Isospin-Breaking Vector Meson Decay Constants From Continuous Families of Finite Energy Sum Rules

Kim Maltman∗
Department of Mathematics and Statistics, York University,
4700 Keele St., Toronto, Ontario, CANADA M3J 1P3
and
Special Research Center for the Subatomic Structure of Matter,
University of Adelaide, Australia 5005

C.E. Wolfe†
Department of Physics and Astronomy, York University.
4700 Keele St., Toronto, Ontario, CANADA M3J 1P3

The isospin-breaking vector meson decay constants are determined from a QCD sum rule analysis of the vector current correlator
⟨O|T(V_{3}\mu V_{8}\nu)|O⟩,
using a recently proposed implementation of the finite energy sum rule approach. The analysis employs the three-loop version of the OPE, and two different families of weight functions. It is shown that the requirement of consistency between results obtained using these two different weight families leads to a rather good determination of the parameter describing the deviation of the D = 6 condensate term in the OPE from its vacuum saturation value, and that the ability to determine this value has non-trivial numerical consequences to the analysis. The phenomenological relevance of the results to the extraction of the strange quark mass and the determination of the 6th order chiral low energy constant, Q, is also briefly discussed.

I. INTRODUCTION

Because the neutral (a = 3, 8) members of the SU(3)F octet of vector currents, J_{a\mu} = \bar{q}\gamma_{\mu}\frac{\lambda^a}{2}q (with \lambda^a the usual Gell-Mann matrices), couple to fermions in the Standard Model, it is possible to use experimental data on the spectral functions associated with correlators involving these currents to determine certain quantities of phenomenological interest. For example, defining the scalar correlators, Π_{ab}(q^2), by means of

i\int d^4x \exp(iqx) \langle 0|T(J_{a\mu}(x)J_{b\nu}(0))|0\rangle \equiv (q_{\mu}q_{\nu} - q^2 g_{\mu\nu})\Pi_{ab}(q^2), \tag{1}

and the corresponding spectral functions, ρ_{ab}(q^2), as usual, by ρ_{ab}(q^2) = \frac{1}{\pi} \text{Im} \Pi_{ab}(q^2), one finds that (1) integrating the difference ρ_{33}(q^2) − ρ_{88}(q^2) with the weight function occurring naturally (due to kinematics) in the finite energy sum rule (FESR) treatment of hadronic τ decays [1] produces a sum rule from which one can, in principle, determine the running strange quark mass, m_s(\mu) [2], and (2) integrating the same difference ρ_{33}(q^2) − ρ_{88}(q^2) with weight function w(s) = 1/s produces a sum rule from which one can extract one of the 6th order low-energy constants (LEC’s), Q, appearing in the 6th order version of the effective chiral Lagrangian [3]. (See Ref. [4] for a discussion of chiral perturbation theory (ChPT) and the method of effective chiral Lagrangians in general, Ref. [5] for the form of the O(q^4) terms in the effective Lagrangian in the most general case, and Ref. [6] for both a discussion of the subset of these terms surviving when one restricts one’s attention to vacuum correlators, and a definition of Q.)

Of course, J_{3\mu} and J_{8\mu} do not couple separately in the Standard Model, but only in the combination

J_{EM\mu} = J_{3\mu} + \frac{1}{\sqrt{3}}J_{8\mu} \tag{2}

which gives the light quark (u, d, s) part of the electromagnetic (EM) current. Thus, what is measured in e^+e^- -> hadrons, is not the desired quantities, ρ_{33} and ρ_{88}, separately, but the combination

∗e-mail: maltman@fewbody.phys.yorku.ca
†e-mail: wolfe@fewbody.phys.yorku.ca
\[ \rho^{EM}(q^2) = \rho^{33}(q^2) + \frac{2}{\sqrt{3}}\rho^{38}(q^2) + \frac{1}{3}\rho^{88}(q^2). \] (3)

In the isospin symmetry limit, \( \rho^{38} \) would vanish and, since one could then classify the final hadronic states according to their G-parity, it would be straightforward to separate the isovector (33) and isoscalar (88) components of the EM spectral function.

In the presence of isospin breaking, however, this process is no longer so straightforward. The most obvious experimental signature of the presence of isospin-breaking in \( e^+e^- \to \text{hadrons} \) is the interference shoulder in the \( e^+e^- \to \pi^+\pi^- \) cross-section in the \( \rho\omega \) region [7]. The \( e^+e^- \to \omega \to \pi^+\pi^- \) contribution to \( \rho^{EM} \) is clearly, to leading order in isospin-breaking, to be associated with \( \rho^{38} \) and hence is usually removed explicitly in analyzing the data. This removal is accomplished by (1) fitting the parameters of a model for the total \( e^+e^- \to \pi^+\pi^- \) amplitude, consisting of \( \rho, \omega \) and possible background contributions, to the experimental data, (2) removing the \( \omega \) contribution once the fit has been performed and (3) squaring the modulus of the remaining \( \rho \) contribution and identifying this result with the \( \rho \) contribution to \( \rho^{33} \) [8].

While this procedure does remove one source of isospin breaking contamination from the nominal \( \rho^{33} \) so extracted, it is easy to see that other such contaminations still remain. Indeed, once one allows isospin-breaking, the physical \( \rho \) and \( \omega \) are admixtures of pure isovector and isoscalar states, the size of the admixture of the “wrong” isospin component being governed by the scale of isospin breaking. As a consequence, the intermediate \( \rho \) contribution to \( \rho^{38} \), for example, does not vanish. In fact, if one denotes the pure isovector \( \rho \) state by \( \rho^{(0)} \) and the pure isoscalar \( \omega \) state by \( \omega^{(0)} \), one expects \( \rho \) contributions to \( \rho^{38} \) from two sources: (1) that due to \( \rho^{(0)} - \omega^{(0)} \) mixing (a one particle reducible contribution, with coupling of the isovector current to the \( \rho^{(0)} \) component and the isoscalar current to the \( \omega^{(0)} \) component of the \( \rho \)), and (2) that due to the “direct” (one particle irreducible, 1PI) coupling of the \( \rho^{(0)} \) component to the isoscalar current (such a coupling being unavoidable in any hadronic effective Lagrangian based on QCD). Thus, removing the contribution due to the intermediate state \( \omega \) from the \( e^+e^- \to \pi^+\pi^- \) cross-section, while removing part of the \( \rho^{38} \) contribution, does not remove it all. One is then left with, not the desired quantity, \( \rho^{33} \), but rather with a combination of \( \rho^{33} \) and the residual part of \( \rho^{38} \) associated with the intermediate \( \rho \) state (plus possible additional such contaminations from elsewhere in the spectrum). Similar isospin-breaking flavor 38 contributions exist for \( e^+e^- \to \omega \to 3\pi \), complicating the extraction of the isoscalar spectral function.

Corrections for such isospin breaking effects, which are unavoidable so long as no process exists in which only one of the two neutral flavor currents couples, are thus necessary if one wishes to perform phenomenological analyses of the type mentioned above. Such corrections would also be important in performing precision tests of CVC, which involve a comparison of \( \rho^{33} \) and the charged isovector spectral function \( \rho^{(3)} \), measured in hadronic \( \tau \) decays (see, for example, Ref. [9]).

