complete form and to investigate the gauge invariance of all spectra found.

2 The one-loop fermion self-energy in the Coulomb gauge

The Coulomb gauge is rather reliable gauge for perturbative calculations within hot gauge Abelian and non-Abelian statistical theories. It does not generate (unlike the \( \alpha \)-gauges) the additional infrared divergencies and requires only the standard ultraviolet regularization. Of course, all calculations in the Coulomb gauge are more complicated than in the Feynman one but these are the technical difficulties which can be overcome without any principle modifications of the theory studied. In what follows our calculations are performed in the Coulomb gauge but often we compare them with the ones made in the Feynman gauge [14-16] and keep the same abbreviations. Some details of these calculations and their cumbrous results are placed in Appendix A.

Within the one-loop approximation the exact decomposition for \( \Sigma(q) \) is given by

\[
\Sigma(q) = i\gamma_{\mu} K_{\mu}(q) + mZ(q) \tag{1}
\]

where three new scalar functions are introduced to find nonperturbatively the function \( G(q) \)

\[
G(q) = \frac{-i\gamma_{\mu} \tilde{q}_{\mu} + K_{\mu}}{(\tilde{q}_{\mu} + K_{\mu})^{2} + m^{2}(1 + Z)^{2}}. \tag{2}
\]

This representation leads to the one-loop dispersion relation (the exact one has the additional functions [22]) for the Fermi excitations which in any gauge has the form

\[
[(iq_{\mu} - \mu - \not{K})^{2} = q^{2} (1 + K)^{2} + m^{2}(1 + Z)^{2} \tag{3}
\]

and after the standard analytical continuation it can be solved analytically or numerically. Here \( K_{\mu} = i\not{K} \) and \( \tilde{q} = \{(q_{\mu} + i\mu), q\} \).
The calculations are rather lengthy and require at first to extract the functions \( Z(q) \) and \( K_\mu(q) \) from \( \Sigma(q) \)

\[
\Sigma^C(q) = -\epsilon^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{n_p^+ \gamma \epsilon_p + i\gamma(p - q) [\{(p - q)p/(p - q)^2 + m]}{q_i + i(\mu + \epsilon_p)^2 + (p - q)^2} + \frac{n_p^B}{|p|} \right.

\]

\[
\left. \frac{\gamma \epsilon_p - i\gamma(p - m)}{q_i + i(\mu + |p|)^2 + (p - q)^2} \left[ h.c.; (m, \mu) \rightarrow -(m, \mu) \right] \right\} \]

and then to calculate them using a number of exact integrals. Our result for the functions \( Z(q) \) and \( K_\mu(q) \) has the form

\[
Z^C(q_i, q) = \epsilon^2 \int_0^\infty \frac{d|p|}{4\pi^2} \left\{ \frac{|p|}{2|q|} \left\{ \frac{1}{\epsilon_p} \left[ \frac{n_p^+ + n_p^-}{2} \left( \ln(a^+_F a^-_F) \right) \right. \right. \right.

\]

\[
\left. \left. \left. - \ln \left( \frac{|p| + |q|}{|p| - |q|} \right)^2 \right) + \frac{n_p^+ - n_p^-}{|p|} \ln \left( \frac{a^+_F}{a^-_F} \right) - \frac{n_p^B}{|p|} \ln \left( a^+_B a^-_B \right) \right\} \right\}

\]

(5)

\[
iK^C_\mu(q_i, q) = \epsilon^2 \int_0^\infty \frac{d|p|}{4\pi^2} \left\{ \left[ \frac{|p|^2}{2|q|} \left( \frac{n_p^+ + n_p^-}{2} \ln \left( \frac{a^+_F}{a^-_F} \right) \right) + \frac{n_p^+ - n_p^-}{2} \ln \left( a^+_B a^-_B \right) \right] \right. \right.

\]

\[
+ \ln \left( \frac{|p| + |q|}{|p| - |q|} \right)^2 \right) + \frac{n_p^B}{|p|} \left[ \ln \left( \frac{a^+_B}{a^-_B} \right) + \frac{|\mu - iq_i|}{|p|} \ln \left( a^+_B a^-_B \right) \right] \right\}

\]

(6)

and the more complicated calculations are necessary to obtain the vector \( K_n(q) \) where \( n = 1, 2, 3 \). Here we use a definition \( K_n(q) = q_n K(q) \) and after that the scalar function \( K(q) \) is calculated to be

\[
q^2 K^C(q_i, q) = \epsilon^2 \int_0^\infty \frac{d|p|}{2|q|} \left\{ \frac{|p|^2}{2|q|} \left( \frac{n_p^+ + n_p^-}{2} \left[ \frac{1}{8|p||q|} \left( h_F \ln \left( \frac{a^+_F}{a^-_F} \right) + \frac{n_p^B}{|p|} \ln \left( a^+_B a^-_B \right) \right) \right] \right) \right.

