Energy and momentum density of thermal gluon oscillations

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Abstract

In the exact propagator for finite temperature gluons the location of the transverse and longitudinal poles in the gluon propagator are unknown functions of wave vector: \( \omega_T(k) \) and \( \omega_L(k) \). The residues of the poles, also unknown, fix the normalization of the one gluon vector potential and thus of the field strength. The naive energy density \( \vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \) is not correct because of dispersion. By keeping the modulations due to the source currents the energy density is shown to be \( \omega_T/V \) and \( \omega_L/V \) regardless of the functional form of \( \omega_T(k) \) and \( \omega_L(k) \). The momentum density is \( k/V \). The resulting energy-momentum tensor is not symmetric.

I. INTRODUCTION

Zero temperature. At zero temperature the canonical operator formalism guarantees that a gluon of frequency \( \omega \) has a field configuration with energy density \( \omega/V \). The same result can be obtained from the propagator because the residue of the pole in the propagator at \( \omega = k \) fixes the potential of a transverse gluon to be

\[
\tilde{A} = \hat{\epsilon} \left( \frac{Z_3}{2k^4} \right)^{1/2} e^{i(\vec{k} \cdot \vec{x} - \hat{\epsilon} k t)}
\]

(1.1)

The resulting electric and magnetic fields have an energy density \( (|\vec{E}|^2 + |\vec{B}|^2)/Z_3 = \omega/V \). In a covariant gauge the factor \( 1/Z_3 \) may be viewed as the constant value of the electric permittivity and magnetic permeability \( 1/\epsilon = \mu = Z_3 \) for on-shell gluons in the vacuum, and the energy density is \( \epsilon|\vec{E}|^2 + |\vec{B}|^2/\mu = \omega/V \). It is important to realize that \( Z_3 \) is a complicated function of the coupling constant and depends on the infrared and ultraviolet regularization. But the statement that the energy density is \( \omega/V \) is exact. It is useful to restate this result. Suppose the full zero-temperature effective action \( \Gamma(A, \psi) \) were known and from this the energy-momentum tensor \( T^{\mu\nu}(A, \psi) \) could be constructed. The energy density \( T^{00} \) would contain terms quadratic in \( A \), as well as cubic and higher powers and also terms containing \( \psi \). If the value of \( A \) is put at the one-gluon value (1.1) it is the quadratic term in the exact \( T^{00} \) that would equal \( \omega/V \). Terms in \( T^{00} \) that contain higher powers of \( A \) do not describe the energy of a single on-shell gluon and should not be included.

Hard thermal loop approximation. At non-zero temperature the analysis of the field energy of thermal gluon oscillations has previously been done within the hard thermal loop
approximation of Braaten and Pisarski [1-3]. This is a consistent high-temperature approximation to QCD in which only the order $T^2$ contributions of all one-loop diagrams are retained. The gluon self-energy is gauge invariant and satisfies $K_{\mu} P_{\mu}^{\nu} = 0$. The poles in the gluon propagator are not on the light cone. For the two polarizations transverse to the wave-vector $k$ the pole is at a frequency $\omega_T(k)$ that is proportional to $g T$ but with a complicated dependence on $k$. There is another pole in the propagator for gluons polarized longitudinally, i.e. parallel to $\vec{k}$, and this pole is at a different frequency $\omega_L(k)$. The energy density of the fields is not proportional to $|\vec{E}|^2 + |\vec{B}|^2$. The analysis of the energy and momentum has been specific to the detailed properties of the hard thermal loop approximation. Because the off-shell amplitudes with any number of external gluons have the remarkable property of being gauge-fixing independent and obeying abelian Ward identities [2,4], it is possible to construct effective actions that generate all the hard thermal loops [3,5-11].

The effective actions are highly non-local and consequently the canonical procedures for constructing the energy-momentum tensor do not apply. The full energy-momentum tensor contains quadratic and all higher all powers of the vector potential $A^\mu$ since they summarize all higher correlations. From the Braaten-Pisarski form of the action [3], I constructed a conserved energy momentum tensor [12] but the quadratic part of that $T^{\mu\nu}$ does not give the physical energy density $\omega_{\alpha}/V$ for the two modes. Other authors have constructed different conserved energy-momentum tensors [13-15], also containing all higher powers of $A^\mu$, and have generalized them to curved space-time [16]. The non-uniqueness of $T^{\mu\nu}$ is expected since a term $\partial^{a}_{\alpha} X^{\alpha\mu\nu}$ will be conserved with respect to $\partial_{\mu}$ for any $X^{\alpha\mu\nu}$ that is antisymmetric under interchange of $\alpha$ and $\mu$. Blaizot and Iancu [15] found an energy-momentum tensor that does give the correct energy densities for the transverse and longitudinal plane waves. It differs from [12] by such a total derivative term. In these analyses the detailed structure of the quadratic, cubic, and all higher correlations are related by the Ward identities. Consequently even in the quadratic approximation the specific $k$-dependence of $\omega_T(k)$ and $\omega_L(k)$ is essential.

The purpose of the present paper is to go beyond the hard thermal loop approximation in computing the field energy of the thermal gluon oscillations. This question is simple enough that it can be answered exactly just from the structure of the exact gluon propagator [17]. The inverse of the propagator gives the quadratic term in effective action. As emphasized above, experience with the hard thermal loop approximation shows that the spatial and temporal non-localities of the quadratic action prevent a unique construction of even the quadratic terms in energy-momentum tensor. In this paper a physical argument will be used along the following lines. First, the quadratic terms in the action are used to obtain the linear field equation $\delta \Gamma(A) / \delta A_\mu(x) = J^\mu(x)$ for the gluon potentials in the presence of an arbitrary color current $J^\mu$. In coordinate space this will turn out to be Maxwell’s equation

$$\partial_\mu H^{\mu\nu} = J^\mu_{\text{free}} + \mathcal{O}(J^2) \tag{1.2}$$

in a media characterized by an electric permittivity $\epsilon(K)$ and a magnetic permeability $\mu(K)$ that are related to the gluon self-energy. Thus even in the exact theory, the gluon oscillations have many features in common with the classical electrodynamics of a classical plasma. Although the color electric field $\vec{E}$ and the color magnetic induction field $\vec{B}$ are produced by all the quarks and gluons, the contribution of the moving quarks and gluons to the self-energy tensor produces polarization and magnetization currents which are incorporated into
the electric displacement field $D_t = \epsilon(K)E_t(K) = H_0t$ and the magnetic field $H_t(K) = B_t(K)/\mu(K) = \epsilon_{\text{en}}H_{mn}/2$. The sources of these macroscopic fields is referred to as the free current $J^\mu_{\text{free}}$. The label ‘free’ does not mean trivial, but rather that the polarization and magnetization currents are not included in this current as that would be double counting. In terms of Feynman diagrams $J^\mu_{\text{free}}$ is one-gluon-irreducible because the self-energy effects of the gluon are contained in the electric permittivity $\epsilon(K)$ and the magnetic permeability $\mu(K)$.