It is easy to see that, to be able to make these corrections (at least in the region below \( s \sim 2 \text{ GeV}^2 \), where the EM spectral function is, experimentally, resonance dominated), it is sufficient to determine the isospin-breaking vector meson decay constants. Let us first clarify notation. We define the flavor 3 and 8 vector meson decay constants via

\[ \langle 0 | J_\mu^V (k) | V(k) \rangle = m_V F_\mu^V \epsilon_\mu(k) \] (4)

where \( V = \rho, \omega, \phi, \cdots, \epsilon_\mu(k) \) is the vector meson polarization vector, and \( a = 3, 8 \). \( F^{(3)}_{\rho}, F^{(8)}_{\omega}, \text{and } F^{(8)}_{\phi} \) are non-zero in the limit of isospin symmetry; \( F^{(3)}_{\rho}, F^{(3)}_{\omega}, \text{and } F^{(3)}_{\phi} \) zero in the absence of isospin breaking. The experimentally determined EM decay constants, \( F_{\rho}^{EM}, F_{\omega}^{EM}, \text{and } F_{\phi}^{EM} \), are then given by

\[ F_{\rho}^{EM} = F_{\rho}^{3} + \frac{1}{\sqrt{3}}F_{\rho}^{8}. \] (5)

Thus, for example, the broad \( \rho \) contribution to \( \rho^{EM} \), usually taken to be associated purely with \( \rho^{33} \), consists not only of a flavor 33 contribution proportional to \( [F_\rho^{3}]^2 \), but also of a flavor 38 contribution proportional to \( \frac{1}{\sqrt{3}}F_\rho^{3}F_\rho^{8} \). The \( \omega \) contribution to \( \rho^{EM} \), similarly, contains both a flavor 88 part proportional to \( [F_\omega^{8}]^2 \) and a flavor 38 part proportional to \( \frac{1}{\sqrt{3}}F_\omega^{3}F_\omega^{8} \). The flavor 38 parts, in both cases, are present only due to isospin breaking, and have to be removed from the experimental \( \rho \) and \( \omega \) contributions to \( \rho^{EM} \) in order to obtain the corresponding \( \rho \) contribution to \( \rho^{33} \) and \( \omega \) contribution to \( \rho^{88} \).

It is important to stress at this point that the conventional “few-percent” rule-of-thumb for estimating the size of isospin-breaking effects, which might lead one to expect such effects to be numerically negligible, is inapplicable in the cases involving \( \rho^{33}(q^2) - \rho^{88}(q^2) \) discussed above. This is true for a number of reasons. First, because the difference of spectral functions is itself flavor-breaking, the relative importance of isospin breaking is enhanced by a factor \( \sim 3 \), characteristic of the inverse of the scale of flavor-breaking. Second, the effect of \( \rho - \omega \) mixing naturally
produces corrections for the ρ contribution to $\rho^{33}$ and ω contribution to $\rho^{88}$ which are opposite in sign; the effects therefore add when the difference is taken. Finally, there is a natural numerical enhancement which makes the size of the correction needed to remove the $\rho^{38}$ part of the ω contribution to $\rho^{EM}$, and hence isolate $\rho^{88}$, larger than naively expected [10]. The latter two points are discussed in somewhat more detail in Section II below.

In what follows, we evaluate the isospin-breaking vector meson decay constants by performing a QCD sum rule analysis of the isospin-breaking vector current correlator $\Pi^{38}$. The vector meson spectral contributions are, in this case, proportional to $F_V^8 F_V^8$, so that a determination of this product, in combination with the experimental determination of $F_V^{33}M$, given in terms of $F_V^{33}$ and $F_V^{88}$ above, allows a separate determination of $F_V^{88}$ and $F_V^{88}$. The rest of the paper is organized as follows. In Section II we discuss qualitative expectations for the pattern of isospin-breaking corrections based on the structure of the leading (chiral) order terms in the vector meson effective chiral Lagrangian, as well as semi-quantitative expectations for their probable scale which can, using this perspective, be obtained from experimental data. In Section III, we discuss briefly the form of QCD sum rules employed (a version of FESR), and the advantages of this approach. In Section IV, we discuss the input used for the hadronic and OPE sides of the sum rules employed, and present our results. Some advantages of the approach, in particular via a vis the handling of the $D = 6$ terms in the OPE of the 38 correlator, will also be discussed here. Finally, in Section V we summarize, and make some brief comments on the phenomenological significance of our results.

II. CHIRAL CONSTRAINTS AND THE SCALE OF ISOSPIN-BREAKING CORRECTIONS

ChPT provides both an underlying conceptual framework and systematic procedure [11] for writing down the most general effective Lagrangian relevant to a given set of hadronic states which fully incorporates the symmetries of QCD and implements the broken symmetries (such as chiral symmetry) with the same pattern of symmetry breaking as occurs in QCD. Although the resulting effective Lagrangian, $\mathcal{L}_{eff}$, is non-renormalizable, and hence contains a infinite number of low-energy constants (LEC’s) (coefficients of those terms allowed by the symmetry arguments that go into constructing $\mathcal{L}_{eff}$), it is possible to formulate the theory in such a way that only a finite number of such terms appear to a given order in the chiral, or low-energy expansion. (For so-called “heavy” fields, whose masses are non-vanishing in the chiral limit, this requires a re-formulation in terms of velocity-dependent fields [12,13].) The leading order terms in this expansion (in which light quark masses, $m_q$, $q = u,d,s$, count as $\mathcal{O}(q^2)$, with $q$ representing some soft external momentum), incorporate the leading constraints associated either with chiral symmetry, or the symmetry pattern of its breaking.

The heavy field implementation of ChPT given in Ref. [13] for the vector mesons and their interactions with the members of the pseudo-Goldstone boson pseudoscalar octet provides some useful information about the pattern of isospin-breaking mixing in the vector meson sector. One must bear in mind that, in general, mixing in field theory is more complicated than in Hamiltonian quantum mechanics. This is because, in an effective field theory, one in general has (and cannot, in fact, avoid) momentum-dependent mixing terms. This means that there are off-diagonal elements of the wave-function renormalization matrix and, as a consequence, the relation between the original, unmixed fields and the final, renormalized and diagonalized fields, involves the product of a symmetric matrix and a rotation (and hence is not itself a rotation). (See, for example, the discussion of the effects of $\pi^0$-$\eta$ mixing in Ref. [4]; an expanded version is also given as part of the discussion of the treatment of the mixed-isospin axial current correlator in Ref. [14].) From the heavy vector meson field formulation, however, one sees that the leading (in chiral order) term in $\mathcal{L}_{eff}$ generating isospin-breaking mixing involves one power of the quark mass matrix and no derivatives [13], and hence produces no off-diagonal contributions to the wave-function renormalization matrix. The leading order mixing effect thus results in a physical ρ and ω basis which is related by a rotation to the original pure isospin $\rho^{(0)}$, $\omega^{(0)}$ basis, just as in the quantum mechanical case. The “wrong” isospin $\omega^{(0)}$ admixture in the physical ρ state is thus equal in magnitude, but opposite in sign, to that of the $\rho^{(0)}$ admixture in the physical ω state, at this order in the chiral expansion. While higher (chiral) order corrections exist, this pattern should remain approximately valid, even at higher order.

Let us now consider the vector meson decay constants. Since the chiral limit is also $SU(3)_F$ symmetric, the leading order term generating the vector meson decay constants is necessarily $SU(3)_F$-symmetric. When one now considers the effects of flavor- and isospin-symmetry breaking (recalling that both are generated by the quark mass matrix, and hence both are produced by the same set of terms in the effective Lagrangian), there are two potential sources of flavor and isospin breaking in the vector meson decay constants. The first is that associated with higher order terms, involving at least one power of the quark mass matrix, coupling the external photon field to the vector meson nonet, the second that induced by the leading quark-mass-dependent term, responsible for mixing, discussed above. The leading order mixing effect simply reproduces the standard leading order $SU(3)_F$ mixing analysis [13], leading to near ideal mixing in the vector meson sector. As is well-known, the combination of ideal mixing and neglect of
flavor-breaking in the EM couplings of the unmixed states leads to the prediction that the vector meson EM decay constants, measured experimentally in $V \to e^+e^-$, [15] should be in the proportions $F_{\rho}^{(0)}: F_{\omega}^{(0)}: F_{\phi}^{(0)} = 3 : 1 : -\sqrt{2}$, where the superscript (0) indicates that the couplings refer to the ideally mixed, but isospin pure vector meson states. That this prediction is borne out by experiment represents empirical evidence that, despite the potential $SU(3)_F$-breaking photon coupling contributions being of the same formal order as effects induced by mixing, the former are numerically suppressed relative to the latter. Since flavor breaking and isospin breaking are generated by the same terms in the effective Lagrangian, this implies that isospin breaking in the vector meson decay constants should also be dominated by mixing effects.