\]

\[
+ \frac{n_p^+ + n_p^-}{2} \left[ 1 + \frac{1}{8|p||q|} \left( h_B \ln \left( a^+_B a^-_B \right) + d_B \ln \left( a^+_B a^-_B \right) \right) \right] \right\}

\]

(7)

where \( h_F = q^2 - m^2 - (iq_i - \mu)^2 \) and \( d_F = 2\epsilon_p (iq_i - \mu) \) (analogously \( h_B = q^2 + m^2 - (iq_i - \mu)^2 \) and \( d_B = 2|p|(iq_i - \mu) \)) are some simple abbreviations and other ones have the more complicated form

\[
a^+_F = \frac{q^2 - m^2 - (iq_i - \mu)^2 + 2\epsilon_p (iq_i - \mu) - 2|p||q|}{q^2 - m^2 - (iq_i - \mu)^2 + 2\epsilon_p (iq_i - \mu) + 2|p||q|}

\]

(8)

\[
a^\pm_B = \frac{q^2 + m^2 - (iq_i - \mu)^2 + 2|p|(iq_i - \mu) - 2|p||q|}{q^2 + m^2 - (iq_i - \mu)^2 + 2|p|(iq_i - \mu) + 2|p||q|}

\]

(9)

Here and everywhere \( \epsilon_p = \sqrt{m^2 + p^2} \). The function \( K^C(q_i, q) \) has the rather cumbersome form and is placed in Appendix A. This function is mainly essential beyond the \( T^2 \)-approximation and below only a few of its terms are exploited to find \( \mu/T \)-corrections.
3 The one-particle spectrum of massive Dirac particles in the $T^2$-approximation

The one-particle spectrum is the perturbative one and corresponds to the calculations when in the leading order the dispersion relation (3) is solved with $\Sigma(q_i, q)$ taken at once on the bare mass shell $i\omega_i = \mu \pm \sqrt{q^2 + m^2}$.

Within $e^2$-approximation this spectrum can be presented as follows:

$$i\omega_i = (\mu + \tilde{K}_i) \pm \sqrt{m^2 \left(1 + Z(q)\right)^2 + q^2 \left(1 + K(q)\right)^2}$$

where all functions being put at once on the bare mass shell are independent on $i\omega_i$. This spectrum is qualitatively different from the collective one and at the beginning is more useful for many applications. In particular, its long wavelength limit ($|q| = 0$-limit) which defines the effective thermal mass is the gauge invariant value [16]

$$i\omega_i = \mu_R \pm m_R = \mu \left(1 + 2 \tilde{I}_B\right) \pm \left[m(1 - 4I_Z) + \frac{I_K}{m}\right]$$

At any rate Eq.(11) is the same in both gauges: the Coulomb and Feynman ones. Here $I_B = \mu \tilde{I}_B$ and other abbreviations are:

$$I_K = I_K^F + I_K^B = e^2 \int_0^\infty \frac{d|p|}{2\pi^2} \epsilon_p \frac{n_p^+ + n_p^-}{2} + e^2 \int_0^\infty \frac{d|p|}{2\pi^2} |p| n_p^B,$$

$$I_B = -e^2 \int_0^\infty \frac{d|p|}{4\pi^2} n_p^+ - n_p^-,$$  

$$I_Z = e^2 \int_0^\infty \frac{d|p|}{4\pi^2} \frac{n_p^+ + n_p^-}{2\epsilon_p}.$$  

The found gauge invariance is a very important property of this spectrum and our task is to investigate whether this property keeps for all momenta or only some separated spectrum limits demonstrate it within the $T^2$-approximation. However, checking this property one should be careful exploiting Eq.(10), since it is given beyond the accepted one-loop accuracy. The more reliable expression has the form

$$i\omega_i = (\mu \pm \epsilon_q) + \tilde{K}_4(q) \pm \left(\frac{n^2 Z(q)}{\epsilon_q} + \frac{q^2 K(q)}{\epsilon_q}\right)$$

where the one-loop corrections to the bare spectrum are given strictly within the $e^2$-accuracy. Below we return to this question once more to demonstrate the important result: the gauge invariance is a priori broken if the perturbative series are considered beyond the accepted accuracy.

3.1 The fermion one-particle spectrum with $m, \mu \neq 0$ in Coulomb and Feynman gauges and its gauge invariance

Here the one-particle Fermi spectrum will be found for all momenta in the most general case $m, \mu \neq 0$ using the expression for $\Sigma(q)$ obtained in the Feynman (F.G) and Coulomb (C.G) gauges. Namely these gauges are preferable in statistics since only they are reliable without any additional regularization.
Going to the $T^2$-approximation, we simplify all logarithms which define the above found integrals, keeping only the leading $T^2$-term and $\mu/T$-corrections. Effectively this operation leads to the following ansatz

\[
\ln\left(\frac{a_F^+}{a_F^-}\right) \simeq \mathcal{L}_\pm^F(q) = \frac{1}{2} \ln \left[ 1 + \frac{2q^2}{m^2} \left(1 \pm \frac{\epsilon_q}{|q|}\right) \right]^2, \quad \ln\left(\frac{a_B^+}{a_B^-}\right) \simeq \mathcal{L}_\pm^B(q) = \ln \left[ 1 + \frac{\epsilon_q}{|q|}\right]^2
\]

(14)

and also \(\ln(a_F^+a_F^-) \simeq -4|q|/|p|\) and \(\ln(a_B^+a_B^-) \simeq 0\). In Eq. (14) and everywhere the bottom signs of \(\mathcal{L}_{\pm}^F/B\)-quantities correspond to the signs in the bare spectrum \(i\epsilon_q = \mu \pm \sqrt{q^2 + m^2}\) which is inserted into all integrals before any expansion. The derived ansatz allows to calculate the $T^2$-terms and $\mu/T$-corrections for all functions involved in Eq.(10) explicitly and to compare the spectra found in different gauges. The results of the calculations made in the Coulomb gauge

\[
Z^C(q) = -4I_B - \frac{\mathcal{L}_\pm^F(q)}{2|q|}I_B, \quad \tilde{K}_i^C(q) = \frac{\mathcal{L}_\pm^F(q)}{4|q|}I_K^B - \frac{\mathcal{L}_\pm^B(q)}{4|q|}I_K^B
\]

\[
q^2K^C_\pm(q) = I_K \mp \epsilon_qK^C_\pm(q)
\]

