To what distances do we know the confining potential?

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Abstract

We argue that asymptotically linear static potential is built in into the common procedure of extracting it from lattice Wilson loop measurements. To illustrate the point, we extract the potential by the standard lattice method in a model vacuum made of instantons. A beautiful infinitely rising linear potential is obtained in the case where the true potential is actually flattening. We argue that the flux tube formation might be also an artifact of the lattice procedure and not necessarily a measured physical effect.

We conclude that at present the rising potential is known for sure up to no more than about 0.7 fm. It may explain why no screening has been clearly observed so far for adjoint sources and for fundamental sources but with dynamical fermions.

Finally, we speculate on how confinement could be achieved even despite the absence of the infinitely rising potential in the pure glue theory.
1 Motivation

In the last two decades it became a common place that confinement is due to a linear rising potential between static probe quarks in the 4-dimensional pure Yang-Mills theory. Being a simple consequence of the strong coupling expansion, an infinitely rising linear potential becomes highly non-trivial in the weak coupling continuum limit. Moreover, it contradicts all previous experience in physics with forces decreasing with distances. Therefore, if proven correct, the linear potential would be a most important discovery.

Unfortunately, a proof of the linear potential in the 4-dimensional pure Yang–Mills theory is missing. Therefore, at present the only source of knowledge about the behaviour of the static potential in the pure glue world are lattice studies. However, lattice measurements have statistical and, in many cases, systematic uncertainties due to finite lattice spacing, volume and methodology used, and thus can never substitute a rigorous mathematical proof of the linear potential. Being a numerical experiment, lattice studies should be addressed by the same questions as real-world experiments. In this case the main questions are:

- To what distances the potential is reliably measured
- To what accuracy it is measured; what are the systematic uncertainties
- What is the best-fit form of the potential in the range it is reliably established

A clear answer to these questions is important for theoretical models of confinement and for phenomenological applications.

An additional motivation for a critical analysis of what is presently known about the static potential comes from the lattice studies themselves. There have been measurements of the potential between static sources belonging to the adjoint representation [1, 2], and also between sources in the fundamental representation but with light dynamical fermions [3, 4]. In both cases, contrary to the case of fundamental sources in the pure glue theory, the rising potential is expected to flatten out at certain separation owing to the screening or the ‘string-breaking’ effect. No clear evidence of flattening has been observed so far in either of the cases. Moreover, the potential between sextet sources in the SU(3) theory does not follow the triplet slope, as expected [2].

The non-observation of screening in cases where it is expected is usually ascribed to a poor overlap of the quark creation and annihilation operators, as given by the Wilson loop, with the ground state at large separation between the probe sources. To override this difficulty, it has been suggested to consider mixing of the Wilson loop with other operators which do saturate at large separations between the sources [5, 6, 7, 8]. No wonder that when one allows for a mixing with such operators the diagonalization will always end up with the lowest eigenvalue flattening at large separations. However, an essential finding of these works is that when the separation between sources becomes large, the Wilson loop effectively decouples from the lowest-energy state characterized by a flattening potential.

What is the physical reason for a miniscule overlap of the adjoint Wilson loop with the ground state of two widely separated sources? How do we know that in the case of fundamental charges the Wilson loop has, on the contrary, a sizeable overlap with the ground state? Unfortunately, the assumption that it is sizeable is important to be able to extract the potential at large separations.
In all cases investigated until now in zero-temperature 4-dimensional theories the potential extracted from measuring Wilson loops is compatible with a linear rise, both when it is expected or unexpected. This fact alone calls for a critical analysis of how the potential is commonly extracted from Wilson loops.

The point of view which we advocate in this paper is that the infinitely rising linear potential is built in by construction in the commonly used procedure, and hence the systematic uncertainty has been underestimated. In fact, the procedure is such that it is sufficient to have the potential to be approximately linear in a limited range of separations (around 0.5 fm) to get it infinitely rising at all distances: the larger \( r \), the better linear it comes out.

The conclusion is that if we do not assume from the start that the potential \textit{ought to be} linear (let us not forget: it is not proved yet), and we do not assume that the overlap of the Wilson loop with the ground-state potential is sizeable (and it is not sizeable at least in two cases about which we know), the only sure statement about the potential is that it is approximately linear up to about 0.7 fm, and still continues to rise. Extraction of the potential above this scale, unfortunately, involves assumptions.