If we take this point of view then, up to sub-leading corrections, we find, for the physical $\rho$ and $\omega$ decay constants, now including isospin breaking and taking into account the relation $F_{\rho}^{(0)} \approx 3F_{\omega}^{(0)}$,

$$
F_{\rho}^{EM} = F_{\rho}^{(0)} - \epsilon F_{\omega}^{(0)} \approx F_{\rho}^{(0)} \left(1 - \frac{\epsilon}{3}\right)
$$

$$
F_{\omega}^{EM} = F_{\omega}^{(0)} + \epsilon F_{\rho}^{(0)} \approx F_{\omega}^{(0)} \left(1 + 3\epsilon\right),
$$

where $\epsilon$ is the leading order mixing angle, defined via

$$
\rho = \rho^{(0)} - \epsilon \omega^{(0)}, \quad \omega = \omega^{(0)} + \epsilon \rho^{(0)}.
$$

We note two relevant features of these results: (1) because of the dominance by mixing, the corrections required to convert the pure isovector $F_{\rho}^{(0)}$ coupling to the experimental $F_{\omega}^{EM}$ coupling is opposite in sign to that required to convert the pure isoscalar $F_{\omega}^{(0)}$ coupling to the $F_{\rho}^{EM}$ and, (2) because of the pattern of ideal mixing and the numerical suppression of the isoscalar current relative to the isovector current in $J_{\mu}^{EM}$, the magnitude of the correction is a factor of 9 larger in the $\omega$ than in the $\rho$ case. This does not mean that, somehow, isospin breaking has become huge in the $\omega$ case, but rather reflects the expected natural suppression of the $\omega$ coupling relative to that of the $\rho$, which is confirmed by experiment.

In general, but particularly in light of the large numerical enhancement just discussed, it is crucial to find a means of estimating the isospin-breaking contributions to the vector meson EM decay constants. In view of the discussion above, although there exist additional effects beyond those associated with mixing at leading order, one can get a rough idea of the size expected for the isospin-breaking vector meson decay constants by considering experimental information on $\rho-\omega$ mixing, and ignoring all non-mixing effects. Although crude, this estimate provides an additional qualitative constraint for our later sum rule analysis.

In order to obtain the parameter $\epsilon$ describing $\rho-\omega$ mixing at leading order, it is sufficient to determine the off-diagonal element, $\Pi_{\rho\omega}$, of the vector meson self-energy matrix. In the past, values for $\Pi_{\rho\omega}$ in the range $\sim -4000$ MeV$^2$ have been quoted, based on simplified analyses of $e^+e^- \to \pi^+\pi^-$ data in the interference region which effectively assume that the one-particle irreducible $\omega^{(0)}\pi^+\pi^-$ vertex is zero, even in the presence of isospin breaking. Since effective operators which generate such a coupling exist in the vector meson effective Lagrangian, however, this assumption is unphysical (in the sense of being incompatible with QCD). Once one includes contributions to the $\omega \to \pi\pi$ amplitude generated both by $\rho-\omega$ mixing and the 1PI vertex (whose strength we will denote by $g_{\omega\pi\pi}$), the analysis of the experimental data is somewhat more complicated but, in principle, allows a separate determination of both $\Pi_{\rho\omega}$ and the isospin-breaking ratio of couplings of the isospin pure states $G = g_{\omega\pi\pi}/g_{\rho\pi\pi}$ [16,17].

An important feature of the analysis framework developed in Ref. [16] is that, from it, one understands that the smallness of previously quoted errors for $\Pi_{\rho\omega}$ is an artifact of the unphysical assumption $G = 0$, and does not survive the more general treatment. It is worth outlining the reason why this is the case since, in doing so, it will become clear that it is a difficult task to improve the experimental situation sufficiently to really pin down the mixing contribution.

The contribution of the physical (i.e., mixed-isospin) $\omega$ to the amplitude for $e^+e^- \to \pi^+\pi^-$ is obtained experimentally by determining the timelike pion form factor, $F_\pi(q^2)$, in the interference region and fitting it to a form

$$
F_\pi(q^2) \propto \left[ \frac{1}{q^2 - m_\rho^2} + \frac{A_{\Delta\rho}}{q^2 - m_{\Delta\rho}^2} \right] + \text{background}
$$

(8)

where $m_\rho^2$ are the complex pole positions, $m_{\Delta\rho}^2 = m_\rho^2 - i m_\rho \Gamma_\rho$, and the fit parameter, $\phi$, is known as the “Orsay phase”. The $\omega$ contribution in Eq. (8) is generated by the coupling of the physical $\omega$ to $\pi^+\pi^-$ which, as discussed above, has two sources: 1PR ($g_{\rho\omega}^{(0)}$ mixing), and 1PI (associated with the $\omega^{(0)}\pi\pi$ vertex). The physical coupling is given, in terms of these contributions, by

$$
g_{\omega\pi\pi} = g_{\omega\pi\pi}^{(0)} + \epsilon g_{\rho\pi\pi}^{(0)}
$$

(9)
where, as usual, the superscript \((0)\) indicates couplings of the unmixed isospin pure states. In the (physically plausible) approximation that one assumes saturation of the imaginary part of \(\Pi \rho \omega\) by \(\pi \pi\) intermediate states, one finds

\[
\text{Im} \Pi \rho \omega (m_\rho^2) = -G m_\rho \Gamma_\rho \tag{10}
\]

and hence, in the narrow \(\rho - \omega\) interference region,

\[
\Pi \rho \omega \sim \tilde{\Pi} \rho \omega - i G m_\rho \Gamma_\rho \tag{11}
\]

where \(\tilde{\Pi} \rho \omega\) is now real. The mixing angle \(\epsilon\) is then given by [16]

\[
\epsilon = \frac{\Pi \rho \omega (m_\rho^2)}{m_\omega^2 - m_\rho^2} = -izT - zG , \tag{12}
\]

where

\[
z = \frac{i m_\rho \Gamma_\rho}{m_\omega^2 - m_\rho^2}, \quad \bar{T} = \frac{\tilde{\Pi} \rho \omega (m_\rho^2)}{m_\rho \Gamma_\rho} . \tag{13}
\]

One then finds, upon substitution of Eq. (12) into Eq. (9), that

\[
g_{\omega \pi \pi} = \left[ G(1 - z) + \frac{\tilde{\Pi} \rho \omega (m_\rho^2)}{m_\rho \Gamma_\rho} \right] g_{\rho \pi \pi}^{(0)} . \tag{14}
\]

In many places in the literature, one finds the approximation \(m_\omega^2 - m_\rho^2 \approx im_\rho \Gamma_\rho\) employed. The deviation of the quantity \(z\) from 1 represents the error made in employing this approximation. Since \(\text{Re} z \approx 1\) and \(\text{Im} z\) is small (\(\sim 0.2\) to \(0.3\)), one might think it rather safe to set \(z = 1\) in the above analysis (this approximation was, in fact, made uniformly in analyses previous to the discussion of Ref. [16]). If it were true that this approximation were reliable, then the effect of \(G\) would cancel exactly [18], and the experimental data would determine the real part of \(\Pi \rho \omega\) in the interference region with the usually quoted errors (\(\tilde{\Pi} \rho \omega(m_\rho^2) = -3844 \pm 271\) MeV\(^2\); see Ref. [17,19,20] and earlier references cited therein).

Unfortunately, it turns out that the approximation is both misleading and unreliable. The reason is that, although \(z\) is approximately real and near 1, \((1 - z)\) is dominantly imaginary. Since the denominator of the second term in Eq. (14) is also dominantly imaginary, the two terms add nearly constructively. Were the phases of these terms to be actually identical, of course, it would be impossible to use experimental information to separate them, no matter how precise that data. Fortunately, there is a small phase difference which, at least in principle, means that a separate determination of the phases of these terms would be possible. If one takes the updated numerical analyses of Ref. [17], for example, one sees that values of \(\tilde{\Pi} \rho \omega\) between \(-4000\) and \(-8000\) MeV\(^2\) are allowed by present data (with a central value \(\sim -6800\) MeV\(^2\)), and that, while the central extraction for \(G\) is moderately large \(\sim 0.1\), \(G = 0\) is only \(2 \pm 2\) \(\sigma\) distant. A significant improvement in this situation would require a significant reduction of the errors in the determination of the Orsay phase. Unfortunately, the prospects for seeing such an improvement in the near future are remote, at present.