The next step is to construct the energy and momentum density of the gluon fields. The energy and momentum of a source-free plane wave cannot be simply identified. Instead it is necessary to examine field configurations that are not completely source-free and thus are not precisely plane waves. The energy and momentum of such modulated waves can be determined from the currents that radiate those fields. The essential physical step is to realize that the power supplied to the fields and the force exerted on the fields is given by the real part of $J^\alpha_{\text{free}}F_{\alpha\nu}^\ast$ for harmonic time dependence. This is expressed in the identity

$$\partial^\mu (H_{\mu\lambda}^\ast F_{\lambda\nu}^\ast) + \frac{1}{2} H_{\lambda\nu}^\ast (\partial^\nu F_{\lambda\alpha}^\ast) = J^\ast_{\text{free}}F_{\alpha\nu}^\ast, \quad (1.3)$$

Since the right hand side gives the rate of energy and momentum input to the fields by the currents, the left hand side must describe the energy and momentum content of the field configurations. To use (1.3) one must solve Maxwell’s equations in the presence of a weak source $J^\alpha_{\text{free}}$ that supplies energy and momentum to the fields. The source produces modulated fields of the form $\exp(ikz - i\omega t) f(z,t)$ where $\vec{E}, \vec{B}, \vec{D}, \vec{H}$ all have different modulation functions $f$ because of the dispersion. Keeping terms to first order in $\partial f/\partial t$ and $\partial f/\partial z$ reduces the left side of (1.3) to a total derivative:

$$\partial_\mu T^{\mu\nu} = \text{Re}(J^\ast_{\text{free}}F^{\alpha\nu}) \quad (1.4)$$

Since there is absorption in the plasma the energy and momentum of any traveling wave will not truly be constant in time. In applying (1.4) it is necessary to neglect the damping or, equivalently, to neglect the imaginary parts of $\epsilon(K)$ and $\mu(K)$. This is a physical limitation. Thus although the poles in the gluon propagator occur at complex values of frequency, only the real parts $\omega_T(k)$ and $\omega_L(k)$ will be retained. In the limit that the source $J^\alpha_{\text{free}} \to 0$ the modulated fields become plane wave fields whose magnitude is completely fixed by the residues of the poles in the propagator. For these plane wave modes of single gluons the energy-momentum tensor in either mode is

$$T^{\mu\nu}_a = \frac{1}{V} \begin{pmatrix} \omega_a & k^n \\ v_a & v^m_a k^n \end{pmatrix} \quad a = T \text{ or } L \quad (1.5)$$

where $v_T = d\omega_T/dk$ and $v_L = d\omega_L/dk$ are the group velocities. By the theorem of Kobes, Kunstatter and Rebhan [18] the functions $\omega_T(k)$ and $\omega_L(k)$ are gauge invariant. The components of (1.4) satisfy simple physical checks. First, the energy density $T^{00} = \omega/V$ and the momentum density $T^{0n} = k^n/V$. Because the energy and momentum are transported at the group velocity, $T^{00} = v^0 T^{00}$ and $T^{0n} = v^n T^{0n}$. No assumption is made about the form of the dispersion relations $\omega_T(k)$ and $\omega_L(k)$. (Note that $T^{\mu\nu}$ would be symmetric for dispersion relations of the form form $\omega = (k^2 + m^2)^{1/2}$ because then $\vec{v} \omega = \vec{k}$. That is certainly not the case here.)
The analysis performed here is intended to be simple and physical. It is a generalization of old results in optics and classical plasmas [19-21]. In those contexts $\epsilon$ is assumed independent of $k$ and $\mu = 1$. Most treatments are rather unphysical in that they omit discussion of the external current that supplies the energy and momentum content of the fields even thought the modulated fields are only possible with external sources.

The remainder of the paper is organized so that all physics is deduced from the exact thermal gluon propagator. Sec 2 uses the poles in the propagator to obtain the normalized transverse and longitudinal plane waves. Sec 3 derives Maxwell’s equations from the gluon propagator and the relation of $\epsilon(K)$ and $\mu(K)$ to the gluon self-energy. Sec 4 deduces the energy and momentum for the transverse mode. Sec 5 does the same for the longitudinal mode.

II. POLES IN THE THERMAL GLUON PROPAGATOR

A. Gluon propagator in general linear gauges

One of the first obstacles in constructing the exact propagator for finite temperature gluons is that $K_\mu \Pi^{\mu \nu} \neq 0$ [22-25]. It was shown in [17] that for any linear gauge-fixing condition the Slavnov-Taylor implies that $\text{Det} | -K^2 \delta^{\mu \nu} - K^\mu K_\nu + \Pi_\mu^{\nu} | = 0$. Local gauge invariance requires one of the four eigenvalues to be zero for all $K^\mu$. This provides a non-linear condition on the self-energy tensor [24-28]. It is easiest to work with a gauge-fixing that preserves the rotational invariance of the plasma rest frame. These include covariant gauges, Coulomb gauges, and temporal gauges. In rotationally covariant gauges the gluon polarizations that are transverse to the spatial momentum $\hat{k}$ are isolated by the projection tensor

$$ A^{\mu \nu}_{\text{rest}} = \begin{pmatrix} 0 & 0 \\ 0 & -\delta^{mn} + k^m k^n \end{pmatrix}. $$