## 2 Standard procedure of extracting static potential from Wilson loops

Let us denote \( W(r, t) \) a rectangular \( r \times t \) Wilson loop averaged over many gauge configurations. The standard transfer-matrix logic says that it can be decomposed as a sum over intermediate states formed by a quark-antiquark pair at separation \( r \):

\[
W(r, t) = \sum_n |C_n(r)|^2 \exp[-V_n(r)t]
\]

(1)

where \( V_n(r) \) are the ‘potentials’ for intermediate states \( n \) and \( C_n(r) \) are the overlaps of these states with the concrete quark pair creation operator.

To get ground-state potential \( V(r) = V_0(r) \) one has to take the limit of large \( t \). To be more quantitative, the ground state is cut out from the sum (1) at \( t \gg 1/\Delta E \) where \( \Delta E \) is the energy splitting between the ground and the next excited state. For a string of length \( r \) this splitting is expected to be \( \Delta E = V_1(r) - V_0(r) \sim 1/r \). Hence, in order to extract the static potential one has to take Wilson loops with \( t \gg r \).

Unfortunately, this key requirement can hardly be achieved for physically interesting separations \( r \geq 1 \) fm. Let us imagine that we want to measure the potential at a moderate separation of \( r = 1 \) fm. The \( t \) side should be much much longer than 1 fm. We take a liberal view and announce that \( 2 \gg 1 \), so let us take \( t = 2 \) fm. The area of the Wilson loop is then \( 2 \text{ fm}^2 \). The expected string tension is \( \sigma \simeq (430 \text{ MeV})^2 \simeq 4.75 \text{ fm}^{-2} \). The expected value of the Wilson loop is then \( W \approx \exp(-4.75 \cdot 2) \approx 10^{-4} \). To what accuracy do we want to measure \( W \)? Let us take a moderate accuracy of 10%, that is we require \( \Delta W \approx 10^{-5} \). Since individual measurements of \( W \) fluctuate wildly in the range from -1 to 1, and the statistical error \( \Delta W \) goes as one over square root of the number of independent measurements, it means that one needs an order of \( 10^{10} \) measurements. On a large lattice one can probably

\[ \Delta E = \pi/r. \]

Recent measurements [9] indicate that certain excitations may be split even less than by the expected \( \Delta E = \pi/r \).
allow as much as $10^4$ measurements of Wilson loops lying in different planes per one gluon configuration, assuming they are statistically independent.

Summarizing this arithmetical exercise, we see that in order to honestly measure the potential at a moderate 1 fm separation with a modest 10% accuracy one needs at least $10^6$ statistically independent gluon configurations! This is beyond any computer capacity either now or in near future: the typical number of configurations used at present is no more than a few thousand. With such statistics one can measure loops of areas no more than $\simeq 1$ fm$^2$, even using the aforementioned liberal assumptions. With this murderous arithmetic one can wonder how any quantitative statements can be made about the static potential at separations beyond 0.7 fm.

To circumvent this difficulty, a link-smearing procedure has been suggested [10, 11] presently used in most lattice studies. The idea is to replace links along the spatial sides of the Wilson loops by links smeared in other spatial directions, or by ‘fat’ links. Through this procedure the average of the Wilson loop increases many times; it is ascribed to the larger overlap $|C_0(r)|$ of fat link operator with the ground state, see eq. (1). We shall see, however, that this increase can be interpreted in another way.

If one is sure that by choosing an appropriate $\bar{Q}Q$ creation operator one selects only the ground state contribution to the decomposition (1), all what one needs is to check that $W(r,t)$ follows a simple one-exponent decay with $t$. This is performed not at $t \gg r$ but rather on the contrary at $t \ll r$. The unfortunate ‘rule of a thumb’ is that the area cannot exceed 1 fm$^2$ (because of the stringent statistics requirements), therefore if one wants to measure the potential at $r = 2$ fm, the $t$ side cannot exceed 0.5 fm, so that the exponential behaviour of $W(r,t)$ in $t$ can be actually checked only up to quite small values of $t$. The hope is that, once established at very low $t$, the same time exponent will prevail at any $t$, therefore measurements at low $t$ can give accurate values for the ground-state potential $V_0(r)$.

Examples of the plateaus in the quantity $-\partial \ln W(r,t)/\partial t$ as function of $t$ for several values of $r$ with $t \ll r$, is given in Fig.1. We show there the data obtained by the Wuppertal group [12] as being probably still a record study. Using the $SU(2)$ gauge theory on lattices of volume up to $48^3 \times 64$ and $\beta$ up to 2.74, Bali, Schlichter and Schilling were able to claim a linearly rising potential up to distances up to 2.3 fm. In addition, a string formation over physical distances up to 2 fm has been reported in this remarkable study.