Although, with present experimental accuracy, the errors in the determination of \(\epsilon\) are disappointingly large, we can, nonetheless, as explained above, use the range of values obtained in Refs. [16,17] to help set a rough scale for the expected size of those corrections required to go from \(F_{\rho \pi}^{EM}\) to \(F_{\rho}^0\), and \(F_{\omega}^{EM}\) to \(F_{\omega}^0\). Using the central values for the four fits given in Table 1 of Ref. [17], one finds that \(F_{\rho}^0\) is less than \(F_{\rho}^{EM}\) by between 0.3 and 3.8\% (the former corresponding to fixing \(G = 0\) by hand, the latter to the MOW and A solutions contained in the Table of Ref. [17]) and \(F_{\omega}^0\) greater than \(F_{\omega}^{EM}\) by between 2.6 and 24.6\%. We will see that the solutions obtained below via the sum rule analysis satisfy these rather loose constraints, and, in fact, provide an alternate determination which is to be favored since it both has considerably smaller errors and is free of the uncertainties associated with effectively truncating at leading chiral order (which goes into the estimate just discussed).
III. QCD SUM RULES AND THE CHOICE OF THE FESR METHOD

As is well-known, the properties of unitarity and analyticity lead to the existence of various (appropriately subtracted) dispersion relations for typical hadronic correlators, \( \Pi(q^2) \). By the term “dispersion relations” we mean here both those relations based on the Cauchy representation theorem, and those based on Cauchy’s theorem (the vanishing of the integral of an analytic function over a closed contour entirely in the region of analyticity). The generic term “QCD sum rules” describes those versions of these relations obtained by making kinematic restrictions which allow one to take advantage of the asymptotic freedom of QCD. In the first case, one simply writes the representation theorem for \( \Pi(q^2) \) with \( q^2 \) large and spacelike; in the second, a choice of the integration contour is made so that at least part of the contour lies in the region of large spacelike momenta. In either case, the point of the kinematic restriction is to take advantage of the fact that, in the region of large spacelike momenta, computational techniques based on the operator product expansion (OPE)/perturbative QCD (pQCD) are applicable.

The generic form of the sum rules generated using the representation theorem is then, up to possible subtractions,

\[
\Pi(q^2) = \int_{s_{th}}^{\infty} ds \frac{\rho(q^2)}{s - q^2},
\]

where the \( s_{th} \) is the lowest physical threshold in the channel in question and the spectral function, \( \rho \), is defined as usual by \( \rho = \frac{1}{2} \text{Im} \Pi \). The LHS is to be computed using the OPE/pQCD, while the RHS is given in terms of measured spectral data and/or some spectral ansatz (involving a limited number of unknown parameters) for the unmeasured part of the spectral function. As first pointed out by Shifman, Vainshtein and Zakharov (SVZ) [21], the utility of such relations is greatly improved by Borel transformation. The effect of the Borel transform is (1) to replace the weight \( 1/(s - q^2) \) in the hadronic spectral integral on the RHS of Eq. (15) with \( \exp(-q^2/M^2) \), where \( M \), the Borel mass, is a parameter of the transformation, (2) to kill all subtraction terms and (3) to create a factorial suppression of the contributions of higher dimensional operators on the OPE side of the equation \( (c/(Q^2)^n \rightarrow c/(n-1)!M^{2n}) \).

Ideally, on the hadronic side, one would like to choose \( M \) as small as possible, in order to suppress the contributions from the large-\( s \) part of the integral, for which the spectral function will typically be complicated, and difficult to model. In contrast, to improve the convergence of the OPE when truncated at operators of relatively low dimension, one wants \( M \) as large as possible. To be usable, SVZ sum rules thus require a compromise: one must hope to find a “stability window” in \( M \), i.e., a region of \( M \) values for which both contributions from the complicated part of the spectral distribution (on the hadronic side) and from the highest dimension operators (on the OPE side) are not too large. Typically, because of the compromise nature of the choice of stability window, this means that neither the contributions from the large-\( s \) part of the spectrum, nor that from the highest dimension operator retained on the OPE side, are negligible [21–24].

For the case of sum rules based on Cauchy’s theorem, a common choice of integration contour (and the one we will employ in the analysis below) is that shown in Fig. 1. The radius, \( s_{0} \), of the circular part of the contour is to be taken large enough that the OPE, to the order available, is reliable in the spacelike region of the circle. The resulting sum rule is then generically of the form

\[
-\frac{1}{2\pi i} \int_{C} dq^2 w(q^2)\Pi(q^2) = \int_{s_{th}}^{s_{0}} dq^2 w(q^2)\rho(q^2),
\]

where \( w(q^2) \) is any function of \( q^2 \) analytic in the region of integration, and \( C \) denotes the circular part of the contour, traversed counterclockwise (from above to below the cut). The OPE is then to be used on \( C \) (see also the discussion to follow), while spectral data and/or a spectral ansatz is to be employed in the “hadronic” integral on the RHS. Such sum rules are called, generically, finite energy sum rules (FESR’s), and have usually been employed with integer-power weights \( w(s) = s^k \), with \( k = 0, 1, 2, \cdots \) [25,26], though the standard theoretical treatment of hadronic \( \tau \) decays, which is of the FESR type, involves a more complicated weight, as determined by kinematics [1].

It is useful, for the discussion which follows, to recall at this point the distinction between “local” and “semi-local” duality. For a typical hadronic correlator, \( \Pi(q^2) \), as above, the OPE is expected to be reliable, not only for \( q^2 \) large and spacelike, but also for those \( q^2 \) on any circle of sufficiently large radius in the complex \( q^2 \)-plane, apart possibly from some region whose size is hadronic in scale about the timelike real axis (where the effects of confinement are expected to be important) [27]. The term “local duality”, in this context, refers to the postulate (underlying, for example, all statements giving the number of subtractions required to make a given dispersion relation convergent in QCD) that, once one is in a region \( q^2 \sim s_{0} \) for which the separation between adjacent resonances is small compared to the typical resonance width, the region of validity of the OPE on the circle of radius \( s_{0} \) extends all the way down to the real timelike axis. This is equivalent to saying that the hadronic spectral function can then be calculated using the OPE. This is what allows, for example, the hadroproduction ratio, \( R \), in \( e^+e^- \) scattering to be computed.
theoretically in the asymptotic region using perturbative methods. Similarly, “semi-local duality” refers to the idea that, at somewhat lower (“intermediate”) scales, where local duality is no longer valid, nonetheless, averaged over some range of (timelike) momenta, the mean values given by using either the actual hadronic spectral function or the OPE version thereof should be the same. It is important to understand that, empirically, the condition that resonance spacing be much smaller than typical resonance widths is crucial to the validity of local duality. Indeed, one can test local duality through the application of various FESR’s in the case of the isovector vector channel, for which the hadronic spectral function is very accurately measured in hadronic $\tau$ decays [9]. One finds that, even at scales as large as $m_{\tau}^2 \sim 3.2 \text{ GeV}^2$, and even though the experimental spectral function appears rather featureless in this region, nonetheless, local duality is rather poorly satisfied [28]. (Note that in this channel, at these scales, the separation of resonances is comparable to their widths.)

In light of the above discussion, we distinguish three (timelike) kinematic regimes: “low” (the region of narrow, well-isolated resonances), “intermediate” (the region of the validity of semi-local duality) and “high” (the region of the validity of local duality, for which the hadronic spectral function can be reliably obtained using pQCD/OPE methods). The distinction is important because of the ways, described above, in which sum rules are practically implemented. Typically, one attempts to use known information, contained in the OPE (i.e., $a(Q^2) = \alpha_s(Q^2)/\pi$, and the values of various vacuum condensates), to place constraints on the parameters of the (hopefully, physically-motivated) ansatz for the corresponding unknown (or not fully-known) spectral function. This attempt will, of course, be successful only if (1) one is able to work at scales for which the OPE, truncated at operators of relatively low dimension, and at a given perturbative order for the Wilson coefficients of these operators, is well-converged, and (2) one has a qualitative form for the hadronic spectral ansatz which is both physically plausible and does not involve a large number of unknown parameters. The latter condition can, realistically, be satisfied only if the scale up to which one needs the spectral ansatz is not too far into the intermediate region. Once the higher resonances start to overlap, and various multiparticle background processes start to become important, the general form of the spectral function will become increasingly difficult to anticipate in advance (until, that is, one is at sufficiently large scales that local duality finally becomes valid).