(15)

and in the Feynman one

\[
Z^F(q) = Z^C(q) - \frac{\mathcal{L}_\pm^F(q)}{2|q|}I_B, \quad \tilde{K}^F_i(q) = K_i^C - 2I_B
\]

\[
q^2K^F_\pm(q) = 2mI_B \left( \frac{m\mathcal{L}_\pm^F(q)}{4|q|} \pm \frac{\epsilon_q}{m} \right) + q^2K^C_\pm(q)
\]

(16)

are very similar but are not equal if $\mu \neq 0$. This means that $\Sigma(q, q)$ and its structure functions are, as a rule, gauge dependent but this is not the case for the one-particle spectrum since within Eq.(13) the algebraic cancellations are possible to restore the spectrum gauge invariance.

Indeed within the Feynman gauge the one-particle spectrum, before algebraic cancellations, has the form

\[
iq_i = (\mu \pm \epsilon_q) + (\tilde{K}_i^C(q) - 2I_B) \pm \left\{ \frac{m^2}{\epsilon_q} \left( Z^C(q) - \frac{\mathcal{L}_\pm^F(q)}{2|q|}I_B \right) \right\}
\]

\[+ \frac{1}{\epsilon_q} \left[ 2mI_B \left( \frac{m\mathcal{L}_\pm^F(q)}{4|q|} \pm \frac{\epsilon_q}{m} \right) + q^2K^C_\pm(q) \right]\]

(17)

but can be easily simplified as follows

\[
iq_i = (\mu \pm \epsilon_q) + \tilde{K}_i^C(q) \pm \left( \frac{m^2Z^C(q)}{\epsilon_q} + \frac{q^2K^C_\pm(q)}{\epsilon_q} \right)
\]

(18)

that is exactly the one-particle spectrum in the Coulomb gauge, and consequently the gauge invariance is restored.
However the cancellations are not complete if the one-particle spectrum is presented by Eq.(10). In this case the spectrum found in the Coulomb gauge

\[ iq_i = (\mu + \tilde{K}_i^C(q)) \pm \sqrt{m^2(1 + Z^C(q))^2 + q^2 \left[ 1 + \frac{1}{q^2} \left( I_K(q) \mp \epsilon_q \tilde{K}_i^C(q) \right) \right]^2} \]  \hspace{1cm} (19)

is essentially different from the one found in the Feynman gauge

\[ iq_i = (\mu - 2I_B + \tilde{K}_i^C(q)) \pm \sqrt{m^2(1 + Z^C(q) - \frac{L_{\pm}^F(q)}{2|q|}I_B)^2 + q^2 \left[ 1 + \frac{1}{q^2} \left( I_K(q) \mp \epsilon_q \tilde{K}_i^C(q) + 2mI_B \left( \frac{mL_{\pm}^F(q)}{4|q|} \pm \frac{\epsilon_q}{m} \right) \right) \right]^2} \]  \hspace{1cm} (20)

since the accepted accuracy is exceeded within this scenario. These spectra are gauge dependent although their low and high energy limits reproduce the gauge invariant results as well. The correct one-particle spectrum should be found using Eq.(13) and has the form

\[ iq_i = (\mu \pm \epsilon_q) \pm \left[ - \frac{m^2}{\epsilon_q} \left( 4I_Z \pm \frac{L_{\pm}^F(q)}{2|q|}I_B \right) \pm \frac{I_K}{\epsilon_q} \right] \]  \hspace{1cm} (21)

which is our main result for this section. To find its limit for small \( q^2 \) one should use the standard expansion of \( L_{\pm}^{FB}(q) \)-quantities

\[ L_{\pm}^{FB}(q) = \mp \frac{4|q|}{m} \pm \frac{2|q|^3}{3m^3} + O(q^5) \]  \hspace{1cm} (22)

that is the same for both functions in the leading order and perform the simple algebra. These calculations yield the following result:

\[ iq_i = \mu \pm 2\tilde{I}_B \pm m_F \pm \left[ (1 + 4I_Z - \frac{I_K}{m^2}) \mp \mu \frac{8\tilde{I}_B}{3m} \right] \frac{q^2}{2m} + O(q^3) \]  \hspace{1cm} (23)

where \( m_F = m(1 - 4I_Z) + I_K/m \) is the effective fermion mass accepted for \( \epsilon^2 \)-approximation.

The high energy limit in the \( T^2 \)-approximation is also easily calculated and has the form

\[ iq_i = \mu \pm |q| \pm \frac{1}{|q|} \left[ I_K - 4m^2I_Z \right] + O\left( \frac{1}{q^2} \ln \frac{q^2}{2m^2} \right) \]  \hspace{1cm} (24)

which is analogous to the bare one. All other terms are small and can be neglected. Comparing the low and high energy limits we see that one spectrum branch (the case of an upper sign) can demonstrate the minimum at finite momentum if the density is rather large.

