The electromagnetic field near a dielectric half–space

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Abstract. We compute the expectations of the squares of the electric and magnetic fields in the vacuum region outside a half–space filled with a uniform non–dispersive dielectric. This gives predictions for the Casimir–Polder force on an atom in the ‘retarded’ regime near a dielectric. We also find a positive energy density due to the electromagnetic field. This would lead, in the case of two parallel dielectric half–spaces, to a positive, separation–independent contribution to the energy density, besides the negative, separation–dependent Casimir energy. Rough estimates suggest that for a very wide range of cases, perhaps including all realizable ones, the total energy density between the half–spaces is positive.

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1. Introduction

In this paper, we investigate the quantum electromagnetic field in the vacuum region outside a half-space filled with a uniform, non-dispersive dielectric. We compute the expectations of the squares of the electric and magnetic fields in this region. We have two motivations for this.

First, the problem is natural in the study of quantum optics. Indeed, other workers have already investigated some aspects of this situation, for example, the effects of a nearby dielectric on atomic transition rates (see, e.g., Khosravi and Loudon 1991, 1992). Here we compute the expectations of the squares of the electric and magnetic fields, thus providing predictions of the Casimir–Polder force on an (electrically or magnetically) polarizable atom near the dielectric. These predictions test the ultraviolet renormalization of the theory at a deeper level than do the transition–rate ones.

Our main motivation, however, comes from the hypothesized uses of negative energy densities to fuel exotic general-relativistic and thermodynamic effects. Serious workers have considered the possibility that negative energy densities might give rise to “worm holes,” “warp drives” and “time machines.” Such predictions depend on being able to generate persistent negative energy densities. At present, the only way that this might be achieved within reasonably well-understood physics is via Casimir–type effects. In the original Casimir (1948) effect, for example, the energy density due to the quantum electromagnetic field between two perfect parallel plane conductors is predicted to be negative. It should immediately be remarked that this negative energy density has never been directly observed. Still, it is this prediction which has generated an enormous amount of theoretical work, because the possible consequences are so spectacular.

We wanted to know what would happen to the prediction of negative energy densities if the plates were no longer idealized as perfect conductors. A realistic treatment of this would require a theory of the quantum electromagnetic field in inhomogeneous absorptive and dispersive media at finite temperature. Such theories are only now under development (see, e.g., Matloob et al. 1995), so it seems wise to consider as a first step the case of a non-absorptive, non-dispersive medium at zero temperature. Thus we shall consider the case of a half-space filled with a material of (frequency-independent) dielectric constant $\epsilon$. While the case of a perfect conductor is formally the limit $\epsilon \uparrow \infty$ of this, an imperfect conductor is not well represented by such a model with $\epsilon$ finite. So we shall not be able to make any positive predictions about the behavior of real conductors.

Still, our results are strong enough to bear on the case of conductors. We shall find that, for dielectrics, finite–$\epsilon$ effects cannot be neglected, especially in computations of

\[ \text{Laboratory experiments measure the force between the plates, that is, the component } \hat{T}_{zz} \text{ of the stress–energy (Sparnaay 1957, 1958; Lamoreaux 1997, Bordag et al. 1998). The energy density is } \hat{T}_{tt}. \text{ These two operators do not commute. There is a connection between them, in that the long–time average of the force is minus the gradient of the energy, but present experiments seem far from being able to measure } \hat{T}_{tt}. \text{ This operator may as a matter of principle not be directly observable; see Helfer 1998.} \]
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the electromagnetic contribution to the energy density. In the case of two parallel half-spaces, these finite-ε corrections do not alter the attractive nature of the Casimir force, but may contribute a positive, separation-independent energy density which dominates the negative, separation-dependent, Casimir energy density. This strongly suggests that only after a careful treatment of the physics of real conductors will we know whether the perfect-conductor idealization is adequate for computing the energy density in such cases.

It is not easy to say accurately and briefly why a finite dielectric constant should modify the energy density to this degree, because the physics is non-local and depends on quantum interference. The presence of a polarizable medium in a region alters the field operators, by causing reflection and refraction of modes at the boundary. If the geometry is particularly simple (a plane interface) and the reflection sufficiently idealized (a perfect conductor), one has a great deal of cancellation. Small deviations from these idealizations can potentially lead to large effects. This is because the energy density and the squares of the field strengths are defined by ultraviolet-divergent integrals (and must be renormalized).