In the SVZ version of QCD sum rules, as explained above, it is only rarely possible to choose the stability window for the analysis in such a way as to avoid contributions from the region where either one does not know the qualitative form of the spectral function, or where, even if one does, to implement it fully would involve the use of more free parameters than could be determined reliably, given the limited amount of information available in the truncated OPE [24]. Conventionally, this problem is dealt with by employing a spectral ansatz in which (1) the low-$s$ region is assumed to be dominated by one or two low-lying resonance contributions and (2) the intermediate- and high-$s$ region is approximated using the local duality version of the spectral function, which one assumes to start at some “continuum threshold”, $s_0$. It is well-known that this form of “continuum ansatz” represents a rather crude approximation, and hence can create significant uncertainties in the analysis if the contributions from the continuum part of the spectral integral are large for $M$ values in the stability window. Typically, one attempts to minimize this problem by avoiding $M$ values for which this is the case, but the cost of doing so is poorer convergence on the OPE side of the sum rule and hence increased uncertainties associated with neglect of higher dimension contributions.

A similar problem exists for the integer-power weighted version of FESR’s. One advantage of the FESR approach is that, in contrast to the SVZ method, the choice of scale appearing on the OPE side of the sum rule ($s_0$) is not constrained by the requirement of working in a stability window. One is, of course, still forced to choose $s_0$ not too far into the intermediate region, since otherwise the spectral ansatz would be too complicated to allow a reliable analysis. The possibility of working at such intermediate scales in the FESR approach can, nonetheless, represent a practical advantage over the SVZ approach in certain cases. This is true of those channels for which the stability window of the SVZ analysis lies at relatively low $M$ (e.g., $M \sim 1 \text{ GeV}^2$, as found for many applications in the literature). In such cases, the larger scale of the FESR analysis leads to an improvement of both the convergence by operator dimension and convergence by perturbative order of the Wilson coefficients on the OPE side of the sum rule. Unfortunately, this advantage is usually more than offset by an increase in the difficulties associated with the use of the local duality approximation in the intermediate region. The reason is obvious: with integer-power weights, the region near the timelike real axis where the circular part of the contour joins the cut, does not have the exponential suppression present for the “continuum” contributions in the SVZ approach. The problems that result can be quantified in the case of the isovector vector channel, where the hadronic spectral function is known experimentally. As shown in Ref. [28], the errors in integer-power weighted FESR’s, even at scales as high as $m_{\tau}^2$, can be very large, despite the fact that the OPE at this scale is both dominated by the leading ($D = 0$) perturbative term, and rather rapidly converging. (A more detailed discussion of why it is that one cannot judge the degree of validity of the local duality approximation simply by looking at the degree of convergence of the OPE in a given region is given in Ref. [28].)

These problems, encountered in employing the conventional, integer-power-weighted version of FESR’s, are, however, not intrinsic to the FESR approach. Indeed, there is at least one example of a non-integer-power-weighted FESR that is very well-satisfied: that giving the hadronic $\tau$ decay widths in terms of an integral over the circle
of radius $s_0 = m_2^2$ of the product of the OPE for the isovector vector current correlator and the weight function $w_r(s) = (1 - s/m_2^2)^2 (1 + 2s/m_0^2)$ (where the dominant input parameter in the OPE representation is $a(m_2^2)$, which can be taken as obtained by running the value measured at the $Z$ mass down to the $\tau$ scale). The reason for the success of this sum rule is well-known: the juncture of the cut and circular portions of the FESR contour corresponds to the edge of hadronic phase space and hence, because of kinematics, the weight function $w_r(s)$ has a (double) zero at $s = m_2^2$. This zero suppresses contributions from the portion of the circular part of the contour near the real timelike axis for which the OPE representation of the correlator is not reliable (at intermediate scales like $m_2^2$) [1]. This immediately suggests that the appropriate way to implement FESR’s in other channels is, not with integer-power weights, but rather with weights having a zero at $s = s_0$. In Ref. [28] it was shown that, in the isovector vector channel, where one can check the procedure explicitly, weight functions of either the form

$$w_s(s) = \left(1 - \frac{s}{s_0}\right) \left(1 + A\frac{s}{s_0}\right),$$

having a single zero at $s = s_0$, or

$$w_d(s) = \left(1 - \frac{s}{s_0}\right)^2 \left(1 + A\frac{s}{s_0}\right),$$

having a double zero at $s = s_0$, produce extremely well-satisfied FESR’s for a wide range of values of $s_0$ and the continuous parameter, $A$. In addition, using only the OPE representation, for a range of $A$ and $s_0$, and fitting the parameters of a sum-of-resonances ansatz to this representation, results in a very good reconstruction of the hadronic spectral function, including a determination of the $\rho$ decay constant accurate to within a few % [28]. In order to have a compact terminology for use in describing the weight families in Eqs. (17) and (18), we will refer to $w_s(s)$ and $w_d(s)$ as single-pinch and double-pinch weights, respectively. The freedom to vary $A$ plays a role analogous to that of the variation of $M$ within the stability window in an SVZ-style analysis, in that it strengthens the sum rule constraints on the parameters of the spectral ansatz by working with a range of different weight profiles. An additional advantage, at least if one wishes to determine not just the parameters of the lowest resonance in the channel but also those associated with higher resonances, is that the weight function can actually be arranged to be larger in the second resonance region than in the first.

In what follows, in light of its success in the isovector vector channel, we will employ the FESR framework, using both the single- and double-pinch weight families, to investigate the isospin-breaking vector current correlator, $\Pi^{38}$, defined above. As usual in the FESR framework, we will work at scales as high as possible, compatible with the constraint of having a tractable and physically sensible spectral ansatz for $s < s_0$. Since little is known about the vector meson resonance spectrum beyond the second excited resonance region, and since including even the second excited resonance region would lead to a spectral ansatz with more parameters than are generally tractable for the present analysis, we are forced to work at scales no higher than $\sim 2.8$ GeV$^2$. Since the separation of the first and second excited vector meson resonance regions is comparable to the resonance widths (the $\rho'$ and $\omega'$ lie at 1419 and 1452 MeV, the $\rho''$ and $\omega''$ at 1723 MeV and 1649 MeV, respectively [15]), it is clear that, at these scales, we are not yet in the region of the validity of local duality. This makes the use of the single- and double-pinch families crucial to the reliability of the FESR analysis. In order to maintain as good convergence as possible on the OPE side of the two sum rule families, while at the same time allowing enough variation in $s_0$ to get a good determination of the parameters of the spectral ansatz, we also restrict our attention to scales, $s_0$, greater than 2 GeV$^2$.

**IV. DETAILS OF THE ANALYSIS**

Since the general framework to be employed in the analysis has been outlined in the previous section, it remains only to discuss the input required on the hadronic and OPE sides of the various sum rules.

We begin with the hadronic side. We take, as our ansatz for the hadronic spectral function, a sum of resonance contributions. For the scales used in the analysis, the resonances present in the region of the hadronic spectral integral are the $\rho$, $\omega$, $\phi$, $\rho'$ and $\omega'$. (Although the tails of the $\rho''$ and $\omega''$ intrude slightly into the hadronic integration region for $s_0$ near 2.8 GeV$^2$, their contributions are strongly suppressed by the zeros in the weight functions. We have checked that including an effective, combined $\rho''-\omega''$ contribution in the spectral ansatz has negligible effect on the extracted $\rho$, $\omega$ and $\phi$ spectral strength parameters.) We thus include contributions, written in terms of Breit-Wigner resonance forms, for all these resonances. Because the separation of the $\rho'$ and $\omega'$ is much smaller than either of their widths, and also to reduce the number of free parameters in the spectral ansatz, we have combined the latter two contributions.
The strong overlap of the two resonances would, in any case, prevent one from being able to sensibly extract separate $\rho'$ and $\omega'$ strength parameters by means of any sum rule analysis of $\Pi^{38}$.