### 3.2 The gauge invariant one-particle spectrum with \( \mu = 0 \) in the \( T^2 \)-approximation

The special case of zero density (when \( \mu = 0 \)) is rather interesting since in this case within the \( T^2 \)-approximation there is no problem with the gauge invariance at all. In this case the
one-particle Fermi spectrum is gauge invariant at the beginning and Eq. (10) can be used to present it as follows

\[ i q_i = \tilde{K}^\pm_i (q) \pm \sqrt{m^2 (1 - 4 I_z)^2 + q^2 \left[ 1 + \frac{I_K + \epsilon q \tilde{K}^\pm_i (q)}{q^2} \right]^2} \quad (25) \]

where

\[ \tilde{K}^\pm_i (q) = -\frac{\mathcal{L}_F^0 (q)}{4 |q|} I_K^0 - \frac{\mathcal{L}_H^0 (q)}{4 |q|} I_K^H = \pm \frac{I_K}{m} \mp \frac{q^2}{6m^3} I_K + O(q^4) \quad (26) \]

Its low and high energy limits repeat the appropriate expressions found above in the Coulomb gauge for the case \( \mu = 0 \) but, and it is more important, this spectrum is gauge invariant at once and, probably, this property is kept within all higher order corrections.

4 The collective Fermi spectrum and its gauge dependence in the \( T^2 \)-approximation

For hot gauge theory (Abelian or non-Abelian) with massless fermions and in the symmetrical case \( (\mu = 0) \) the Fermi excitations have collective nature and appear as "massive" quasiparticles (or quasi-holes) under the physical vacuum. In the simplest case their thermal masses (or more exactly their thermal gaps) are the same for each spectrum branch and they arise dynamically in spite of the exact chiral symmetry inherent to initial Lagrangian on the operator level. Such excitations in hot QCD were first found in [3] and then studied in many other papers. Their spectrum [3,5] was found analytically in the \( T^2 \)-approximation for all momenta

\[ \omega^2_\pm (\xi) = \xi^2 \omega^2_0 \left( \frac{\eta}{\xi - \eta} + \frac{\eta}{2} \ln \frac{\xi - 1}{\xi + 1} \right), \quad \eta = \pm 1 \quad (27) \]

and is gauge invariant in any case within one-loop calculations. Here the variable \( \xi \) runs over the range \( 1 < \xi < \infty \) and the long wavelength spectrum limit (i.e. \( q = 0 \)-limit) corresponds to the case \( \xi \to \infty \). In QCD the thermal gap is: \( \omega^2_0 = g^2 T^2 / 6 \), and the spectrum is split since \( \eta = \pm 1 \). All its branches have the finite gap at zero momentum (the finite thermal mass) and \( \omega^2_\pm (\xi) \) quasi-hole branches have a very specific minimum at finite \( q \). This minimum always exists when \( m, \mu = 0 \) and is a very interesting subject for discussion [3-9]. For small \( q \) this spectrum has the linear limit

\[ \omega^2_\pm (q) = \omega^2_0 \left[ 1 \pm \frac{2}{3} \frac{|q|}{\omega_0} + \frac{7}{9} \frac{q^2}{\omega_0^2} + O(|q|^3) \right] \quad (28) \]

since all branches have the finite gap, but no real mass. In the high momentum region the quasi-particle and quasi-hole branches have qualitatively different limits. The quasi-particle branches demonstrate the standard powerful behaviour

\[ \omega^2_\pm (q) = q^2 + 2 \omega^2_0 - \left( \frac{\omega^4_0}{q^2} \right) \ln \frac{q^2}{\omega^2_0} + O \left( \frac{1}{q^2} \right) \quad (29) \]
but this is not the case for the quasi-hole ones

$$\omega_\omega(q) = q^2 \left[ 1 + 4 \exp\left( -\frac{2q^2}{\omega_0^2} \right) + O\left( \exp\left( -\frac{4q^2}{\omega_0^2} \right) \right) \right] \quad (30)$$

which more quickly (exponentially) approach the line $\omega = q$.

However this scenario changes drastically when $m$ or $\mu$ is not equal to zero. In the first turn this spectrum becomes gauge dependent and more complicated: its mass (or gap) is split and its minimum inherent to the quasi-hole branches, as a rule, disappears. We study this situation below using the Coulomb gauge and compare our calculations with the results found in the Feynman one from the papers [14,16] where these excitations were also obtained analytically for all momenta with nonzero $m, \mu$-parameters.

### 4.1 The collective spectrum of massless Dirac particles in the Coulomb gauge for the case $\mu \neq 0$

The scenario with $m = 0$ but $\mu \neq 0$ is very similar to the previous one where $m, \mu = 0$ and keeps many its properties although the symmetry $(iq_4 \rightarrow -iq_4)$ is lost. Here, as previously, all spectrum branches have the finite gap and the linear limit for small $q$, but there are two different gaps one of which, sometimes, can be equal to zero.