The gluon four-momentum is $K^\mu$ and a related vector in which the energy and momentum components are reversed is denoted by $\tilde{K}^\mu$:

$$ K^\mu = (\omega, \vec{k}) \quad \tilde{K}^\mu = (k, \omega \hat{k}) $$

The new vector satisfies $\tilde{K}^2 = -K^2$ and $\tilde{K} \cdot K = 0$. The same construction applies to any vector whose spatial component is along $\hat{k}$:

$$ V^\mu = (v_0, v \hat{k}) \quad \tilde{V}^\mu = (v, v_0 \hat{k}) $$

These satisfy $\tilde{V}^2 = -V^2$ and $\tilde{V} \cdot V = 0$. For a linear gauge-fixing condition that preserves the rotational invariance the inverse of the free propagator has the form

$$ \left[ D_{\text{free}}^{-1}(K) \right]^{\nu \mu} = -K^2 \delta^{\nu \mu} + K^{\mu} K^{\nu} - \frac{F^{\mu} F^{\nu}}{\xi} $$

where the gauge-fixing vector $F^{\mu}$ has spatial components parallel to $\hat{k}$. The free propagator is
\[ D_{\mu\nu}(K) = -\frac{A_{\mu\nu}}{K^2} + \frac{\bar{F}_{\mu}^{\nu} \bar{F}_{\nu}^{\mu}}{(F \cdot K)^2} - \xi \frac{K_{\mu} K_{\nu}}{(F \cdot K)^2} \]  
(2.5)

Familiar examples of this include covariant gauges \( [\bar{F}_{\mu} = \bar{K}_{\mu}] \), Coulomb gauges \( [\bar{F}_{\mu} = (k, 0, 0, 0)] \), and temporal gauges \( [\bar{F}_{\mu} = (0, \hat{k})] \). Including interactions gives a proper self-energy tensor of the form

\[ \Pi^{\mu\nu} = \Pi_{T} A^{\mu\nu} + \Pi_{L} \frac{\bar{K}_{\mu} \bar{K}_{\nu}}{K^2} + \Pi_{C} \frac{K_{\mu} \bar{K}_{\nu} + \bar{K}_{\mu} K_{\nu}}{K^2} + \Pi_{D} \frac{K_{\mu} K_{\nu}}{K^2} \]  
(2.6)

because of rotational invariance [24-28]. (For QED and also for QCD in the hard thermal loop approximation \( \Pi_{C} = \Pi_{D} = 0 \).) The Slavnov-Taylor condition leads to \( \Pi_{C} = \sigma \sqrt{(K^2 - \Pi_{L}) \Pi_{D}} \) and the full propagator is

\[ D^{\mu\nu}(K) = -\frac{A_{\mu\nu}}{K^2 - \Pi_{T}} + \frac{H_{\mu} H_{\nu}}{(F \cdot H)^2} \]  
(2.7)

where \( H_{\mu} = K_{\mu} \sqrt{K^2 - \Pi_{L}} + \sigma \bar{K}_{\mu} \sqrt{\Pi_{D}} \). See also [22,27]. The second denominator may be written directly in terms of the self-energy tensor as

\[ \frac{(F \cdot H)^2}{K^2} = (F \cdot K)^2 + \bar{F}_{\mu} F_{\nu} \Pi^{\mu\nu} \]  
(2.8)

The longitudinal denominator for three simple gauge choices are as follows:

- **Covariant**: \( \frac{(F \cdot H)^2}{K^2} = (K^2)^2 + \bar{K}_{\mu} K_{\nu} \Pi^{\mu\nu} \)  
(2.9)

- **Coulomb**: \( \frac{(F \cdot H)^2}{K^2} = k^2(k^2 + \Pi^{00}) \)  
(2.10)

- **Temporal**: \( \frac{(F \cdot H)^2}{K^2} = \frac{k_0^2}{\hat{k}} + \hat{k}^m \hat{k}^n \Pi^{mn} \)  
(2.11)

It is sometimes convenient to parametrize the gauge-fixing vector as \( F_{\mu} = f_1 K_{\mu} + f_2 \bar{K}_{\mu} \) so that \( F \cdot H = K^2(f_1 \sqrt{K^2 - \Pi_{L}} - \sigma f_2 \sqrt{\Pi_{D}}) \).

**B. Transverse plane waves**

The poles in the propagator (2.7) determine the freely propagating plane wave solutions. For the transverse mode the value of \( \omega \) for which \( K^2 = \Pi_{T} \) defines the complex energy \( \omega_T = \omega_T - i\gamma_T \). At this pole

\[ \omega \rightarrow \omega_T^c : \quad D^{mn}(K) \rightarrow \frac{Z_T}{2\omega_T^c(\omega - \omega_T^c)} (\delta^{mn} - \hat{k}^m \hat{k}^n), \]  
(2.12)

where the wave function renormalization factor is defined by

\[ \frac{2\omega_T^c}{Z_T} = \left. \frac{\partial}{\partial \omega}(K^2 - \Pi_{T}) \right|_{\omega_T^c}. \]  
(2.13)
The residue of the pole fixes the normalization of the vector potential for one transverse

gluon to be

\[ \vec{A} = -i \hat{\epsilon} \left( \frac{Z_T}{2 \omega_T^5 V} \right)^{1/2} e^{i (\vec{k} \cdot \vec{r} - \omega^c_T t)} \]  

(2.14)

with \( V \) the quantization volume. The phase factor \( i \) has been chosen for later convenience.

As usual \( \hat{\epsilon} \) is a unit vector satisfying \( \vec{k} \cdot \hat{\epsilon} = 0 \). For definiteness the direction of propagation

will be chosen as the \( z \) axis and the direction of polarization as the \( x \) axis:

\[ A_x = -i \left( \frac{Z_T}{2 \omega_T^5 V} \right)^{1/2} e^{i (kz - \omega^c_T t)} \]  

(2.15)

The corresponding fields are

\[ E_x = \omega^c_T \left( \frac{Z_T}{2 \omega_T^5 V} \right)^{1/2} e^{i (kz - \omega^c_T t)} \quad B_y = k \left( \frac{Z_T}{2 \omega_T^5 V} \right)^{1/2} e^{i (kz - \omega^c_T t)} \]  

(2.16)

It will be shown later that these fields propagate freely in that they satisfy Maxwell’s equations with an electric permittivity \( \epsilon \) and a magnetic permeability \( \mu \) but with with no free current.