One can see from Fig.1 that the quality of the plateaus at $r \leq 1$ fm is quite good though there is trend of the data to slope down as $t$ increases. At larger $r$ this trend becomes more pronounced, until the error bars explode so that one can hardly get to any conclusions about the plateaus. The procedure of extracting the potential is detailed in ref.[12] but basically it follows from the data in Fig.1. Having no objections to the measurements per se, we still have doubts in their interpretation, which we share below.

3 The danger of misinterpretation

We shall now explain why using the $t \ll r$ data is dangerous and may result in large systematic errors in determining the potential, despite the use of the smeared or fat spatial

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2 This is an optimistic estimate based on the correlation length of about 1 fm. However, the Wilson loop is a peculiar object for which the correlation length is infinity if one implies linear confinement.
Plateaus for the effective potential

Figure 1: Effective potential $V_{eff}(r,t) = \ln \left[ W(r,t)/W(r,t+a) \right]$ as function of $t$ at different values of $r$ from ref. [12]. The data points correspond (from bottom to top) to $r = 0.65, 0.98, 1.30, 1.62, 1.95$ and $2.25$ fm. The inverse coupling used $\beta = 2.635$ corresponds to the lattice spacing $a = 0.0541$ fm. The data points have been slightly scattered in the horizontal direction to resolve the error bars. Courtesy G.Bali.

First, it is known from numerical experience that smearing spatial links leads to an essential increase of time-independent prefactor in the average Wilson loops, but it is not clear why that necessarily means an increase of the overlap with precisely the ground state and not with some higher state or, even more probably, with some complicated superposition of various states.

Second, even with the fat links one never gets the prefactor to be exactly unity, hence one inevitably measures a contamination with many states $V_n(r)$. One can, theoretically, imagine a case where the true ground state has a tiny overlap with the smeared Wilson loops: then it is next to hopeless to extract it. That such a case is not totally academic is exemplified by the fact that the above procedure, being applied to Wilson loops in the adjoint representation or to the fundamental representation but with dynamical fermions, does not show clear signals of the expected screening of the ground-state potential [2, 4].

Third, most dangerous of all, when taking $t \ll r$, it can be questioned why isn’t it possible to turn the head by 90° and call the long side “$t$” (instead of $r$) and the short side “$r$” (instead of $t$). Then the exponential falloff of the Wilson loop with the long side length $r$ (i.e. the linear potential) is automatically guaranteed, because now it is time, and the time is large. Taking fat links along the longer side only, though it formally destroys the Euclidean $t \leftrightarrow r$ symmetry, does not override the danger of misinterpretation.

The interpretation of the Wilson loop with $t \ll r$ would be the following. One creates a $\bar{Q}Q$ pair by the short (unsmearered) side of the rectangular and separates them to the distance
By using a superposition of smeared links along the long side one actually considers not absolutely static \( \bar{Q} \) and \( Q \) but rather oscillating around fixed positions. Since smearing is a kind of random walk in the transverse directions [10, 11], the actual positions of \( \bar{Q} \) and \( Q \) can be said to be Gaussian-distributed, with a width depending on the details of the link-smearing procedure. Since Gaussian distribution is a wave function for an oscillator potential, one can say that, by ways of smearing, one studies the \( \bar{Q}Q \) system in the normal gluon vacuum but with quarks put in superficial oscillator potential wells centered at fixed positions. Else, one can say that quarks are not infinitely heavy anymore but experience normal zero-point oscillations in the gluon vacuum. In either case one can introduce a hamiltonian effectively describing the system. Its lowest-energy state depends on the \( \bar{Q}Q \) separation (as given by the short side \( t \)) and is tested when one measures the Wilson loop at large times, i.e. at large \( r \).

Anyhow, for \( r \) larger than \( t \) and than the effective smearing radius the average Wilson loop should behave as

\[
W(r, t) \simeq \exp \left[ -\bar{V}_0(t) \cdot r \right],
\]

where \( \bar{V}_0(t) \) is the ground-state energy of the \( \bar{Q}Q \) pair oscillating somewhat around their separation \( t \). If one now determines the effective potential in the conventional way by differentiating \( \ln W(r, t) \) in respect to \( t \), one finds

\[
V_{\text{eff}}(r, t) = -\frac{\partial \ln W(r, t)}{\partial t} \simeq \frac{d\bar{V}_0(t)}{dt} \cdot r,
\]

i.e. a potential which is asymptotically linear in \( r \). In order to reproduce the plateau region in \( t \) all one needs is to have \( \bar{V}_0(t) \) approximately linear in \( t \) in the limited range of rather small values of \( t \) where the ‘plateaus’ are actually checked, see Fig.1. Hence, from Fig.1 one finds that the potential is approximately linear up to the endpoint of the plateau region, that is up to 0.5 fm. The linear \( r \) dependence of the potential extracted from \( V_{\text{eff}} \) at \( t \ll r \) is then a triviality: the larger \( r \) is, the more exact eq. (2) becomes, the better-quality ‘linear potential’ one gets.