The dispersion relation (3) is simplified to be

$$[(iq_4 - \mu) - \tilde{K}_4]^2 = q^2 (1 + K)^2 \quad (31)$$

and its solution can be written as follows

$$(iq_4 - \mu) - \tilde{K}_4 = \eta|q|(1 + K) \quad (32)$$

Here $\eta = \pm 1$ and we use the simple redefinition $K_4 = i\tilde{K}_4$. All functions involved into Eq.(32) are calculated using the $T^2$-approximation where the essential simplifications are possible due to the simple ansatz

$$\ln(a^+a^-) = \frac{-2|q|}{|p|} + O\left(\frac{1}{T^2}\right), \quad \ln \frac{a^+}{a^-} = 2 \ln \frac{\xi - 1}{\xi + 1} + O\left(\frac{1}{T^2}\right) \quad (33)$$

which keeps only $T^2$-terms and $\mu/T$-corrections in Eq.(32). With the ansatz (33) the further calculations are easily performed and their results are given by

$$\tilde{K}^C(q_4, q) = \frac{I_K}{q^2} \left( 1 + \frac{\xi - 1}{\xi + 1} \right) - I_B \left( \xi - \frac{1}{2} \left( 1 - \xi^2 \right) \ln \frac{\xi - 1}{\xi + 1} \right) \frac{1}{|q|}$$

$$\tilde{K}_4^C(q_4, q) = I_B - \frac{I_K}{2|q|} \ln \frac{\xi - 1}{\xi + 1} \quad (34)$$

where all numerical integrals $I_K$ and $I_B$ are defined by Eq.(12). Now one should plug the expressions found above into Eq.(32) and perform a simple algebra to find $\omega = \xi|q|$. Here $\omega = (iq_4 - \mu)$ and the variable $\xi$ is more convenient than $|q|$. The latter should be excluded from Eq.(32). The result is the quadratic equation with respect to $\omega(\xi)$

$$\omega^2 (\xi - \eta) - \omega \xi I_B (1 - \eta b(\xi)) - I_K \xi^2 A(\xi) = 0 \quad (35)$$
where the functions $A(\xi)$ and $b(\xi)$ are given by

$$A(\xi) = \eta \left( 1 + \frac{\xi - \eta}{2} \ln \frac{\xi - 1}{\xi + 1} \right)$$

$$b(\xi) = \xi - \frac{1}{2}(1 - \xi^2) \ln \frac{\xi - 1}{\xi + 1}$$

Keeping the $\epsilon^2$-accuracy our solution of Eq.(35) is found to be

$$\omega_{\pm}(\xi|\eta) = \xi \left[ \frac{I_B}{2}(1 - \eta b(\xi)) \right] \pm \sqrt{I_K \left( \frac{\eta}{\xi - \eta} + \frac{\eta}{2} \ln \frac{\xi - 1}{\xi + 1} \right)}$$

and presents the one-loop spectrum in medium for the case $\mu \neq 0$. Here $\eta = \pm 1$ as usual and $\omega = iq_4 - \mu$. The spectrum (37) has four branches which are split for all momenta, excepting their $|q| = 0$ limit, where they are combined by pairs demonstrating two different gaps

$$E_{\pm} = \frac{I_B}{2} \pm \sqrt{I_K}$$

Their asymptotic behaviour for small $|q|$ has the form

$$\omega_{\pm}(q|\eta) = E_{\pm} + \frac{\eta}{3} |q| + \frac{1}{6} \left( I_B \pm \frac{8}{3} \sqrt{I_K} \right) \frac{q^2}{E_{\pm}^2} + O(q^3)$$

where one branch always has the minimum at finite momentum. Two branches, if $E_- = 0$, lose their gap, but this can occur only under the special constrain on $\mu, T$-parameters.

Unfortunately this scenario is gauge dependent and spectrum (37) differs from the one found in the Feynman gauge [14]. Nevertheless, some mathematical correspondence takes place: the spectrum in the Feynman gauge arises from Eq.(37) if $\omega(\xi)$ will be replaced to $[-\omega(\xi)]$. This correspondence is rather strange and is inherent only to this simple scenario where the mixing is absent between $\mu$ and $m$ parameters. If $\mu, m \neq 0$ the situation is more complicated and the gauge invariance in this case seems to be very problematical.

### 4.2 The collective spectrum of massive Dirac particles for the case $\mu = 0$ in the Coulomb gauge

It is another scenario which can be considered analytically for all $|q|$ in the $T^2$-approximation. Here $I_B = 0$ identically and symmetry $(iq_4 \rightarrow -iq_4)$ is restored. But now the dispersion relation has the additional mass term

$$[(iq_4 - \mu) - K_4]^2 = q^2 (1 + K)^2 + m^2 (1 + Z)^2$$

and all calculations are more complicated. To solve Eq.(40), taking into account only the leading $m/T$-corrections, we use the functions from Eq.(34) with $\mu = 0$

$$\tilde{K}^C(q_4, q) = \frac{I_K}{q^2} \left( 1 + \frac{\xi}{2} \ln \frac{\xi - 1}{\xi + 1} \right) , \quad \tilde{K}_4^C(q_4, q) = -\frac{I_K}{2|q|} \ln \frac{\xi - 1}{\xi + 1}$$