The spectral ansatz then has the form

$$\rho^{38}(q^2) = \frac{1}{4\sqrt{3}} \left[ f_{\rho} \delta(q^2 - m_\rho^2) - f_{\omega} \delta(q^2 - m_\omega^2) + f_{\phi} \delta(q^2 - m_\phi^2) + f_{\rho'\omega'} \delta(q^2 - m_\rho' \omega') \right],$$  

where

$$\hat{\delta}(q^2 - m^2) \equiv \frac{1}{\pi} \frac{m\Gamma}{(q^2 - m^2)^2 + m^2\Gamma^2}$$

(with $\Gamma$ the width of the resonance in question). This expression reduces to $\delta(q^2 - m^2)$ in the narrow width approximation (NWA). The minus sign in front of $f_{\omega}$ and the factor of $1/4\sqrt{3}$ are conventional; inclusion of the former ensures that $f_{\omega}$ and $f_{\rho}$ become equal in the limit that the spectral contributions in the $\rho-\omega$ region are generated entirely by leading order $\rho-\omega$ mixing.

For the combined $\rho'-\omega'$ contribution we have taken average values for the effective mass and width, $f_{\rho'}, f_{\omega}', f_{\phi}$ and $f_{\rho'\omega'}$ are free parameters, to be determined from the matching of hadronic and OPE sides of the single- and double-pinch sum rules, for a range of $s_0$, $A$ values.

A few comments are in order concerning the form of the ansatz above, and the physical meaning of the parameters to be extracted from the analysis which follows.

The first concerns the need for the inclusion of a $\phi$ contribution. Note that the correlator $\Pi^{38}$ is very closely related to that, $\Pi^{38\phi}$, obtained by dropping the strange part of the hypercharge current from $\Pi^{38}$ (the OPE’s for the two correlators are, in fact, identical to three-loop order). The latter correlator has been studied in a number of earlier SVZ-style analyses [29–32]. In the earliest of these, the NWA was employed for all resonances, and no $\phi$ contribution was included in the spectral ansatz [29,30]. As pointed out in Ref. [31], however, the existence of significant cancellations between the NWA $\rho$ and $\omega$ contributions (which would be exact in the limit of mixing dominance and equality of $\rho$ and $\omega$ masses) means that a $\phi$ contribution, even if significantly smaller than the individual $\rho$ and $\omega$ contributions, could nonetheless be important. Performing the sum rule analysis with a $\phi$ contribution included shows that this is, indeed the case [31]. Including the $\phi$ contribution in the spectral ansatz also cured an unphysical feature of the solutions obtained earlier, which did not include it [31]. The analysis of Ref. [31], however, still employed the NWA for all resonances.

The second point concerns the need to incorporate the $\rho$ width into the analysis. Because, again, of the high degree of cancellation between the NWA $\rho$ and $\omega$ contributions, it was pointed out that the precise degree of this cancellation might well be sensitive to whether or not the difference between the $\rho$ and $\omega$ widths was retained in the spectral ansatz [32]. The subsequent analysis of Ref. [32] showed that this is, indeed, the case: the spectral parameters, $F_{\phi}$, decrease by factors $\sim 6$ when one employs the physical widths in place of the NWA.

The third point concerns the interpretation of the higher resonance strength parameters, $f_{\phi}$ and $f_{\rho'\omega'}$. It is, of course, very natural to take the spectral function to be resonance dominated. Moreover, the near-threshold region of the spectral function has been computed to two loops in ChPT [33], and one can see from this result that the corresponding low-$s$ background contribution to the relevant spectral integrals is tiny compared to that from the $\rho-\omega$ region. The case of the $\phi$, however, is less clear, since background contributions above the $\rho-\omega$ region are not amenable to as reliable estimates, and hence might not be similarly negligible. In the ansatz as written, such physical contributions, if present, could only be mocked up (approximately) by additional effective contributions to the $\phi$ and $\rho'-\omega'$ strengths. Thus, one must use some caution in interpreting, for example, the extracted $f_{\phi}$ in terms of the physical resonance parameters $F_{\phi}$ and $F_{\phi}'$ – some portion of $f_{\phi}$ could actually correspond to an averaged version of background contributions in the region between $\rho-\omega$ and $\rho'-\omega'$. The quality of the agreement between the hadronic and OPE sides of our sum rules is, however, post facto evidence in favor of resonance dominance, and hence also in favor of the possibility of interpreting $f_{\phi}$ in terms of $F_{\phi}$ and $F_{\phi}'$.

Let us turn, then, to the input on the OPE side of the sum rules. We will discuss the contributions, in turn, by operator dimension.

Since the correlator in question is isospin-breaking, the only dimension $D = 0$ contribution to $\Pi^{38}$ is electromagnetic (we adhere, here, to common usage, according to which the leading mass-dependent perturbative terms are labelled $D = 2$). We retain only the leading order (2-loop) graph in this case.

The $D = 2$ contributions are dominated by the strong interaction terms proportional to $(m_d - m_u)^2$. To 3-loop order, the results for these terms follow from the 3-loop expressions for the correlator involving a flavor-non-diagonal current and its conjugate [34], since the perturbative contributions involving two quark loops and a purely gluonic intermediate state (present for flavor diagonal currents but not for flavor-non-diagonal currents) do not enter until 4-loop order. The resulting expressions are given in the Appendix. To evaluate them, we require the running masses,
\( m(Q^2) \), and running strong coupling, \( \alpha_s(Q^2) \). These can be obtained once the values are determined at any fixed scale, \( \mu_0 \). Since the 4-loop \( \gamma \) [35] and \( \beta \) [36] functions for QCD are now known, we have employed these when running the masses and coupling (explicitly, we solve the RG equations exactly, using the truncated 4-loop \( \gamma \) and \( \beta \) functions as input).

As input for the running coupling, we take \( \mu_0 = m_\tau \) and use the latest (1998) value for \( \alpha_s(m_\tau^2) \) obtained by the ALEPH Collaboration in their analysis of non-strange hadronic \( \tau \) decays [37]. (The analysis of the strange decays employed previous theoretical results for the \( D = 2 \) terms, proportional to \((m_s - m_u)^2\), which turn out to be in error \([38-40]\); the value obtained in the global ALEPH analysis must, therefore, be excluded.)

The situation with the light quark mass difference \( \delta m \equiv (m_d - m_u) \) is somewhat more complicated. We first write

\[
\delta m = \left( \frac{m_d - m_u}{m_d + m_u} \right) (m_d + m_u) \equiv r(m_d + m_u) .
\]

The isospin-breaking mass ratio, \( r \), is known, from a number of ChPT analyses, to be \( r = 0.288 \pm 0.037 [41] \), which would allow one to determine \( \delta m \) if \( m_d + m_u \) were known. The most recent determination of \( m_d + m_u \) is that based on an integer-power-weighted FESR analysis of the isovector pseudoscalar channel [26,42]. In this analysis, the pion pole contribution to the spectral function is known experimentally but the continuum contribution is not. The authors of Refs. [26,42], therefore, constructed an ansatz for the unmeasured continuum contribution. It turns out that the continuum portion of the resulting model spectral function provides roughly 3/4 of the contribution to the extracted value of \( (m_d + m_u)^2 \). Unfortunately, it has recently been shown, using the FESR framework discussed above, that this continuum ansatz is unphysical [28], so one cannot employ the values of Refs. [26,42].