The ‘increase of the overlap with the ground state’ assumed to be achieved when one considers smeared spatial links, gets a most natural explanation from this point of view. The effective overlap is defined as (see, e.g., [12])

\[
c_{\text{eff}}(r, t) = W(r, t) \exp [V_{\text{eff}}(r, t) \cdot t] \simeq \exp \left[ -r \left( \bar{V}_0(t) - t \frac{d\bar{V}_0}{dt} \right) \right].
\]

If \( \bar{V}_0 \) is approximately linear in \( t \) in the ‘plateau’ range of observations, the linear term in the parenthesis cancels, however the constant part of \( \bar{V}_0 \) does not. The constant part is the quark self-energy. In perturbation theory it diverges linearly for static quarks (corresponding to the unsmeared spatial links) but only logarithmically for fluctuating sources (corresponding to smeared links). Therefore, the constant part of \( \bar{V}_0 \) is much smaller for smeared links. This seems to be the real reason why the ‘overlap’ defined by eq. (4) is increased by one to two orders of magnitude when one goes from unsmeared to smeared spatial links – a fact which is not easy to explain in the standard logic.
To conclude, there is a danger that the linear rising potential is built in by construction into the procedure of extracting it from Wilson loop measurements at $t < r$, whereas it is exactly what is so demanding to prove.

### 4 Instanton ensemble

We decided to check to what extent does the standard procedure work by applying it to a model gluon vacuum for which the static potential is known theoretically, namely to the random instanton ensemble \(^3\). We take the simplest superposition ansatz of $N_+ = N_- = N/2$ instantons and antiinstantons ($I$'s and $\bar{I}$'s for short) in the singular gauge,

$$A^a_\mu(x) = \sum_{I=1}^{N_+} A^I_\mu^a(x) + \sum_{I=1}^{N_-} A^{\bar{I}}_\mu^a(x), \quad A^{I\mu}_a(x) = \frac{O^{ai}_\mu \eta_i(x-z)_\nu 2\rho^2}{(x-z)^2 [(x-z)^2 + \rho^2]}. \quad (5)$$

(for $\bar{I}$'s the 't Hooft symbol $\bar{\eta}$ is replaced by $\eta$). The $SO(3)$ orientation matrices $O^{ai}_\mu$ are taken to be random, as well as the centers $z_\mu$. The sizes of $I$'s and $\bar{I}$'s are distributed according to the probability

$$P(\rho) = \int_0^\rho d\rho \nu(\rho) = \frac{(\rho/\rho_1)^{b-4}}{[1 + (\rho/\rho_1)\nu^{-1}]^{b-3}}, \quad b = 22/3, \quad 0 \leq P(\rho) \leq 1. \quad (6)$$

This distribution function follows 't Hooft’s $\nu(\rho) \sim \rho^{b-5}$ regime at small sizes and falls off as $\nu(\rho) \sim 1/\rho^b$ at large sizes. The parameter $\rho_1$ is related to the maximum of the distribution at $\rho_0 = \rho_1[(b-5)/\nu]^{1/(\nu-1)}$. We choose the ‘conformal’ $\nu = 5$ power in numerics.

We have computed the averages of Wilson loops with various $r, t$ in this random instanton ensemble. We have used $N_+ + N_- = 128 + 128$ and $256 + 256$ $I$'s and $\bar{I}$'s put in a 4-dim cubic box of volume $V$. The number of instanton configurations over which averaging was performed varied from 800 for small loops to 1600 for larger ones. The ratio of the most probable size $\rho_0$ to the average separation between pseudoparticles $R = (N/V)^{-1/4}$ was fixed to be $\rho_0/R = 0.4$. With this ratio fixed, the measured potential appears to be proportional, within errors, to the density $N/V$ which, therefore, sets the scale both for the potential and for the units in which the distances $r$ and $t$ are measured. To be specific, we choose $R = 0.645$ fm, so that $\rho_0 = 0.258$ fm.