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and \( Z_C(q, q) = -3I_Z \). With these functions Eq.(40) is transformed to be

\[
\omega^4 \left( \xi^2 - 1 \right) - \omega^2 \xi^2 \left[ m^2(1 - 3I_Z)^2 + 2I_K \right] - I_K^2 \xi^4 \left( 1 + \xi \ln \frac{\xi - 1}{\xi + 1} \right) = 0
\]  
(42)

and becomes the quadratic equation in respect to \( \omega^2(\xi) \) (not to \( \omega(\xi) \) as previously). However, this is not the case when \( m, \mu \neq 0 \) simultaneously (see [15] and Appendix B). In this case the dispersion relation generates the full equation of the fourth degree in respect to \( \omega(\xi) \) and requires the numerical calculations. The analytical solution found within Eq.(42) has the form

\[
\omega^2(\xi) = \frac{\xi^2(2I_K + m^2_R)}{2(\xi^2 - 1)} \pm \sqrt{\frac{\xi^4}{(\xi^2 - 1)^2} \left( \frac{b(\xi) I_K^2}{m^2_R(m^2_R + 4I_K)} \right) |q|^2 + O(|q|^4)}
\]  
(43)

and presents the collective Fermi spectrum for all \( |q| \) in the Coulomb gauge. Here \( m^2_R = m^2(1 - 3I_Z)^2 \) and, namely, this quantity differs the Coulomb and Feynman gauges, where \( m^2_R = m^2(1 - 2I_Z)^2 \). Excepting this quantity the found spectra are completely the same in the both gauges, and, in some sense, are gauge invariant. It is not excluded that the additional resummation is necessary to improve the situation, but today this ansatz is unknown. The found spectrum is rather complicated and presents four completely separated branches: two branches (when the plus sign is taken in Eq.(43)) correspond to quasi-particle excitations and the other two (when the minus sign is taken) to quasi-hole ones. These spectrum branches differ in their asymptotic behavior and in many other properties.

The long-wavelength behavior of these spectrum branches (when \( \xi \to \infty \)) has the form

\[
\omega_{\pm}(|q|^2) = M_{\pm}^2 + \left( M_{\pm}^2 \pm \frac{4}{9} \sqrt{\frac{I_K}{m^2_R(m^2_R + 4I_K)}} \right) |q|^2 + O(|q|^4)
\]  
(44)

where the squares of the effective masses are given by

\[
M_{\pm}^2 = \frac{m^2_R}{2} + I_K \pm \sqrt{m^2_R \left( \frac{m^2_R}{4} + I_K \right)}
\]  
(45)

These masses are different for all branches \( M_{\pm} = \frac{1}{2}(\eta m_R \pm \sqrt{m^2_R + 4I_K}) \) and are in agreement with the results found in many other papers. Here \( \eta = \pm 1 \). It is also important that the quasi-hole branches \( \omega_{-}(|q|^2) \) are very sensitive to the choice of the parameters \( m, T \). In many cases these branches are monotonic functions for small \( |q|^2 \), and the well-known minimum [3] disappears. Although this minimum always exists for massless particles, the special conditions are necessary to generate it when \( m \neq 0 \).

In the high-momentum region the asymptotical behaviors found for the quasi-particles and quasi-hole excitations are also completely different. The quasi-particle branches are approximated within a rather usual expression

\[
\omega_{+}(|q|^2) = |q|^2 + (2I_K + m^2_R) - \frac{I_K^2}{|q|^2} \ln \frac{4|q|^2}{2I_K + m^2_R} + O\left( \frac{1}{|q|^2} \right)
\]  
(46)

where the nonanalytic term is not essential. The situation is different for the quasi-hole excitations, which do not exist in the vacuum (when \( T \) and \( \mu \) are equal to zero). They
disappear very rapidly, and their asymptotic behaviour is found to be
\[
\omega_{\pm}(|q|^2) = |q|^2 \left[ 1 + 4 \exp(-\frac{|q|^2 (2I_K + m_R^2)}{I_K^2}) + O(\exp(-\frac{2|q|^2 (2I_K + m_R^2)}{I_K^2})) \right]
\] (47)

In the high momentum region these spectrum branches approach the line \(\omega^2 = |q|^2\) more quickly than (46).

4.3 The collective thermal mass in the Coulomb and Feynman gauges and its gauge dependence

In the most general case when \(\mu, T\) and \(m\) are nonzero simultaneously, the effective mass can be also calculated analytically [14-16]. Here we compare it with the results found above and discuss its gauge dependence. In the Coulomb gauge this mass (see [16] and Appendix B) is given by
\[
\omega^C_{\pm}(0) = \frac{1}{2} \left[ \eta \ m^C_R + I_B \right] \pm \sqrt{\frac{[\eta \ m^C_R + I_B]^2}{4} + (I_K + 2\eta m I_B)}.
\] (48)

where \(m^C_R = m(1 - 3I_Z)\). The mass found is split and has the well-separated mass spectrum, where two values of them pertain to the quasi-particle excitations and other two present the quasi-hole ones. This is evident from Eq.(48) where \(\eta = \pm 1\). However, this spectrum is not the same as in the Feynman gauge [14]
\[
\omega^F_{\pm}(0) = \frac{1}{2} \left[ \eta \ m^F_R - I_B \right] \pm \sqrt{\frac{[\eta \ m^F_R - I_B]^2}{4} + (I_K + 4\eta m I_B)}.
\] (49)

where \(m^F_R = m(1 - 2I_Z)\) and the \(\mu\)-dependence is different. Thus, these masses, excepting the case \(m, \mu = 0\), are gauge dependent and this is true although Eqs.(48),(49) are given beyond the accepted calculation accuracy. If one strictly keeps the \(\epsilon^2\)-accuracy these equations are:
\[
\omega^C_{\pm} = \frac{1}{2} (\eta \pm 1) m^C_R \pm \frac{I_K}{m} \pm \frac{I_B}{2} (5\eta \pm 1), \quad \omega^F_{\pm} = \frac{1}{2} (\eta \pm 1) m^F_R \pm \frac{I_K}{m} \pm \frac{I_B}{2} (7\eta \mp 1)
\] (50)