### C. Longitudinal plane waves

When \((F \cdot H)^2 = 0\) the propagator (2.7) has a different pole which will be called longi-
dudinal because the resulting electric field is along the spatial wave vector \( \vec{k} \) and there is

no magnetic field. The value of \( \omega \) at which the denominator \((F \cdot H)^2\) vanishes defines the

complex energy \( \omega_L^c \), which has real and imaginary parts \( \omega_L^c = \omega_L - i \gamma_L \). The propagator at

this pole behaves as

\[ \omega \to \omega_L^c : \quad D^{\mu\nu}(K) \to \frac{Z_L}{2 \omega_L^c (\omega - \omega_L^c)} \frac{\tilde{F}^{\mu} \tilde{F}^{\nu}}{K^2} \]  

(2.17)

where the \( \xi \) dependent tensor in (2.7) is omitted. The wave function renormalization factor

for the longitudinal mode is

\[ \frac{2 \omega_L^c}{Z_L} = \frac{\partial}{\partial \omega} \left( \frac{F \cdot H}{K^2} \right)^2 \bigg|_{\omega_L^c} \]  

(2.18)

The coefficient of the pole fixes the normalization of the vector potential for one longitudinal

gluon to be

\[ \vec{A}^\mu = -i \left( \frac{Z_L}{2 \omega_L^c V K_L^2} \right)^{1/2} \tilde{F}^\mu \ e^{-i (\vec{k} \cdot \vec{r} - \omega_L^c t)} \]  

(2.19)

where the phase factor \(-i\) has been chosen for later convenience. Since

\[ \tilde{F}^\mu = f_1 \vec{K}^\mu + f_2 K^\mu = -f_1 \frac{K^2}{k} u^\mu + (f_1 \frac{\omega}{k} + f_2) K^\mu \]  

(2.20)
the potential is
\[
A^\mu = i \frac{f_1}{k} \left( \frac{Z_L K_L^2}{2 \omega_L^c V} \right)^{1/2} u^\mu e^{-i K_L \cdot x + (\cdots) K^\mu}.
\] (2.21)

The omitted terms proportional to \( K^\mu \) do not contribute to the abelian field strength. Only the potential along \( u^\mu \) is non-trivial and in the plasma rest frame the only non-vanishing field is
\[
E_z = f_1 \left( \frac{Z_L K_L^2}{2 \omega_L^c V} \right)^{1/2} e^{i(kz - \omega_L t)}. \] (2.22)

Since there is no magnetic field, the way in which this mode transports momentum is particularly subtle.

**III. MAXWELL’S EQUATIONS FROM THE GLUON PROPAGATOR**

The previous section displayed the plane waves that come from the propagator poles. This section will show that the off-shell gluon propagator implies Maxwell’s equations in a media.

**A. Electric and magnetic phenomena**

*Current conservation.* The gluon propagator determines the quadratic term in the generating functional for one-particle-irreducible diagrams in the presence of a colored source current \( J_\mu^a \):
\[
W(J) = \frac{1}{2} \int d^4x d^4y \ J_\mu^a(x) D^{\mu\nu}(x - y) J_\nu^a(y) + \mathcal{O}(J^3).\] (3.1)

All four components of \( J_\mu \) are independent variables, not constrained by current conservation. In order to obtain the equation of motion for the gauge potentials it is necessary to impose color conservation, \( D_\mu^{ac} J_\mu^a = 0 \), where
\[
D_\mu^{ac} \equiv \delta^{ac} \partial_\mu - g f^{abc \gamma} \frac{\delta W}{\delta J_\gamma^b(x)}.
\] (3.2)

To impose the constraint it is convenient to introduce a Lagrange multiplier function \( \psi(x) \) and vary all four components of \( J_\mu \) independently in the functional
\[
W'(J) = W(J) - \frac{1}{2} \int d^4x \ \psi(x) (D^\mu J_\mu)^a (D^\nu J_\nu)^a\] (3.3)

The resulting vector potential \( A^\mu = \delta W'/\delta J_\mu \) is
\[
A_\mu^a(x) = \int d^4y \ D^{\mu\nu}(x - y) J_\nu^a(y) + \psi(x) \partial^\mu J_\mu^a(x) + \mathcal{O}(J^2).\] (3.4)
In momentum space
\[ A^\mu(K) = \left[D^\mu(K) - \Psi K^\mu K^\nu\right] J^\nu(K) + O(J^2), \]  
(3.5)
where \( \Psi \equiv \psi(-K) \). From now on the color indices will be omitted. To solve for the Lagrange multiplier function, contract both sides of the above with the vector \( F_\mu \) and use the fact that (2.7) satisfies \( F_\mu D^\mu = -\xi H^\nu J^\nu / (F \cdot H) \) to get:

\[ F_\mu A^\mu(K) = -\frac{\xi}{F \cdot H} H^\nu J^\nu(K) - \Psi (F \cdot K) K^\nu J^\nu(K) \]  
(3.6)
Substituting this into (3.5) and rearranging gives

\[ A^\mu(K) = \left[D^\mu(K) + \xi \frac{K^\mu H^\nu}{(F \cdot K)(F \cdot H)}\right] J^\nu(K), \]  
(3.7)
where
\[ A^\mu(K) \equiv \left[g^\mu - \frac{K^\mu F^\nu}{K \cdot F}\right] A^\nu(K). \]  
(3.8)
This potential satisfies the gauge condition \( F_\mu A^\mu(K) = 0 \). In coordinate space (3.7) is the integral equation for \( A^\mu \). By multiplying from the left by the tensor shown below, (3.7) can be converted into the differential equation