These values are compatible with the characteristics of the instanton ensemble obtained from smearing the vacuum gluon configurations by the RG mapping method \([14]\) though these authors find the ensemble to be more dilute. However, at the moment we are concerned not by the accurate description of the instanton ensemble but by the methodological problem of extracting the static potential from the Wilson loop measurements. Sufficient to say that a portion of closely situated $I$'s and $\bar{I}$'s may be lost by the smearing procedure, so that the above choice of parameters is not totally unrealistic.

To ensure statistical independence of individual measurements we made only one measurement per configuration of the Wilson loop placed in the $(zt)$ plane in the middle of an open box of length 2.62 or 3.11 fm (for the chosen instanton density). The path-ordered

\(^3\) The first comparison of Wilson loop measurements with the instanton gas predictions has been made in ref. \([13]\).
exponents along rectangular loops were computed by solving differential equations, or by taking products of ‘links’ introduced by hand to mimic the lattice procedure. With fields given by a continuum formula, the first method is faster than the latter (for given accuracy) since any standard routine of solving differential equations makes the discretization in a more clever way than just taking equal spacing independent of the field. We have found that one needs ‘lattice spacing’ not less than 0.06 fm to reproduce Wilson loops for a typical configuration to an accuracy better than 5%.

Choosing the lattice spacing to be 1/10 of the average separation between instantons, i.e. 0.065 fm, we have performed a standard link smearing procedure for the spatial sides of the loop, replacing each link by a U-shaped ‘staple’ lying in the transverse spatial directions,

\[ U_\mu(n) \rightarrow U_\mu(n) + \alpha \sum_{\nu=\pm x, \pm y} U_\nu(n) U_\mu(n + \nu) U_\nu^\dagger(n + \mu) \]

with a variable weight \( \alpha \).

We have found that this smearing has no effect, within statistical errors, on the average of the Wilson loops, for any weight \( \alpha \) varying between 0 and 1. The reason is that the instanton ensemble is already smooth enough, so that no smearing of links is needed, unless a small-size instanton happens to get inside a staple, which is statistically a negligible effect. This should be contrasted with real lattice calculations which are overwhelmed by ultraviolet noise, so that smearing links has a dramatic effect. In any case our ‘overlaps’ \( |C(r)| \) were not small, with fat links or without them. According to the common reasoning, all what one needs then is to check that \( W(r, t) \) falls exponentially with \( t \), even though \( t \) is not much larger than \( r \).

We show the data for \( W(r, t) \) in Fig.2 for \( t \) ranging from 0.13 to 0.52 fm and \( r \) ranging from 0.52 to 1.81 fm. For each value of \( r \) the quantity \(-\ln[W(r, t)]\) can be well fitted by a linear dependence on \( t \) even though \( t \) is less than \( r \). There are no signs that the curves wish to level off as \( t \) increases. These are the famous ‘plateaus’ for the quantity \(-\partial \ln(W(r, t))/\partial t\) which persuade optimists that the asymptotics in \( t \) is already reached, and that one can read off the static potential \( V(r) \) as the slopes of the straight lines in \( t \), and the overlap \( |C(r)|^2 \) as their intercepts.

Following this common practice we plot \( V(r) \) and \(-\ln|C(r)|^2 \) in Fig.3. A linear fit to \( V(r) \) is quite impressive; it gives the value of the ‘string tension’ \( \sigma \simeq (430 \text{ MeV})^2 \). Naturally, there is no Coulomb \( 1/r \) term at small \( r \), which emerges from Gaussian quantum fluctuations of gluon field about whatever background.

The potential \( V(r) \) resulting from these Wilson loop measurements is plotted again in Fig. 4, together with the theoretically-known heavy-quark potential induced by instantons, which we explain below.

5 Heavy-quark potential induced by instantons

The leading term (in the density of instantons) of the instanton-induced potential was given in ref. [15] without a derivation. A derivation was presented in ref. [16], which allows generalization to higher orders in density as well as to potentials induced by objects different from instantons. A further generalization to arbitrary groups and the representations for probe quarks has been derived in ref. [17] which we cite here.
Figure 2: Seems to demonstrate that Wilson loops can be well fitted by simple time exponents even at small values of $t$.