To explain the situation more quantitatively, we first review some aspects of the Casimir effect, and then discuss the idealizations that have been made and how they might be expected to be modified in a more realistic treatment.

Between two perfect parallel plane conductors, one finds that the renormalized energy density is given by

$$\langle \hat{T}_{00} \rangle_{\text{ren}} = \frac{1}{2} \langle \hat{E}^2 + \hat{B}^2 \rangle_{\text{ren}} = -\frac{\pi^2 \hbar c}{720 \lambda^4},$$

where λ is the distance between the plates. That this is independent of position can be shown on invariance grounds (and relies on the ideal, perfect-conductor boundary conditions). However, the electric and magnetic fields are not position-independent; one finds

$$\langle \hat{E}^2 \rangle_{\text{ren}} = -\frac{\pi^2 \hbar c}{720 \lambda^4} + \frac{\pi^2 \hbar c}{16 \lambda^4} \frac{3 - 2 \sin^2(\pi z/\lambda)}{\sin^4(\pi z/\lambda)}$$

$$\langle \hat{B}^2 \rangle_{\text{ren}} = -\frac{\pi^2 \hbar c}{720 \lambda^4} - \frac{\pi^2 \hbar c}{16 \lambda^4} \frac{3 - 2 \sin^2(\pi z/\lambda)}{\sin^4(\pi z/\lambda)}$$

at distance z from one plate. Near one plate, as z ↓ 0, we find the asymptotic forms

$$\langle \hat{E}^2 \rangle_{\text{ren}} \sim +\frac{3 \hbar c}{16 \pi^2 z^4}$$

$$\langle \hat{B}^2 \rangle_{\text{ren}} \sim -\frac{3 \hbar c}{16 \pi^2 z^4}.$$

In other words, the renormalized expectations of \( \hat{E}^2 \) and \( \hat{B}^2 \) both diverge near a perfectly conducting plate, but there is a perfect cancellation between the divergent terms, leaving only a finite result.

Several comments on this are in order. First, the negative expectation of \( \hat{B}^2 \) occurs because it is a renormalized quantity, and means that the fluctuations of \( \hat{B} \) are less than those of the Minkowski vacuum. Second, the divergences of \( \langle \hat{E}^2 \rangle_{\text{ren}} \) and
\( \langle \hat{B}^2 \rangle_{\text{ren}} \) as \( z \downarrow 0 \) are not expected to be physical, but rather arise from the idealized boundary conditions used. A real conductor would not be well approximated by a perfect conductor within atomic distances, and probably not within its plasma wavelength. Thus the expressions (2)–(5) are really only expected to be valid when one is sufficiently far from the conductor to neglect atomic structure and finite skin–depth.

Still, one is led to ask what would happen if the antisymmetry between the divergent parts of \( \langle \hat{E}^2 \rangle_{\text{ren}} \) and \( \langle \hat{B}^2 \rangle_{\text{ren}} \) could be disturbed. Could one produce energy densities much greater in magnitude than the Casimir expression (1)? A natural way to try to do this is to replace the perfect conductor by a dielectric, and this is what we have done here. Of course, our model is not expected to be accurate within atomic distances or even scales of the order of a skin–depth. Still, we shall be able to draw some interesting conclusions.

We are able to compute \( \langle \hat{E}^2 \rangle_{\text{ren}}, \langle \hat{E}^2 \rangle_{\text{ren}}, \langle \hat{B}^2 \rangle_{\text{ren}} \) and \( \langle \hat{B}^2 \rangle_{\text{ren}} \) explicitly, as functions of the distance \( z \) from the dielectric boundary and of the dielectric susceptibility \( \chi \). The expressions have the form

\[
\frac{\eta \hbar c}{z^4}
\]

where the \( \eta \)s are transcendental functions of \( \chi \). (See equations 28, 30, 32, 34.) We find in particular that the energy density in the vacuum half–space has the form \( \eta \rho \hbar c / z^4 \), where \( \eta \rho \) is a positive function of \( \chi \). This means that the total energy per unit surface area of the electromagnetic field on the vacuum side,

\[
\int_0^{\infty} (\eta \rho \hbar c / z^4) \, dz ,
\]

is divergent. This is unphysical and again can be ascribed to the oversimplification of our model, where all modes, of whatever frequency, are equally affected by the dielectric. In a more realistic model, the dielectric’s atomic structure would be taken into account. This would mean that at small distances (of the order of the skin depth probably and at the atomic scale certainly) the energy density would not be given by \( \eta \rho \hbar c / z^4 \), but by some other, presumably finite, expression. Correspondingly, we ought really to think of our theory as an effective field theory valid only up to frequencies corresponding to wavelengths of order the skin depth or so.