\[ \left[(-K^2 + \Pi_T)A^{\alpha \mu} + \left(\frac{F \cdot H}{f_1 K^2}\right) \frac{\tilde{K}^\alpha K^\mu}{K^2}\right] A^\nu(K) = J^\nu_{\text{free}}(K), \]  
(3.9)
where the free current is defined as
\[ J^\alpha_{\text{free}}(K) \equiv J^\alpha(K) + \xi \Pi_D \frac{\tilde{K}^\alpha K^\mu J^\nu(K)}{(f_1 K^2)^2} \]  
(3.10)
and satisfies \( K_\alpha J^\alpha_{\text{free}}(K) = 0 \). The transverse plane waves of Sec 2.2 at \( K^2 - \Pi_T = 0 \) propagate with \( J_{\text{free}} = 0 \). Similarly the longitudinal plane waves of Sec 2.3 at \( (F \cdot H)^2 = 0 \) propagate with \( J_{\text{free}} = 0 \). It is sometimes useful to have the inverse form of (3.9):

\[ A^\nu_{\mu}(K) = -\frac{A^{\mu \nu}}{K^2 - \Pi_T} + K^2 \frac{F^\mu F_\nu}{(F \cdot H)^2} J^\nu_{\text{free}}(K) \]  
(3.11)

Identification of \( \epsilon \) and \( \mu \). Equations (3.9) are the inhomogeneous Maxwell equations in a media characterized by an electric permittivity \( \epsilon \) and a magnetic permeability \( \mu \). To see this one needs to separate the electric and magnetic component of \( \tilde{K}^\mu A^\nu_{\mu}(K) \) by using

\[ \tilde{k} \hat{K}^\mu A^\nu_{\mu}(K)(K) = i\tilde{E}^\nu(K) + \omega (\hat{e}^{\epsilon \mu \nu} - \hat{e}^{\epsilon \mu \nu}) A^{\mu \nu}(K), \]  
(3.12)
where \( \tilde{E}(K) = -i\tilde{k} A^0(K) + i\omega \tilde{A}(K)(K) \). The \( \alpha = 0 \) component of (3.9) is

\[ \epsilon(k, \omega) \hat{\tilde{k}} \cdot \tilde{E}(K) = J^0_{\text{free}}, \]  
(3.13)
with a gauge-dependent electric permittivity given by

$$\epsilon(k, \omega) = \frac{1}{K^2} \left( \frac{F \cdot H}{f_1 K^2} \right)^2 = 1 + \frac{\tilde{E}_\mu \tilde{E}_\nu \Pi^{\mu\nu}}{(F \cdot K)^2}. \quad (3.14)$$

The electric displacement field is $\tilde{D}(K) = \epsilon \tilde{E}(K)$. Next comes the Ampère-Maxwell equation, which is contained in the spatial components of (3.9):

$$(K^2 - \Pi_T)(-\delta^{tm} + \hat{k}^t \hat{k}^m)A^{tm}(K) + \epsilon(k, \omega)\omega \hat{k}^t \tilde{E}_\mu A^t_\mu(K) = J^t_{\text{free}}. \quad (3.15)$$

Using (3.12) this becomes

$$\frac{1}{\mu(k, \omega)} i \hat{k} \times \tilde{B}(K) + \epsilon(k, \omega) i \omega \tilde{E}(K) = \tilde{j}^{\text{ext}}, \quad (3.16)$$

where $\tilde{B}(K) = i \hat{k} \times \tilde{A}(K)$ is the magnetic induction field and the gauge-dependent magnetic permeability is

$$\frac{1}{\mu(k, \omega)} = \frac{1}{k^2} [-K^2 + \Pi_T + \omega^2 \epsilon(k, \omega)]. \quad (3.17)$$

The magnetic field is $\tilde{H}(K) = \tilde{B}(K)/\mu$.

**B. Energy and Momentum Conservation**

In coordinate space the inhomogeneous equations (3.13) and (3.16) take the familiar form

$$\nabla \cdot \tilde{D} = J^0_{\text{free}} \quad \nabla \times \tilde{H} - \frac{\partial \tilde{D}}{\partial t} = \tilde{j}^{\text{free}}. \quad (3.18)$$

The two homogeneous equations, $\nabla \cdot \tilde{B} = 0$ and $\nabla \times \tilde{E} + \partial \tilde{B}/\partial t = 0$, are true for any $A^\mu$.

These results can be expressed in terms of the tensors

$$H^{\mu\nu} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (3.19)$$

Maxwell’s equations in the rest frame of the plasma can be written

$$\partial^\mu H^{\mu\nu}_\text{free} = J^{\nu}_\text{free} \quad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0. \quad (3.20)$$

In order to identify the energy and momentum in the fields, the essential physical step is to recognize that the work done on the fields by the currents and the force exerted on the fields by the currents are given by the real part of $J^{*}_\alpha_{\text{free}} F^{\alpha\nu}$ for harmonic time dependence. This is expressed in the identity

$$\partial^\mu (H^{*}_{\mu\lambda} F^{\lambda\nu}) + \frac{1}{2} H^{*}_{\lambda\alpha} (\partial^\nu F^{\lambda\alpha}) = J^{*}_{\alpha_{\text{free}}} F^{\alpha\nu}, \quad (3.21)$$
which follows directly from (3.20). The $\nu = 0$ component is

$$
- \vec{\nabla} \cdot (\vec{H}^* \times \vec{E}) + \frac{\partial \vec{D}^*}{\partial t} \cdot \vec{E} + \vec{H}^* \cdot \frac{\partial \vec{B}}{\partial t} = - \vec{j}_{\text{free}}^* \cdot \vec{E}.
$$

(3.22)

The right hand side is the power supplied to the fields by the color charges in the source current. The left hand side contains the spatial and temporal changes in the field energy as will later be shown. The $\nu = 1, 2, 3$ components of (3.21) are

$$
\frac{\partial}{\partial t} (\vec{D}^* \times \vec{B}) + (\vec{\nabla} \vec{H}^*)^* B_j - B_j \nabla_j \vec{H}^* + D_j^* \vec{\nabla} E_j

= - \vec{j}_{\text{free}}^* \vec{E} - \vec{j}_{\text{free}}^* \times \vec{B}.
$$

(3.23)

The right hand side is the force acting on the fields by the color charges. This external force is responsible for changes in the field momentum that are contained on the left hand side.