If \( m_u \) were known (at some scale), then one could straightforwardly determine \( m_d + m_u \) (at that same scale) using the known (scale-independent) ratio of masses, \( r_s = \frac{m_s}{m_u + m_d} \), obtained by Leutwyler [41] using ChPT. Unfortunately, the situation is also somewhat complicated for \( m_s \). A number of recent analyses produce values of \( m_s(1 \text{ GeV}^2) \) (in the \( \overline{\text{MS}} \) scheme) ranging from \( \sim 110 \text{ MeV} \) to \( \sim 210 \text{ MeV} \), often with rather large errors \([43-47,38,39,48]\). Because the analyses based either on flavor breaking in hadronic \( \tau \) decays [38,39] or Narison’s \( \tau \)-decay-like sum rule for \( \Pi^3 - \Pi^{\overline{3}} \) [46,47] have rather large errors resulting from experimental uncertainties which are unlikely to be significantly improved in the near future, the most favorable approach would appear to be that based on various sum rule treatments of the strange scalar channel, where the dominant \( K\pi \) part of the spectral function is in principle determined, via the Omnès representation of the timelike scalar \( K\pi \) form factor, in terms of experimental \( K\pi \) phase shifts and \( K_{\overline{3}} \) data [43]. The most recent analyses of this channel [45,48] employ the SVZ framework, and produce values \( m_s(1 \text{ GeV}^2) = 125 - 160 \text{ MeV} [45] \), and \( 160 \pm 30 \text{ MeV} [48] \). (The same low-s part of the spectral function is used in both analyses; the only difference between the two lies in the treatment of the “continuum”. The results of Ref. [48], in addition, show no stability window for \( m_s \).) Preliminary work using the FESR framework discussed above indicates that residual errors associated with the use of the local duality approximation in the continuum region remain, for this channel, when one uses the SVZ approach. (See, e.g., the results of Ref. [28]. From these one can see (1) that using the central values for the parameters describing the fit to the \( K\pi \) phases from Refs. [43,45], together with the central value from the \( m_s \) range from Ref. [45], one obtains rather poorly satisfied families of FESR’s, and (2) that using the spectral function of Refs. [43,45,48], again with central values for the fit parameters, the FESR analysis, in fact, produces \( m_u \) values larger by \( \sim 20 \text{ MeV} \) than those obtained in the analysis of Ref. [45]). Although work on the extraction of \( m_s \) is still in progress [49], we conclude already from the preliminary results noted above, that \( m_s(1 \text{ GeV}^2) \sim 165 \text{ MeV} \), probably with errors \( \sim \pm 20 \text{ MeV} \) or less. The ChPT ratio then produces \( (m_d + m_u)(1 \text{ GeV}^2) \sim 13.5 \text{ MeV} \), with errors \( \sim \pm 2 \text{ MeV} \). For any value in this range it turns out that the \( D = 2 \) contributions are at the \( \sim 15\% \) (or less) level of the \( D = 4 \) contributions, and the resulting errors lead, therefore, to very small (\% level) uncertainties in the final results. Since these uncertainties are much smaller than those generated by the uncertainty in the isospin-breaking mass ratio \( r \), we have employed the central value \( (m_d + m_u)(1 \text{ GeV}^2) = 13.5 \text{ MeV} \), and retained only the uncertainty in \( r \), in the analysis which follows.

The \( D = 4 \) contributions are much more straightforward. Although in principle both those \( D = 4 \) terms proportional to the isospin-breaking mass difference, \( \delta m \), and those proportional to the isospin-breaking condensate difference, \( < \bar{d}d > - < \bar{u}u > \), appear in the OPE of \( \Pi^3 \), the latter are numerically tiny compared to the former. The dominant \( D = 4 \) contribution can then be written in terms of \( r \) and the combination \( (m_d + m_u) < \bar{q}q > \), which we can take from the GMO relation

\[
(m_d + m_u) < \bar{q}q > = -\frac{m_s^2}{2} .
\]

The dominant uncertainties for the \( D = 4 \) terms thus result from those on \( r \).

The phenomenological situation is not so favorable in the case of the \( D = 6 \) condensates. Usually, in the absence of pre-existing determinations of the relevant condensates, one makes estimates based on the vacuum saturation approximation.
approximation (VSA). It is well known that, in situations where it has been possible to perform phenomenological
checks by extracting the total $D = 6$ contribution from data, the VSA has proven to significantly underestimate these
ccontributions [50]. Usually one simply replaces the factor $\alpha_s <\bar{q}q>$, which is produced by the VSA, by an effective
scale-independent factor, written $\rho'\alpha_s <\bar{q}q>^2$. The parameter, $\rho'$, then represents the deviation from the VSA.
Ideally, it should either be possible to determine $\rho'$ from data, or the $D = 6$ contributions should be small, for the
sum rule in question. In our case, neither of these conditions holds. In particular, because we are forced to work at
scales as low as 2 GeV$^2$ in order to constrain the spectral parameters, the $D = 6$ contributions can, for $s_0 \sim 2$ GeV$^2$,
and certain values of $A$ employed in our analysis, approach $\sim 40\%$ of the leading $D = 4$ term. Fortunately, it turns
out, as we will see explicitly below, that by working with both the single- and double-pinch weight families, we can
actually obtain a rather good determination of the $D = 6$ contribution to the correlator (albeit it as a function of $r$)
by insisting on the consistency of the results obtained from the two different sum rule families.

In the Appendix, it is shown that the VSA leads to an expression for the $D = 6$ contribution to $\Pi^{38}$ proportional to
\begin{equation}
\alpha_s \left( \langle \bar{d}d \rangle^2 - \langle \bar{u}u \rangle^2 \right) = \gamma \left( \alpha_s \langle \bar{q}q \rangle^2 \right), \tag{23}
\end{equation}
where $\langle \bar{q}q \rangle$ is the average of the $u$ and $d$ condensates, and
\begin{equation}
\gamma \equiv \frac{\langle \bar{d}d \rangle}{\langle \bar{u}u \rangle} - 1, \tag{24}
\end{equation}
describes isospin-breaking in the light quark condensates. In order to compare the deviation from the VSA in the
isospin-breaking channel with that in the analogous isospin-conserving isovector vector ($ab = 33$) channel, we write the
re-scaled version of the RHS of Eq.(23) in the form
\begin{equation}
\rho' \alpha_s \left( \langle \bar{d}d \rangle^2 - \langle \bar{u}u \rangle^2 \right) = \rho_{\text{red}} \gamma \left( \rho \alpha_s \langle \bar{q}q \rangle^2 \right), \tag{25}
\end{equation}
where $\rho$ is the parameter describing the deviation from the VSA in the 33 channel, and one has, phenomenologically,
[51]
\begin{equation}
\rho \alpha_s \langle \bar{q}q \rangle^2 = (5.8 \pm 0.9) \times 10^{-4} \text{ GeV}^6. \tag{26}
\end{equation}
With this definition, $\rho_{\text{red}}$ reduces to 1 in the limit that the deviation from the VSA is the same in the 33 and 38
channels.

The consistency procedure for fixing the $D = 6$ contribution to $\Pi^{38}$, together with the phenomenological input of
Eq. (26), of course, determines only the product $\rho_{\text{red}}\gamma$. In presenting our results for $\rho_{\text{red}}$ below, we have taken
$\gamma \simeq -0.008$, which represents an average of the previous determinations listed in Ref. [30], bar one. (We omit the value
based on an analysis of baryon splittings because it implies (via the 1-loop ChPT relations between flavor-breaking
and isospin-breaking in the light quark condensate [4]) $(\bar{s}s)/(\bar{u}u) > 1$, which appears unphysical). We will discuss the
determination of $\rho_{\text{red}}$ in more detail below when we present the results of the analysis.

The last point in need of discussion concerns the way in which we handle the integrals on the OPE side of the
various FESR’s. Two options exist in the literature. The first, sometimes called the “fixed order expansion”, involves
first expanding $\alpha_s(Q^2)$ and the mass factors, generically $m(Q^2)$, in terms of $\alpha_s(s_0)$ and $m(s_0)$. The coefficients of the
perturbative expansions in powers of $\alpha_s(s_0)$ are then polynomials in $\ln(s/s_0)$ [52], and the desired contour integrals can
thus be written in terms of elementary integrals involving logarithms and powers of $s$. The integrated OPE expressions
which result involve $m(s_0)^2$, multiplied by a power series in $\alpha_s(s_0)$. There is, of course, in this expression, the usual
residual dependence on the choice of scale $s_0$ for the expansions discussed above, which results from truncating the
full perturbative series at fixed order. The second alternative, often referred to as “contour improvement”, involves
numerically integrating the factors $[m(Q^2)]^k [\alpha_s(Q^2)]^j s^n$ around the circular contour in the $s = -Q^2$ plane [53]. It
is known that this has the effect of simultaneously improving the convergence of the perturbative series and reducing
the residual scale dependence [53,26,42]. As a result, we have evaluated all the integrals on the OPE sides of our sum
rules using this approach.