In the next section, we outline the technical details of the computations. In Section 3, we summarize the asymptotic behaviors of the squares of the \( \eta \)s for the squares of the fields, and present the graphs of these functions. Section 4 summarizes the behavior of the expectation of the stress tensor. Section 5 contains discussions of the significance of our results, and Section 6 recapitulates the main conclusions.

2. The Computation

2.1. The Orthonormal Eigenmodes

The case of a half–space uniformly filled with a dielectric has been studied earlier, and we shall use the orthonormal eigenmodes as given by Carniglia and Mandel (1971).
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We shall take the $z$ axis to be normal to the interface, with $z$ increasing in the vacuum region. We take advantage of the translational symmetries in time and in the $x_T = (x, y)$ directions to resolve all modes by Fourier transforms in these variables, with Fourier transform variables $\omega$ and $k_T$. These Fourier transform variables thus retain their senses on both sides of the interface.

The dielectric constant is $\epsilon = 1 + \chi$. The wave number in the $z$–direction is $k$ in the vacuum and $\hat{k}$ in the dielectric. Thus we have

\[ \hat{k}^2 + k_T^2 = \epsilon \omega^2 \quad \text{for} \quad z < 0 \quad \text{(dielectric)} \]
\[ k^2 + k_T^2 = \omega^2 \quad \text{for} \quad z > 0 \quad \text{(vacuum)}. \]

In what follows $\hat{k}_T$ and $\hat{e}_z$ are the unit vectors in the $k_T$ and $z$–directions. In later sections, hats will indicate field operators, too, but no confusion should arise.

We shall only need the modes on the vacuum side of the interface. The transverse electric component of the ‘electric’ field mode (where $E$ is normal to the plane of incidence) incident from the left ($\hat{k} > 0$), is

\[ E^E_{k_T} = (2\epsilon)^{-1/2} \frac{2\hat{k}}{k + k} e^{ikz} e^{ik_T x_T} (k_T \times \hat{e}_z). \]

The transverse magnetic component of the ‘electric’ field mode incident from the right ($\hat{k} > 0$) is

\[ E^M_{k_T} = (\sqrt{2} \omega)^{-1} \frac{2\hat{k}}{k + \epsilon k} e^{ikz} e^{ik_T x_T} (k_T \hat{e}_z - k k_T). \]

The transverse electric component of the ‘electric’ field mode incident from the right ($k > 0$) is

\[ E^E_{k_T} = \sqrt{2}^{-1} \left( e^{-ikz} + \frac{k - \hat{k}}{k + k} e^{ikz} \right) e^{ik_T x_T} (k_T \times \hat{e}_z). \]

The transverse magnetic component of the ‘electric’ field mode incident from the right ($k > 0$) is

\[ E^M_{k_T} = (\sqrt{2} \omega)^{-1} \left( e^{-ikz} (k_T \hat{e}_z + k k_T) + \frac{\epsilon k - \hat{k}}{\epsilon k + k} e^{ikz} (k_T \hat{e}_z - k k_T) \right) e^{ik_T x_T}. \]

The transverse electric component of the ‘magnetic’ field mode incident from the left ($\hat{k} > 0$) is

\[ B^E_{k_T} = (2\epsilon \omega^2)^{-1/2} \frac{2\hat{k}}{k + k} e^{ikz} e^{ik_T x_T} (k k_T - k_T \hat{e}_z). \]