IV. TRANSVERSE GLUON OSCILLATIONS

A. Modulated fields radiated from a current

The transverse plane waves (2.16) that correspond to poles in the propagator at $K^2 = \Pi_T$ are exact solutions of Maxwell’s equations (3.9) with $J_{\text{free}}^\mu = 0$. As discussed in Sec 1, to identify the energy and momentum it is necessary to consider a modulated wave of the form

$$
E_x = e^{i(kz - \omega_T t)} E_T(z, t),
$$

(4.1)

where the amplitude function $E_T(z, t)$ is almost constant. The only significant variation in the amplitude function should be when the wave is emitted or absorbed (i.e. at large $|z|$ and large $|t|$). In Fourier space the amplitude is

$$
E_T(z, t) = \int d\alpha d\beta C_T(\alpha, \beta) e^{i(\alpha z - \beta t)},
$$

(4.2)

where $C_T(\alpha, \beta)$ should have support only for very small wave vector $\alpha$ and very small frequency $\beta$. Note that this is not a superposition of source-free fields, which would be the case if $k + \alpha$ produced a pole at $\omega_T + \beta$. Here $\alpha$ and $\beta$ are independent variables; the wave is slightly off the mass-shell. The total wave vector and total frequency in (4.1) will be denoted

$$
k' \equiv k + \alpha \qquad \omega' \equiv \omega_T + \beta.
$$

(4.3)

Faraday’s law requires that the magnetic induction field be

$$
B_y = e^{i(kz - \omega_T t)} \int d\alpha d\beta \frac{k'}{\omega} C_T(\alpha, \beta) e^{i(\alpha z - \beta t)}.
$$

(4.4)

The constitutive relations then fix the electric displacement field and the magnetic field to be

$$
D_x = e^{i(kz - \omega_T t)} \int d\alpha d\beta \epsilon(k', \omega') C_T(\alpha, \beta) e^{i(\alpha z - \beta t)}
$$

(4.5)
\[ H_y = e^{i(kz - \omega t)} \int \Delta \rho \beta \frac{k'}{\omega' \mu(k', \omega')} C_T(\alpha, \beta) e^{i(\alpha z - \beta t)}. \] (4.6)

The current which radiates these modulated fields can be easily computed from (3.18). The only non-vanishing component of \( J_{\text{free}}^x \) is \( J_x \) with

\[ J_x \text{ free} = e^{i(kz - \omega t)} \int \Delta \rho \beta \left[ \frac{k'^2}{\omega' \mu(k', \omega')} - \omega' \epsilon(k', \omega') \right] C_T(\alpha, \beta) e^{i(\alpha z - \beta t)}. \] (4.7)

In the plane wave limit \( C_T(\alpha, \beta) \to \delta(\alpha) \delta(\beta) E_0 \) and the source will vanish, i.e. \( J_{\text{free}}^x \to 0 \).

The fields \( E_x, D_x, B_y, H_y \) each have slightly different modulations multiplying the \( \exp(ikz - i\omega t) \) factor. It is these slight differences that will account for the energy balance. For example, in the integrand of \( B_y \) the factor \( k'/\omega' \) can be expanded to first order as

\[ B_y = e^{i(kz - \omega t)} \int \Delta \rho \beta \left( \frac{k}{\omega T} + \frac{1}{\omega T} \alpha - \frac{k}{\omega^2 T} \beta + \cdots \right) C_T(\alpha, \beta) e^{i(\alpha z - \beta t)}. \] (4.8)

Only these terms are important since \( C_T(\alpha, \beta) \) has support only for very small values of \( \alpha \) and \( \beta \). Using (4.2) this can be written as

\[ B_y = e^{i(kz - \omega t)} \left( \frac{k}{\omega T} E_T - \frac{i}{\omega T} \frac{\partial E_T}{\partial z} - \frac{i k}{\omega^2 T} \frac{\partial E_T}{\partial t} + \cdots \right). \] (4.9)

In the same fashion

\[ D_x = e^{i(kz - \omega t)} \left( \epsilon E_T - \frac{i}{\epsilon k} \frac{\partial E_T}{\partial z} + \frac{i}{\epsilon} \frac{\partial E_T}{\partial t} + \cdots \right); \] (4.10)

\[ H_y = e^{i(kz - \omega t)} \left( \frac{k}{\omega T \mu} E_T - i \frac{\partial}{\partial k} \left( \frac{k}{\omega T \mu} \right) \frac{\partial E_T}{\partial z} + i \frac{\partial}{\partial \omega} \left( \frac{k}{\omega T \mu} \right) \frac{\partial E_T}{\partial t} + \cdots \right). \] (4.11)

Note that \( \epsilon, \mu \), and the partial derivatives are to be evaluated at \( \omega = \omega_T \). If \( \epsilon \) and \( \mu \) were constants, then the above relations would reduce to \( D_x = \epsilon E_x \) and \( H_y = B_y/\mu \).

**B. Energy Conservation**

For the TEM mode with \( \vec{E} \) polarized along the \( x \) axis, the work-energy theorem (3.22) is

\[ \frac{\partial}{\partial z} \text{Re}(H_y^* E_x) + \text{Re}(\frac{\partial D_x^*}{\partial t} E_x) + \text{Re}(H_y \frac{\partial B_y}{\partial t}) = -\text{Re}(J_{\text{free}}^x E_x) \] (4.12)

As discussed in Sec 1, it is necessary to treat \( \epsilon \) and \( \mu \) as real in order to ultimately obtain a time-independent energy. For harmonic fields and currents the real part of each product gives one-half the instantaneous value as will be seen below. To first order in the derivatives of \( E_T \), the spatial divergence term gives

\[ \frac{\partial}{\partial z} \text{Re}(H_y^* E_x) = \frac{k}{\mu \omega T} \frac{\partial}{\partial z} |E_T|^2, \] (4.13)
as expected. However the time derivatives in (4.12) become

$$
\text{Re}(\frac{\partial D_x^*}{\partial t} E_x) = \frac{\partial}{\partial t} \frac{\partial |E_T|^2}{2} - \frac{\partial \epsilon}{\partial k} \frac{\partial |E_T|^2}{2} - \frac{\partial \epsilon}{\partial k} \frac{\partial |E_T|^2}{2}
$$