Figure 3: Effective potential $V_{\text{eff}}(r)$ (crosses) and ‘overlap’ $-\ln |C(r)|^2$ (circles) extracted from the data in Fig. 2. For the largest $r = 1.81$ fm the values of $V_{\text{eff}}(r)$ and of $-\ln |C(r)|^2$ obtained from fat links are also shown for illustration: the results coincide within errors with those obtained from unsmeared links.
Let probe quarks belong to the representation $R$ of a gauge group $G$ whose dimensions are $d(R)$ and $d(G)$, respectively. For example, for a fundamental representation of the $SU(N_c)$ group $d(R) = N_c$ and $d(G) = N^2_c - 1$. Since instantons are essentially $SU(2)$ objects one has first of all to decompose the given representation $R$ in respect to its $SU(2)$ content. For example, if one takes probe quarks from the $SU(3)$ octet, it has two $SU(2)$ doublets with $J = 1/2$, one triplet with $J = 1$ and one singlet with $J = 0$, so that $\sum_J (2J + 1) = d(R) = 8$. A fundamental representation of any $SU(N_c)$ group has only one doublet ($J=1/2$), all the rest ‘particles’ are $SU(2)$ singlets. The instanton-induced potential can be also decomposed in contributions of the $SU(2)$ multiplets,

$$V(r) = 4\pi \frac{N}{V} \int_0^{\infty} d\rho \, \nu(\rho) \frac{1}{d(R)} \sum_{J \in R} (2J + 1) F_J(x), \quad x = \frac{r}{2\rho}.$$  \hspace{1cm} (8)

Here $N/V$ is the $I$'s and $\bar{I}$'s density, $\nu(\rho)$ is their size distribution normalized to unity and $F_J(x)$ are dimensionless functions depending on the quark separation $r$ measured in units of $2\rho$, they depend on the spin $J$ of the $SU(2)$ multiplet inside the given representation $R$. These functions are given by integrals over dimensionless variables $y = |z|/\rho$ and $t$ where $|z|$ is the distance of an instanton from the axis drawn in the middle between the two sources, and $t$ is the cosine of the angle between $-\vec{r}$ and $-\vec{z}$:

$$F_J(x) = \int_0^{\infty} dy \, y^2 \int_0^1 dt \left( 1 - \cos \phi_+ \cos \phi_- - \frac{y^2 - x^2}{\sqrt{x^2 + y^2 + 2xyt} \sqrt{x^2 + y^2 + 2xyt}} \sin \phi_+ \sin \phi_- \right),$$

$$\phi_{\pm} = 2\pi \sqrt{\frac{J(J+1)}{3}} \left( \frac{\sqrt{x^2 + y^2 \pm 2xyt}}{\sqrt{x^2 + y^2 \pm 2xyt + 1}} - 1 \right).$$  \hspace{1cm} (9)

The functions $F_J(x)$ behave as $\sim x^2$ at small $x$; at large $x$ they tend to constants depending on $J$. If the third moment of the instanton size distribution is convergent the potential $V(r)$ flattens out asymptotically to twice the renormalization of the heavy quark mass [16]

$$\Delta M = 16\pi \cdot 0.552 \cdot \frac{N}{V N_c} \rho^2 \quad \text{(for quarks in the fundamental representation).}$$  \hspace{1cm} (10)

If the size distribution happens to fall off as $\nu(\rho) \sim 1/\rho^3$ at large $\rho$ one gets a linear infinitely rising potential [17]. However, such a size distribution means that large instantons inevitably overlap, and the sum ansatz (5) is not reasonable.

Eq. (8) is actually the first term in the expansion of the potential in the instanton density $N/V$ [16]. We have evaluated the next term and checked that it is much smaller than the first one in the range of parameters of interest. Taking the size distribution (6) with the same set of parameters as in the numerical simulations we get the potential shown in Fig.4. If one wants to change the average size $\bar{\rho}$ one has to rescale the $r$ axis; the instanton density $N/V$ is just a scale factor of the potential as a whole.

The correct instanton-induced potential starts to rise quadratically with the separation, then in a rather long interval it remains approximately linear but asymptotically it approaches $2\Delta M \simeq 2 \cdot 1.37 \text{ GeV}$, for the chosen instanton distribution.
Figure 4: Static potential $V(r)$ as measured from Wilson loops with $t < r$ (open triangles) versus theoretical prediction from the instanton ensemble (solid line).

We see that the potential extracted by standard procedure from Wilson loop measurements at $t < r$ reproduces the theoretical expectation reasonably well at $r < 1.3$ fm; at larger separations the former continues to rise linearly while the latter flattens out.

This exercise illustrates that it might be dangerous to extract the potential from Wilson loops with $t < r$ even though one observes nice plateaus in $t$.