The transverse magnetic component of the ‘magnetic’ field mode incident from the left ($\hat{k} > 0$) is

\[ B^M_{k_T} = 2^{-1/2} \frac{2\hat{k}}{k + \epsilon k} e^{ikz} e^{ik_T x_T} (k_T \times \hat{e}_z). \]
The transverse electric component of the ‘magnetic’ field mode incident from the right 
\((k > 0)\) is
\[
B_{kk'T}^E = -2^{-1/2} \omega^{-1} \left( e^{-ikz} \left( k \hat{k}_T + k_T \hat{e}_z \right) - \frac{k - \tilde{k}}{k + \tilde{k}} e^{ikz} \left( k \hat{k}_T - k_T \hat{e}_z \right) \right) e^{ikT \cdot x} .
\] (16)
The transverse magnetic component of the ‘magnetic’ field mode incident from the right 
\((k > 0)\) is
\[
B_{kk'T}^M = 2^{-1/2} \left( e^{-ikz} + \frac{ek - k}{k + k} e^{ikz} \right) e^{ikT \cdot x} \left( \hat{k}_T \times \hat{e}_z \right) .
\] (17)

The electric and magnetic field operators are thus given by
\[
\hat{E}(x, t) = \frac{1}{(2\pi)^3} \int_{k > 0} d^3 \tilde{k} \sum_{\lambda = E, M} \sqrt{\omega} \left( \hat{\alpha}^\lambda_{kk'T} E^\lambda_{kk'T} e^{-i\omega t} + \text{h.c.} \right)
\]
\[
+ \frac{1}{(2\pi)^3} \int_{k > 0} d^3 k \sum_{\lambda = E, M} \sqrt{\omega} \left( \hat{\alpha}^\lambda_{kk'T} B^\lambda_{kk'T} e^{-i\omega t} + \text{h.c.} \right) ,
\] (18)
and
\[
\hat{B}(x, t) = \frac{1}{(2\pi)^3} \int_{k > 0} d^3 \tilde{k} \sum_{\lambda = E, M} \sqrt{\omega} \left( \hat{\delta}^\lambda_{kk'T} \tilde{E}^\lambda_{kk'T} e^{-i\omega t} + \text{h.c.} \right)
\]
\[
+ \frac{1}{(2\pi)^3} \int_{k > 0} d^3 k \sum_{\lambda = E, M} \sqrt{\omega} \left( \hat{\delta}^\lambda_{kk'T} \tilde{B}^\lambda_{kk'T} e^{-i\omega t} + \text{h.c.} \right) ,
\] (19)
where the creation and annihilation operators satisfy the commutation relations
\[
\left[ \hat{\alpha}^\lambda_{kk'T}, \hat{\delta}^\lambda_{kk'T} \right] = 4\pi^3 \hbar \delta_{\lambda\lambda'} \delta(\tilde{k} - k') \delta(k_T - k_T') ,
\] (20)
and
\[
\left[ \hat{\delta}^\lambda_{kk'T}, \hat{\delta}^\lambda_{kk'T} \right] = 4\pi^3 \hbar \delta_{\lambda\lambda'} \delta(k - k') \delta(k_T - k_T') .
\] (21)

2.2. Computation of \( \hat{E}_z^2 \)

We shall outline the computation of \( \hat{E}_z^2 \). Computations of the squares of the other field components follow the same pattern.

We use a standard point-splitting in imaginary time, and set \( i\tau = t' - t \). Then we have
\[
\langle \hat{E}_z^2 \rangle = \frac{1}{(2\pi)^3} \int_{k > 0} d^3 \tilde{k} \frac{k_T^2}{2\omega} \left( \frac{2\tilde{k}}{k + \tilde{k}} \right) \left( \frac{2\tilde{k}}{k + \tilde{k}} \right)^* e^{i(k-k')z} e^{-\omega \tau}
\]
\[
+ \frac{1}{(2\pi)^3} \int_{k > 0} d^3 k \frac{k_T^2}{2\omega} \left( e^{-ikz} + \frac{ek - \tilde{k}}{ek + \tilde{k}} e^{ikz} \right) \left( e^{-ikz} + \frac{ek - \tilde{k}}{ek + \tilde{k}} e^{ikz} \right)^* e^{-\omega \tau} .
\] (22)
We rewrite the integral over \( \tilde{k} > 0 \) as an integral over \( k > 0 \) (representing plane waves) plus an integral over \( 0 < k < \omega \sqrt{\lambda} \) (representing evanescent waves):
\[
\langle \hat{E}_z^2 \rangle = \frac{1}{(2\pi)^2} \int_0^\infty d\omega \int_0^\omega dk (\omega^2 - k^2) \left( 1 + \frac{ek - \tilde{k}}{ek + \tilde{k}} \cos 2kz \right) e^{-\omega \tau} .
\]
Integrating over renormalized expectation value:

\[ + \frac{1}{(2\pi)^2} \int_0^\infty d\omega \int_0^\infty d\kappa (\omega^2 + \kappa^2) \frac{2e\kappa \tilde{k}}{k^2 + e^2\kappa^2} e^{-2e\kappa \varepsilon - \omega \tau}, \]

where we have used the relationships

\[ k\tau^2 = \begin{cases} \omega^2 + \kappa^2 & \text{for } \tilde{k} < \omega \sqrt{\chi} \\ \omega^2 - \kappa^2 & \text{for } \tilde{k} > \omega \sqrt{\chi} \end{cases} \]

and have performed the simple polar angle integration.

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we obtain

Thus we have reduced the problem of finding the expectation value of the square of the \(z\)-component of the electric field to a one-dimensional integral. The integral can be evaluated using contour integration in the complex plane and by exploiting Cauchy's residue theorem. After integrating and extensive algebra we obtain the following formally divergent expression for the expectation value:

\[ \langle \hat{E}_z^2 \rangle = \int_0^\infty d\omega \int_0^1 d\xi \omega^3 (1 - \xi^2) \left( 1 + \frac{\xi - \sqrt{\chi + \xi^2}}{\xi + \sqrt{\chi + \xi^2}} \cos 2\omega \xi \tau \right) e^{-\omega \tau} \]

\[ + \frac{1}{(2\pi)^2} \int_0^\infty d\omega \int_0^1 d\xi \omega^3 (1 + \sqrt{\chi \xi^2}) \frac{2e\xi \sqrt{\chi + \xi^2}}{1 + (\xi^2 - 1)\xi^2} e^{-2e\sqrt{\chi} \xi \tau} e^{-\omega \tau}. \]

Integrating over \(\omega\) gives

\[ \langle \hat{E}_z^2 \rangle = \frac{hc}{(2\pi)^2} \int_0^1 \left[ \frac{6 - \xi^2 - 2(1 - \xi^2)}{\tau^2} + 6(1 - \xi^2) \frac{\xi - \sqrt{\chi + \xi^2}}{\xi + \sqrt{\chi + \xi^2}} \frac{16z^4 \xi^4 - 24z^2 \xi^2 \tau^2 + \tau^4}{(4z^2 \xi^2 + \tau^2)^4} \right. \]

\[ + \left. \frac{12e\sqrt{\chi}(1 + \chi \xi^2)\sqrt{1 - \xi^2}}{(1 + (\xi^2 - 1)\xi^2)(2z\sqrt{\chi} \xi + \tau)^4} \right] d\xi. \]

Thus we have reduced the problem of finding the expectation value of the square of the \(z\)-component of the electric field to a one-dimensional integral. The integral can be evaluated using contour integration in the complex plane and by exploiting Cauchy’s residue theorem. After integrating and extensive algebra we obtain the following formally divergent expression for the expectation value:

\[ \langle \hat{E}_z^2 \rangle = \lim_{\tau \to 0} \left( \frac{hc}{\pi^2 \tau^4} + \frac{hc}{(2\pi)^2 \tau^4} \left[ \frac{1}{16 \chi^{3/2}} \left( 2\sqrt{\chi}(6e^2 - 3e^{3/2} - 2\chi) \right. \right. \right. \]

\[ + 6e(1 - 2e^2 + 2\chi) \ln(\sqrt{\tau + \sqrt{\chi}}) \]

\[ + \left. \left. \left. + \frac{6e^2(e^2 - \chi - 1)}{\sqrt{\varepsilon^2 - 1}} \ln \left( \frac{\sqrt{\varepsilon + 1} - 1}{\sqrt{\varepsilon + 1} + 1} (\sqrt{\varepsilon + 1} + \sqrt{\varepsilon})^2 \right) \right] + O(\tau) \right) \right). \]