(4.14)

$$
\text{Re}(H_y \frac{\partial B_y}{\partial t}) = - \frac{\partial}{\partial \omega} \left( k^2 \frac{\partial}{\partial t} \frac{|E_T|^2}{2} + \frac{k^2}{\omega_T} \frac{1}{\partial \mu} \frac{\partial |E_T|^2}{\partial z} \right).
$$

(4.15)

It is because of spatial dispersion, specifically because $\epsilon$ and $\mu$ depend on $k$, that the right hand sides of (4.14) and (4.15) contain spatial derivatives of $|E_T|^2$. The sum of (4.13)-(4.15) is

$$
\left[ \frac{2k}{\mu \omega_T} - \frac{\partial \epsilon}{\partial k} + \frac{k^2}{\omega_T} \frac{1}{\partial \mu} \frac{\partial |E_T|^2}{\partial z} \right] \frac{\partial |E_T|^2}{\partial t} + \left[ \frac{\partial}{\partial \omega} (\omega \epsilon) - \frac{\partial}{\partial \omega} (k^2 \mu) \frac{\partial |E_T|^2}{\partial t} \right] = -\text{Re}(E_x J_{x,\text{ext}}^*).
$$

(4.16)

The energy flux and energy density are

$$
T^{00}(z, t) = -\frac{\partial f}{\partial k} |E_T|^2 \quad T^{00}(z, t) = \frac{\partial f}{\partial \omega} |E_T|^2.
$$

(4.17)

Relation (3.17) allows $f$ to be expressed entirely in terms of the transverse self-energy: $f(k, \omega) = (K^2 - \Pi_T) / \omega$. The $k$ and $\omega$ derivatives in (4.16) are to be evaluated at $\omega_T$, i.e. at $f = 0$. Therefore the energy is transported at the group velocity of the wave:

$$
T^{00}(z, t)/T^{00}(z, t) = -\left( \frac{\partial f}{\partial k} / \partial \omega \right)_{f=0} = \frac{d\omega_T}{dk}.
$$

(4.18)

Because $\partial f / \partial \omega$ is related to the residue $Z_T$ through (2.13), the energy density can be written

$$
T^{00}(z, t) = \frac{2}{Z_T} |E_T|^2.
$$

(4.19)

*Plane Wave Limit.* The final test of $T^{00}$ is that it give the correct energy density in the plane wave limit: $J_{\text{ext}}^\mu \to 0$. The electric field magnitude of the plane wave TEM gluon oscillation is fixed by the propagator to be $|E_T|^2 = \omega_T Z_T / 2 V$ in (2.16). Thus the energy density of one transverse gluon is $T^{00} = \omega_T / V$ as it should be.

### C. Momentum Conservation

For the TEM mode the equation for momentum conservation (3.23) is

$$
\frac{\partial}{\partial t} (D_x^* B_y) + \frac{\partial H_y^*}{\partial z} D_x + D_x \frac{\partial E_x}{\partial z} = -J_{\text{free}}^* B_y
$$

(4.20)
Substituting the fields (4.9)-(4.11) and taking the real part gives

\[-\frac{k}{\omega_T} \frac{\partial f}{\partial k} \frac{\partial}{\partial z} \frac{|E_T|^2}{2} + \frac{k}{\omega_T} \frac{\partial f}{\partial \omega} \frac{\partial}{\partial t} \frac{|E_T|^2}{2} = -\text{Re}(J_{\text{free}}^t B_y). \tag{4.21}\]

The momentum flux and momentum density are therefore

\[T^{33} = -\frac{k}{\omega_T} \frac{\partial f}{\partial k} |E_T|^2, \quad T^{03} = \frac{k}{\omega_T} \frac{\partial f}{\partial \omega} |E_T|^2 \tag{4.22}\]

The momentum is transported at the group velocity of the wave:

\[T^{33}(z,t)/T^{03}(z,t) = -\left( \frac{\partial f}{\partial k}/\frac{\partial f}{\partial \omega} \right)_{f=0} = \frac{d\omega_T}{dk}. \tag{4.23}\]

Using (2.13) one can write the momentum density as

\[T^{03} = \frac{2k}{\omega_T Z_T} |E_T|^2 \tag{4.24}\]

With the plane wave value of \(E_T\) this gives the correct value \(T^{03} = k/V\).

V. LONGITUDINAL GLUONIC OSCILLATIONS

A. Modulated fields radiated from a current

The longitudinal plane wave (2.22) comes from a pole in the propagator at \((F \cdot H)^2 = 0\) satisfying Maxwell’s equations (3.9) with \(J_{\text{free}} = 0\). To identify the energy and momentum content one needs a modulated wave

\[E_z = e^{i(kz - \omega_L t)} E_L(z,t), \tag{5.1}\]

where the amplitude \(E_L(z,t)\) is slowly varying in space and in time. The only significant variation should be at \(|z| \to \infty\) and \(|t| \to \infty\). It has the Fourier representation

\[E_L(z,t) = \int d\alpha d\beta C_L(\alpha, \beta) e^{i(\alpha z - \beta t)}, \tag{5.2}\]

where the function \(C_L(\alpha, \beta)\) has support only for small \(\alpha\) and small \(\beta\). The electric field has an associated displacement field

\[D_z = e^{i(kz - \omega_L t)} \int d\alpha d\beta \epsilon(k', \omega') C_L(\alpha, \beta) e^{i(\alpha z - \beta t)}, \tag{5.3}\]

where \(k' \equiv k + \alpha\) and \(\omega' \equiv \omega_L + \beta\) as before. Both \(\vec{H}\) and \(\vec{B}\) are identically zero. The modulated wave requires an external gluon current with components

\[J_{\text{free}}^z = e^{i(kz - \omega_L t)} \int d\alpha d\beta \dot{\epsilon}(k', \omega') C_L(\alpha, \beta) e^{i(\alpha z - \beta t)} \tag{5.4}\]
\[ J_0 \text{free} = e^{i(kz - \omega_L t)} \int d\alpha d\beta \ i k' \epsilon(k', \omega') C_L(\alpha, \beta) e^{i(\alpha z - \beta t)} \]  

(5.5)

In the plane wave limit \( C_L \to \delta(\alpha) \delta(\beta) \) and therefore \( \epsilon(k', \omega') \to \epsilon(k, \omega_L) = 0 \). This means that \( D_z \to 0, J_z \to 0, \) and \( J_0 \to 0 \) eventually.