where we propose that \(m \neq 0\). If at once \(m = 0\) one has the simpler result
\[
\omega^C_{\pm} = \frac{I_B}{2} \pm \sqrt{I_K}, \quad \omega^F_{\pm} = -\frac{I_B}{2} \pm \sqrt{I_K}
\] (51)

which repeats Eq.(38). Eq.(50) leads to the following mass spectrum
\[
m_p^{(C/F)} = \eta \left( m^C_R + \frac{I_B}{m} \right) + 3I_B
\] (52)
\[
m^C_R = \eta \frac{I_K}{m} - 2I_B, \quad m^F_R = \eta \frac{I_K}{m} - 4I_B
\] (53)

which demonstrates that the gauge invariance for the quasi-hole thermal masses in (53) is destroyed more sharply than for the quasi-particle masses in (52). It is very likely, that in this situation the futher resummation easily restores the gauge invariance of the quasi-particle thermal mass even if \(\mu \neq 0\) but the problem with the quasi-holes one, probably, remains.
5 Conclusion

To summarize we have established two kinds of fermion spectra in statistical QED and the same in QCD: the one-particle spectrum and the collective one. The one-particle spectrum is qualitatively the same as the bare one and is gauge invariant, in any case within the one-loop approximation. It has two branches $i q_4 = \mu_R \pm m_R$ and their chemical potential and effective mass are only quantitatively changed due to the interaction with the medium. It is not excluded that within QED this spectrum is gauge invariant in all higher orders as well, if the calculation accuracy is kept strictly. Moreover in hot QED this spectrum seems to be gauge invariant exactly but this is not the case, a priori, for hot QCD where the strong infrared divergencies spoil this scenario.

The collective fermion spectrum was calculated in the Coulomb gauge and was compared with the one found in the Feynman gauge to investigate its gauge dependence. This spectrum is additionally split (usually for all momenta) and its branches always have the nozero thermal masses different for all the spectrum ones, if $m \neq 0$. These masses arise dynamically and are nozero even if the exact chiral symmetry forbids them on the operator level. They are not correlated with bare mass and are generated always: but only for the case $m, \mu = 0$ this thermal mass and the full collective spectrum are gauge invariant. For any other cases the collective spectrum is gauge dependent and its connection with the real excitations should be investigated separately.

However one can see (comparing our results in the Feynman and Coulomb gauges) that gauge invariance is broken variously in different scenarios. This fact demonstrates itself more sharply when $\mu \neq 0$ [due to asymmetry $(i q_4 \to -i q_4)$] and is smoothly expressed (only slightly shifts all spectrum branches) for the symmetric case where $\mu = 0$ and $\omega^2$ (not $\omega$) presents all the spectrum ones. In the last case (namely, when $\mu = 0$), the additional resummation, probably, can change the situation to restore the gauge invariance. But this is very unlikely for the scenario with $\mu \neq 0$ since in this case the spectra found (especially the quasi-hole ones) are rather different and their rearrangement seems to be difficult. Moreover this problem is more aggravated if all calculations are performed beyond the $T^2$-approximations. In this case one at once encounters many difficulties arising due to the ill-infrared behaviour inherent to hot QCD (or QED as well) and all calculations become very ambiguous. The coexistence of the infrared peculiarities of hot gauge theories (like QCD) with the gauge invariance of their results is very problematical and this remains a very serious problem for any calculations performed today (especially for nonperturbative ones).

The additional problems arise if the damping is taken into account [20,21], especially when these calculations are performed beyond the $T^2$-approximation. Here the selfconsistent calculations are only acceptable, where, in the first turn, the calculation accuracy is strictly kept, and the resummation scheme is checked to be, at least, gauge covariant (the Ward identities should be satisfied between the effective propagators and vertices).

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APPENDIX A

The one-loop fermion self-energy in the Coulomb gauge, after the standard summation over the spinor indices and frequencies is presented by Eq.(4). Its calculation strongly differs from the calculations performed in the Feynman gauge only on the last stage when integration over angles is performed. This difference demonstrates itself more pronounced only for the vector \( K_n(q_, q) \) which has the form

\[
K_n^C(q_, q) = e^2 \int \frac{d^3 p}{(2\pi)^3} \left\{ -\frac{p_n \cdot \left( \frac{p^+}{2\epsilon_p} \right)}{2\epsilon_p (p - q)^2} \right\}
+ p_n \left( 1 + \frac{pq}{p^2} \right) \frac{\epsilon_p}{|p|} \left( \frac{1}{(i\epsilon_q q + \epsilon_p^+ + \epsilon_p^-)^2} + \frac{1}{(i\epsilon_q q + \epsilon_p^-)^2 + (p - q)^2} \right)
- \left( 1 + \frac{p - q|q|}{(p - q)^2} \right) \frac{\epsilon_p}{(i\epsilon_q q + \epsilon_p^-)^2 + (p - q)^2} \right\}
\]

and we propose that \( K_n^C(q_, q) = q_n K_n^C(q_, q) \). Then the very lengthy but standard calculations lead to the result presented in Eq.(7) where the function \( K_n^C(q_, q) \) is given by

\[
q^2 K_n^C(q_, q) = I_1 + I_2 + I_3,
\]