6 Interpretation

When one measures Wilson loops at $t \ll r$ the linear dependence of $\log W(r, t)$ on $r$ is built in, because $r$ is the long side of the rectangular; the larger $r$ is the better linear dependence in $r$ will be seen by default. In order to get a ‘plateau’ of $\partial \log W(r, t)/\partial t$ in $t$ when $t$ is relatively small, all one needs is the true potential to grow approximately linearly in a limited range of separations corresponding to the ‘plateau’ region of $t$. Such a behaviour is exemplified by a dense instanton ensemble, as seen from Fig.4. Measurements with $t < r$ pick up the ‘string tension’ (i.e. the derivative $dV/dr$) from the steep part of the potential at small to moderate separations, and continues it, by construction, to arbitrarily large separations.

To discriminate between a hypothetical case of a first steeply rising and then flattening potential, and a case of an infinitely rising linear potential one has really to make measurements with $t > r$ and not vice versa.

It has been shown recently in ref. [18] that one can extract the instanton-induced potential from Wilson loops only when one takes $t \simeq (2$ to $3) \cdot r$, as it should be expected from general considerations. In Fig.5 we plot the results of ref. [18] in comparison to eq. (8). We see, first, that the instanton-induced potential is, to a good accuracy, proportional to
the instanton density $N/V$, which justifies the use of the first virial expansion term (8); second, that this formula reproduces well the potential extracted from Wilson loops with $t \approx (2$ to $3) \cdot r$.

It may be argued that the instanton model does not correspond to any well-defined Hermitean hamiltonian, and hence one cannot, generally speaking, write the spectral decomposition for Wilson loops, eq. (1). We find, however, that the average $W(r,t)$ is a positive function monotonously decreasing with $t$ at fixed $r$ and with $r$ at fixed $t$. The same has been observed in ref. [18]. Therefore, $W(r,t)$ can, in fact, be decomposed as in eq. (1) with positive coefficients and with positive effective potential $V_{\text{eff}}(r,t)$, see. eq. (3). In that respect the instanton model does not differ from the full Yang–Mills theory.

Summarizing, measurements of Wilson loops with $t \gg r$ do reliably reproduce the true potential. This is not the case for loops with $t < r$.

A very important phenomenon whose observation would strongly support the string picture in general and the infinitely rising linear potential in particular, is a formation of a flux tube as the separation between source quarks increases. This phenomenon has been also studied in much detail and with unprecedented precision in ref. [12] by ways of measuring correlations of Wilson loops with plaquettes placed inside the loops $^4$.

As in the case of the potential, an irreproachable way to extract the fields created by a pair of static quarks would be to use Wilson loops with $t \gg r$. Unfortunately, in this case the statistics requirements are even more disastrous than in the case of extracting $V(r)$, since the signal-to-noise is smaller than for the Wilson loop itself. Therefore, the authors of

$^4$To our knowledge, such measurements were first performed in refs. [19, 20].
ref. [12] are forced to consider the opposite limit, \( t \ll r \), again assuming that the ground state is cut out by using fat spatial links. In fact, the side called \( t \) has been taken below 0.5 fm while the long side, called \( r \), was taken up to more than 2 fm. As the long side increases, the authors observe a certain flattening of the fields extent in the transverse plane, which is interpreted as a flux tube formation.

Unfortunately, this interpretation suffers from the same ambiguity as the extraction of the potential. With \( t \ll r \) one can view the fields as created by quarks oscillating about the moderate separation \( t \) (oscillations are the result of the link-smearing procedure) but existing during a long time \( r \). It is then a triviality that at large \( r \) the average fields become constant in time, i.e. in \( r \). To check that there is a real string formation and not a misinterpretation one has to make sure that the flux tube is not thinning away as \( t \) gets much larger than \( r \). This seems to be a formidable task, in view of the arithmetic presented in section 2.

7 Discussion

There is a paradox in lattice measurements mentioned in the first section: at zero temperatures no screening of the rising potential has been clearly observed so far in situations where screening is expected. This is the case of a pure glue theory with adjoint sources [1, 2] and the case of fundamental sources but with dynamical fermions [3, 4]. This lack of screening is deduced from measuring Wilson loops with ‘fat’ links at \( t \ll r \) since it is statistically impossible to study the opposite limit. An obvious resolution of this paradox would be that in neither of the cases one measures the true potential but rather an automatic continuation of the potential from the region with maximal \( dV/dr \) to larger separations. It should be mentioned that quite recently a clear indication of screening has been observed with dynamical fermions in \( d = 3 \) in ref. [21] where rather large values of \( t \) for the Wilson loop were shown to be needed, however, at the cost of using coarse lattices.