Before taking the plane wave limit, the modulations in \( D_z \) are different than those of \( E_z \) because the former includes the field of the polarization charges. To approximate \( D_z \), expand \( \epsilon(k', \omega') \) in a Taylor series:

\[ D_z = e^{i(kz - \omega_L t)} \int d\alpha d\beta \left( \alpha \frac{\partial \epsilon}{\partial k} + \beta \frac{\partial \epsilon}{\partial \omega} + \cdots \right) C_L(\alpha, \beta) e^{i(\alpha z - \beta t)}. \]  

(5.6)

In terms of \( E_L \) this means that

\[ D_z = e^{i(kz - \omega_L t)} \left( -i \frac{\partial \epsilon}{\partial k} \frac{\partial E_L}{\partial z} + i \frac{\partial \epsilon}{\partial \omega} \frac{\partial E_L}{\partial t} + \cdots \right). \]  

(5.7)

The derivatives of \( \epsilon \) are evaluated at \( \epsilon = 0 \).

**B. Energy Conservation**

Since the only nonvanishing fields for the modulated longitudinal mode are \( E_z \) and \( D_z \), the work-energy theorem (3.22) is

\[ \Re\left( \frac{\partial D_z^*}{\partial t} E_z \right) = -\Re(J_{\text{free}}^* E_z). \]  

(5.8)

Despite its rather trivial appearance this equation does describe the flow of energy in the plasma. Because of (5.7)

\[ \frac{\partial D_z^*}{\partial t} E_z = E_L \left( -\omega_L \frac{\partial \epsilon}{\partial k} \frac{\partial E_L}{\partial z} + \omega_L \frac{\partial \epsilon}{\partial \omega} \frac{\partial E_L}{\partial t} + \cdots \right). \]  

(5.9)

Taking the real part, with \( \epsilon \) assumed real, gives

\[ -\omega_L \frac{\partial \epsilon}{\partial k} \frac{\partial |E_L|^2}{\partial z} + \omega_L \frac{\partial \epsilon}{\partial \omega} \frac{\partial |E_L|^2}{\partial t} = -\Re(E_z J_{\text{free}}^*). \]  

(5.10)

Thus the energy flux and the energy density are

\[ T^{30} = -\omega_L \frac{\partial \epsilon}{\partial k} |E_L|^2, \quad T^{00} = \omega_L \frac{\partial \epsilon}{\partial \omega} |E_L|^2, \]  

(5.11)

with the derivatives evaluated at \( \epsilon = 0 \). The energy is transported at the group velocity of the longitudinal wave:

\[ \frac{T^{30}(z, t)}{T^{00}(z, t)} = -\left( \frac{\partial \epsilon}{\partial k} / \frac{\partial \epsilon}{\partial \omega} \right)_{\epsilon=0} = \frac{d\omega_L}{dk}. \]  

(5.12)

Because of (2.18) and (3.14)

\[ \frac{\partial \epsilon}{\partial \omega} \bigg|_{\epsilon=0} = \frac{1}{f^2 K^2 Z_L} \frac{2\omega_L}{Z_L}. \]  

(5.13)
The energy density is therefore

\[ T^{00} = \frac{2\omega_T^2}{f_1^2 K_L^2 Z_L} |E_L|^2 \] (5.14)

**Plane Wave Limit.** Whenever the external current is present the space and time derivatives of \( |E_L(z, t)|^2 \) prescribe the energy transport within the plasma. When the external current is removed, the value of \( T^{00} \) remains constant. The value of that constant is the energy contained in the wave. The pole in the propagator fixes the amplitude for the free longitudinal gluon mode to be \( |E_L|^2 = f_1^2 Z_L K_L^2/(2\omega_L V) \) as shown in (2.22). The energy density of this mode is therefore \( T^{00} = \omega_L/V \).

**C. Momentum conservation**

For the longitudinal mode the force equation (3.23) is

\[ \frac{\partial D_z^*}{\partial z} E_z = -J_{\text{free}}^0 E_z. \] (5.15)

Note that although \( J_{\text{free}} \) is non-zero, it exerts no force on the plasma charges in this mode because \( \vec{B} = 0 \). Substituting (5.7) and taking the real part gives

\[ -k \frac{\partial \epsilon}{\partial k} \frac{\partial |E_L|^2}{\partial z} + k \frac{\partial \epsilon}{\partial \omega} \frac{\partial |E_L|^2}{\partial t} = -\text{Re}(J_{\text{free}}^0 E_z). \] (5.16)

Thus the momentum flux and momentum density are

\[ T^{33} = -k \frac{\partial \epsilon}{\partial k} |E_L|^2 \quad T^{03} = k \frac{\partial \epsilon}{\partial \omega} |E_L|^2. \] (5.17)

Momentum is transported at the group velocity of the wave

\[ T^{33}(z, t)/T^{03}(z, t) = -\left( \frac{\partial \epsilon}{\partial k} / \frac{\partial \epsilon}{\partial \omega} \right)_{\omega = 0} = \frac{d\omega_L}{dk}. \] (5.18)

The momentum density can also be written

\[ T^{03} = \frac{2k\omega_L}{f_1^2 K_L^2 Z_L} |E_L|^2 \] (5.19)

In the plane wave limit \( |E_L|^2 = f_1^2 Z_L K_L^2/(2\omega_L V) \) so that \( T^{03} = k/V \), which is the correct result.

**VI. CONCLUSION**

Regardless of the functional dependence of \( \omega_T \) and \( \omega_L \) on momentum, temperature, and quark masses the energy density carried by the field of a single gluon of either type has been shown to be \( \omega_T/V \) and \( \omega_L/V \), respectively.

**ACKNOWLEDGMENTS**

This work was supported in part by National Science Foundation grant PHY-9630149.
REFERENCES


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