\[
I_1 = -e^2 \int_0^\infty \frac{d(|p| p^2}{4\pi^3} \frac{n^+_p - n^-_p}{\epsilon_p} \frac{1}{24|p||q|(\epsilon_p^2 + q_i^2)^2} \right\}
\]

\[
\left\{ -\frac{(p^2 - q^2)^2}{d_F} \ln \frac{(p + q)^2}{(p - q)^2} + h_F d_F \left[ \left( \epsilon_p^2 + q_i^2 \right) \ln(a_p^+ a_F^-) - d_F \ln \frac{a_p^+}{a_F^-} \right]
+ \frac{1}{2} (h_F^2 + d_F^2 - 4p^2 q^2) \left[ \left( \epsilon_p^2 + q_i^2 \right) \ln \frac{a_p^+}{a_F^-} - d_F \ln \frac{a_F^-}{a_F^-} \right] \right\}
\]

\[
I_2 = -e^2 \int_0^\infty \frac{d(|p| p^2}{4\pi^3} \frac{n^+_p + n^-_p}{\epsilon_p} \frac{1}{24|p||q|(\epsilon_p^2 + q_i^2)^2} \right\}
\]

\[
\left\{ \left( \epsilon_p^2 + q_i^2 \right) (p^2 - q^2)^2 \right\}
\]

\[
+ \frac{1}{2} (h_F^2 + d_F^2 - 4p^2 q^2) \left[ \left( \epsilon_p^2 + q_i^2 \right) \ln(a_p^+ a_F^-) - d_F \ln \frac{a_F^-}{a_F^-} \right] \right\}
\]

\[
I_3 = -e^2 \int_0^\infty \frac{d(|p| p^2}{4\pi^3} \frac{n^B_p}{|p|} \left\{ h_B + \frac{1}{8|p||q|} \left[ (h_B^2 + d_B^2) \ln(a_H a_B^-) + 2d_B h_B \ln \frac{a_B^+}{a_B^-} \right] \right\}
\]

The abbreviations are the same as in Eq.(7). The integrals found are essential for the next-to-leading orders beyond \( T^2 \)-approximation. Here only the first integral in Eq.(55) is useful to find \( \mu/T \)-corrections.
APPENDIX B

Our starting point is the dispersion relation (3)

\[
[(iq_i - \mu) - \tilde{K}_4]^2 = q^2 (1 + K)^2 + m^2(1 + Z)^2
\]

(56)

with \(m \neq 0\) and we solve it in the \(T^2\)-approximation with \(\mu \neq 0\). All calculations are performed in the Coulomb gauge taking into account only the leading \(T^2\)-terms and \(\mu/T\)-corrections. In this case the functions within Eq.(56) are:

\[
K(q_i, q) = \frac{I_K}{q^2} \left(1 + \frac{\xi}{2} \ln \frac{\xi - 1}{\xi + 1}\right) - I_B \left(\frac{1}{2} (1 - \xi^2) \ln \frac{\xi - 1}{\xi + 1}\right) \frac{1}{|q|}
\]

\[
-\tilde{K}_4(q_i, q) = \frac{I_K}{2|q|} \ln \frac{\xi - 1}{\xi + 1} - I_B, \quad -Z(q_i, q) = 3I_Z + \frac{I_B}{|q|} \ln \frac{\xi - 1}{\xi + 1}
\]

(57)

making it possible to essential simplify its solution. Here the variable \(\xi = \omega/|q|\) is more convenient than \(|q|\) and as usual \(\omega = (iq_i - \mu)\). All integrals are defined by Eq.(12).

After the simple algebra has been performed within Eq.(56) our result is the equation of the fourth degree with respect to \(\omega(\xi)\)

\[
\omega^4 [\xi^2 - 1] - 2\xi^3 \ln \frac{\xi - 1}{\xi + 1} - I_B [\xi - b(\xi)] + \omega^2 [I_B (1 - b(\xi)^2) - m_R^2 - 2I_K] + 2\omega \xi I_B [I_K (1 + d(\xi)) b(\xi) + \xi d(\xi) (2m_R - I_K)] - I_K^2 + \xi^2 [I_K - 4m^2] = 0
\]

(58)

where \(m_R = m(1 - 3I_Z)\) is the renormalized fermion mass, and the functions \(d(\xi)\) and \(b(\xi)\) are given by

\[
d(\xi) = \frac{\xi}{2} \ln \frac{\xi - 1}{\xi + 1}
\]

\[
b(\xi) = \xi - \frac{1}{2} (1 - \xi^2) \ln \frac{\xi - 1}{\xi + 1}.
\]

(59)

The dispersion relation (58) being very complicated is not solved analytically. However in the long wavelength limit (when \(\xi \to \infty\)) it can be simplified as

\[
\left[\omega^2 - \omega(I_B - \eta m_R) - (I_K - 2\eta I_B)\right] \cdot \left[\omega^2 - \omega(I_B + \eta m_R) - (I_K + 2\eta I_B)\right] = 0
\]

(60)

and one finds the rather simple solution

\[
\omega(0) = \frac{1}{2} \left[\eta m_R + I_B\right] \pm \sqrt{\frac{[\eta m_R + I_B]^2}{4} + (I_K + 2\eta I_B)}
\]

(61)

which demonstrates four well-separated effective masses: two of them pertain to quasi-particle excitations and other two to quasi-holes. Here \(\eta = \pm 1\), and the parameters \(m\) and \(\mu\) are nonzero.

15
References