Subtracting the (again divergent) vacuum (Minkowski space) expectation value \( \langle \hat{E}_z^2 \rangle_{\text{Minkowski}} = \lim_{\tau \to 0} \frac{hc}{\pi^2 \tau^4} \) and taking the limit as \(\tau \to 0\) gives the exact renormalized expectation value:

\[ \langle \hat{E}_z^2 \rangle_{\text{ren}} = \frac{hc}{(2\pi)^2 \tau^4} \left[ \frac{1}{16 \chi^{3/2}} \left( 2\sqrt{\chi}(6e^2 - 3e^{3/2} - 2\chi) \right. \right. \right. \]

\[ + 6e(1 - 2e^2 + 2\chi) \ln(\sqrt{\tau + \sqrt{\chi}}) \]

\[ + \left. \left. \left. + \frac{6e^2(e^2 - \chi - 1)}{\sqrt{\varepsilon^2 - 1}} \ln \left( \frac{\sqrt{\varepsilon + 1} - 1}{\sqrt{\varepsilon + 1} + 1} (\sqrt{\varepsilon + 1} + \sqrt{\varepsilon})^2 \right) \right] \right) \]

\[ = \frac{hcn^{E}_z}{z^4}, \]
The renormalized expectations of the squares of the other components can be calculated by the same techniques. They are given by:

\[ \langle E^2 \rangle_{\text{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[ \frac{1}{16\chi^{3/2}} \left( 2\sqrt{\chi}(6 - 3\sqrt{\epsilon} - 2\chi) - 6(1 - 2\epsilon\chi) \right) \right. \\
\left. \cdot \ln(\sqrt{\epsilon} + \sqrt{\chi}) - \frac{6\epsilon^2\chi}{\sqrt{\epsilon^2 - 1}} \ln \left( \frac{\sqrt{\epsilon + 1} - 1}{\sqrt{\epsilon + 1} + 1 + \sqrt{\epsilon}} \right) \right] \]

\[ = \frac{\hbar c \eta^E_T}{z^4}; \quad (30) \]

\[ \langle B^2_z \rangle_{\text{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[ \frac{1}{16\chi^{3/2}} \left( 2\sqrt{\chi}(12 - 9\sqrt{\epsilon} - 2\chi) \right) \right. \\
\left. - 6(1 - 2\epsilon) \ln \left( \sqrt{\epsilon} + \sqrt{\chi} \right) \right] \]

\[ = \frac{\hbar c \eta^B_z}{z^4}; \quad (33) \]

and

\[ \langle B^2_T \rangle_{\text{ren}} = \frac{\hbar c}{(2\pi)^2 z^4} \left[ \frac{1}{16\chi^{3/2}} \left( 2\sqrt{\chi}(6 + 6\epsilon^2 - 2\chi - 3(\epsilon + 2)\sqrt{\epsilon}) \right) \right. \\
\left. + 6(\epsilon - 2\epsilon^3 + 2\chi) \ln(\sqrt{\epsilon} + \sqrt{\chi}) \right. \\
\left. + 6^2\sqrt{\epsilon^2 - 1} \ln \left( \frac{\sqrt{\epsilon + 1} - 1}{\sqrt{\epsilon + 1} + 1 + \sqrt{\epsilon}} \right) \right] \]

\[ = \frac{\hbar c \eta^B_T}{z^4}. \quad (35) \]

These expressions are very complicated, and the characters of the functions \( \eta(\chi) \) will be investigated in the next section. For the present, we remark that the successful renormalization provides a very strong check on the computations, since the term of order \( \tau^{-4} \) must cancel perfectly against the term from Minkowski space, and the remaining potential poles in \( \tau \) (of orders \( \tau^{-3}, \tau^{-2} \) and \( \tau^{-1} \)) must vanish identically. Another check is provided by the vanishing of \( \langle \hat{T}_{zz} \rangle_{\text{ren}} \), as will be discussed in Section 4.

3. The Squares of the Fields

In the previous section, we found the expectations of the squares of the fields explicitly. In each case the result had the form \( \hbar c\eta/z^4 \), where \( z \) was the distance to the interface and \( \eta \) was a complicated transcendental function of the susceptibility \( \chi \). In this section, we present the graphs of the functions \( \eta \), as well as their limiting behaviors for \( \chi \downarrow 0 \) and \( \chi \uparrow \infty \).

The functions \( \eta^E_T, \eta^B_T \) (scaled to have a common limiting value) are presented in figure 1. They are in each case monotonic and approach a constant value asymptotically, but the approach is extremely slow.

For \( \chi \downarrow 0 \), we have

\[ \langle E^2_z \rangle_{\text{ren}} = \frac{\hbar c}{(2\pi)^2} \frac{9}{80z^4} \chi + O(\chi^2), \quad (36) \]