Would a saturation of the static potential for fundamental sources in pure glue theory at some finite value of \( V_\infty \) mean that there is no confinement? Not necessarily. Quarks are confined in full QCD despite there is no long distance force, and, by the way, so are gluons despite the adjoint sources should be also screened. One has to understand why it is so.

The real world has quarks, both light and heavy ones. The physics of light quarks is strongly dominated by the effect of spontaneous chiral symmetry breaking. It results in light quarks acquiring a dynamical (or constituent) mass of about \( M_{\text{const}} \approx 350 \) MeV with Goldstone pions becoming the lowest excitations in the spectrum. Therefore, light quarks might not exist as asymptotic states: instead of producing a light quark-antiquark pair it is energetically favourable to produce one or several pions. Mathematically, it would correspond to the quark propagator with momentum-dependent mass, having singularities only on the ‘second Riemann sheet’ under the cut starting from the pion threshold.

As to heavy quarks, if \( \Delta M = V_\infty/2 \) happens to be larger than approximately \( M_{\text{const}} \) the heavy quarks would be unstable under a decay to \( B \) or \( D \) mesons. The case is to some extent

\[ ^{} \text{In ref. [12] stability of the fields between sources as function of } t \text{ is illustrated in Figs. 17-19 where a rather moderate separation } r = 0.5 \text{ fm has been used, and with } t \text{ varying also up to } 0.5 \text{ fm. Despite moderate parameters of the loop the error bars exceed 50\% at the largest value of } t \text{ shown there, i.e. 0.5 fm.} \]

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extent similar to electrodynamics with charges $Z > 137$: such particles are unstable under a production of $e^+ e^-$ pairs and therefore cannot exist as asymptotic states. The heavy quarks might be thus confined too.  

For the instanton vacuum it is possible to quantify the condition that the renormalization of heavy quark mass is larger than the light constituent quark mass. The quantity $\Delta M$ has been given above in eq. (10) while the constituent quark mass is given by the equation [24]

$$M_{\text{const}} \simeq 1.45 \cdot 2\pi \sqrt{\frac{N}{V N_c}} \sqrt{\rho^2}.$$  

(11)

Notice that $\Delta M$ is linear in the instanton density while $M_{\text{const}}$ is proportional to its square root. This important circumstance is due to the fact that nonzero $M_{\text{const}}$ is an order parameter for chiral symmetry breaking. The condition that $\Delta M > M_{\text{const}}$ reads

$$\frac{N}{V N_c} \left(\frac{\rho}{\rho^2}\right)^2 > 0.1,$$

(12)

meaning that the instanton medium should be sufficiently dense but not necessarily very dense. With this condition fulfilled, there is a good chance of getting confinement of quarks even in the case where the static potential levels off.

Finally, we would like to draw attention to persistent warnings by Grady that certain phenomena usually associated with confinement in pure glue theory might be, in fact, relics of the strong coupling regime. These include the density of abelian monopoles and center vortices [25], the formation of percolating monopole clusters [26] and the value and the linearity of the static potential itself [25, 27].

8 Conclusions

The answer to the question put in the title depends, unfortunately, on the standpoint of a person. A pessimist would say that we are still in a kind of strong coupling regime in certain lattice measurements, and that it is still too early to draw any conclusions about the behaviour of the static potential.

An optimist would say that the potential is proven to be linear up to an enormous 4 fm separation [9], however it implies that the asymptotics of Wilson loops is reached at incredibly small $t < 0.25$ fm. We remind the reader that at present one cannot measure Wilson loops with areas exceeding $\simeq 1$ fm$^2$ for statistical reasons.

We think that a more weighted conclusion which can be made from lattice measurements is that the static potential is still rising at distances about 0.5 – 0.7 fm but its precise form is unknown beyond that separation. Similarly, we would avoid making definite conclusions on string formation.

The link-smearing procedure being quite useful for measuring point correlation functions seems to be extremely dangerous for measuring nonlocal quantities such as Wilson loops since there is a real risk of getting the string and the linear potential by mere construction of the procedure, if one restricts oneself to measurements at $t << r$. We have illustrated

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\textsuperscript{6}The “$Z > 137$” scenario of confinement has been advocated for many years by V.N.Gribov [22, 23].
that by applying the procedure to the model gluon vacuum made of instantons for which the potential is known theoretically.

A force that is not decreasing with the distance is a feature never before encountered in 3+1 dimensional physics. If correct, this statement is so important that it deserves to be demonstrated beyond any reasonable doubt.

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