Let us now turn to the results, which are presented in the Table. As explained above, the dominant uncertainty is
due to that in the ChPT determination of $r$. We have, therefore, tabulated the results for the range of values corresponding
to the errors on $r$ quoted by Leutwyler [41]. All results are based on matching the hadronic and OPE sides of the two sum rule families for $s_0$ in the range 2.0 to 2.8 GeV$^2$, and with $A$ in the range 2 to 5 for the single-pinch
case and 3 to 6 in the double-pinch case. The choice of range of $A$ in each case has been made so as to keep the
convergence of the perturbative series for the $D = 2$ term under control. It is worth mentioning that the quality of the
match between the OPE and hadronic sides which results after the fitting of the spectral parameters is significantly better for Leutwyler’s central value of $r$.

The value of the $D = 6$ VSA-violating parameter, $\rho_{\text{red}}$, given in the Table, is determined by requiring that the values of $f_\rho$ obtained using the single- and double-pinch weight families are the same. The sensitivity of $f_\rho$ to variations in $\rho_{\text{red}}$ (true also of the other $f_V$), as well as the difference in the $\rho_{\text{red}}$ dependence of $f_\rho$ for the single- and double-pinch analyses, is shown in Figure 2. The fact that, once $\rho_{\text{red}}$ has been determined by the requirement of the consistency of the two output $f_\rho$ values, all the rest of the spectral parameters, determined using either the single- or double-pinch weights, also become consistent is strong evidence in favor of the reliability of the analysis. Note that (1) the possibility of determining the correction to the VSA for the $D = 6$ operators, and (2) the inclusion of both the $D = 2$ terms and the $O(\alpha_s, \alpha_s^2)$ contributions to the Wilson coefficient of the $D = 4$ term, are features not present in previous analyses of the analogous isospin-breaking $\Pi^{\omega\omega}$ correlator. Although the value of $\rho_{\text{red}}$, determined as just described, depends somewhat on $r$, this dependence is not strong, and we obtain $\rho_{\text{red}} = 1.15 \pm 0.15 \pm 0.2$. The first error corresponds to that in Eq. (26), the second to that on $r$. We see that the violation of the VSA is very similar in both the 33 and 38 channels. The importance, in reducing the errors on the determinations of the spectral parameters, $f_V$, of being able to determine $\rho_{\text{red}}$ is also evident from Fig. 2.

Having determined the $D = 6$ contributions by self-consistency, the errors on the extracted values of $f_V$ are determined solely by those on $r$, and are $\sim 10 - 15\%$, completely correlated with $r$.

Having extracted the parameters $f_V$, it is straightforward to determine the isospin violating decay constants. One finds

\begin{align*}
F_\rho^8 &= 2.4 \pm 0.3 \text{ MeV} \\
F_\omega^3 &= -3.4 \pm 0.4 \text{ MeV} \\
F_\phi^3 &= 0.33 \pm 0.02 \text{ MeV},
\end{align*}

where the errors reflect those on the input isospin-breaking mass ratio, $r$.

V. SUMMARY AND DISCUSSION OF PHENOMENOLOGICAL CONSEQUENCES

A number of useful general observations follow from the analysis above. First, we have found that the violation of the VSA for the $D = 6$ condensates is very similar in the isospin-breaking (38) and isospin-conserving (33) vector current channels. Second, we have seen that, although the analysis can be performed successfully for any $r$ in the range given by Leutwyler, the central value of that range is preferred, in the sense of giving the best match between OPE and hadronic sides of both the single- and double-pinch families of sum rules. Finally, we have demonstrated that the FESR method, particularly when implemented using both the single- and double-pinch weight families, is very effective, allowing a determination of the VSA method, particularly when implemented using both the single- and double-pinch weight families, is very effective, allowing a determination of the VSA coefficient of the $D = 4$ term, are features not present in previous analyses of the analogous isospin-breaking $\Pi^{\omega\omega}$ correlator. Although the value of $\rho_{\text{red}}$, determined as just described, depends somewhat on $r$, this dependence is not strong, and we obtain $\rho_{\text{red}} = 1.15 \pm 0.15 \pm 0.2$. The first error corresponds to that in Eq. (26), the second to that on $r$. We see that the violation of the VSA is very similar in both the 33 and 38 channels. The importance, in reducing the errors on the determinations of the spectral parameters, $f_V$, of being able to determine $\rho_{\text{red}}$ is also evident from Fig. 2.

Having determined the $D = 6$ contributions by self-consistency, the errors on the extracted values of $f_V$ are determined solely by those on $r$, and are $\sim 10 - 15\%$, completely correlated with $r$.

Having extracted the parameters $f_V$, it is straightforward to determine the isospin violating decay constants. One finds

\begin{align*}
F_\rho^8 &= 2.4 \pm 0.3 \text{ MeV} \\
F_\omega^3 &= -3.4 \pm 0.4 \text{ MeV} \\
F_\phi^3 &= 0.33 \pm 0.02 \text{ MeV},
\end{align*}

where the errors reflect those on the input isospin-breaking mass ratio, $r$.

V. SUMMARY AND DISCUSSION OF PHENOMENOLOGICAL CONSEQUENCES

A number of useful general observations follow from the analysis above. First, we have found that the violation of the VSA for the $D = 6$ condensates is very similar in the isospin-breaking (38) and isospin-conserving (33) vector current channels. Second, we have seen that, although the analysis can be performed successfully for any $r$ in the range given by Leutwyler, the central value of that range is preferred, in the sense of giving the best match between OPE and hadronic sides of both the single- and double-pinch families of sum rules. Finally, we have demonstrated that the FESR method, particularly when implemented using both the single- and double-pinch weight families, is very effective, allowing a determination of the $\rho^\omega$ spectral parameters, $f_V$, with rather small errors. These errors (between 10 and 15\% for $f_\rho$, $f_\omega$ and $f_\phi$) are a factor of 3 smaller than those obtained in the earlier analysis of Ref. [10] based on results of an SVZ analysis of $\Pi^{\omega\omega}$ [32]. In order to obtain this level of reduction, the ability to self-consistently determine $\rho_{\text{red}}$ was crucial.

Let us now turn to the phenomenological consequences of our results. First note that the corrections required to convert the measured contribution of the vector meson, $V$, to the EM spectral function, $\rho_{\text{EM}}$, into the corresponding contribution to either $\rho^3$ (for $V = \rho$) or $\rho^8$ (for $V = \omega, \phi$) are given by the ratios

\begin{align*}
\left[ \frac{F_\rho^3}{F_\rho^8} \right]^2 &= 0.982 \pm 0.0021 \\
\left[ \frac{F_\omega^3}{\sqrt{3} F_{\text{EM}}^\omega} \right]^2 &= 1.154 \pm 0.017 \\
\left[ \frac{F_\phi^3}{\sqrt{3} F_{\text{EM}}^\phi} \right]^2 &= 1.009 \pm 0.001,
\end{align*}

where the numerical values follow from those in Eq. (27). The size of the deviations of the $\rho$ and $\omega$ corrections from 1 are reduced by $\sim 15 - 20\%$ from those obtained in the earlier analysis [10]; that for the $\phi$ is increased, but remains small. In all cases the errors have been reduced by a factor of 3 or more. Note that the first of these corrections is the one relevant to precision tests of CVC. Note also that, as claimed above, the corrections given in Eqs. (28), for both the $\rho$ and $\omega$, lie in the corresponding ranges produced by the estimate of Section II.
With the results given in Eq. (28), it is now possible to correct the EM data used as input to the inverse moment chiral sum rule for the 6th order LEC, \( Q \). The sum rule is given by [6,54]

\[
\int_{4m_{\pi}^2}^{\infty} \frac{ds}{s} (\rho_{33} - \rho_{88}(s)) = \frac{16(m_{\pi}^2 - m_{\pi}^2)}{3F^2} Q(\mu^2) + \frac{1}{48\pi^2} \log \left( \frac{m_{\pi}^2}{m_{\pi}^2} \right) + \frac{(L_6(\mu^2) + L_{10}(\mu^2))}{2\pi^2 F^2} \left( m_{\pi}^2 \log \left( \frac{m_{\pi}^2}{m_{\pi}^2} \right) - m_{\pi}^2 \log \left( \frac{m_{\pi}^2}{m_{\pi}^2} \right) \right),
\]

where \( \mu \) is the renormalization scale of the effective chiral theory and \( L_i \) are the usual renormalized 4th order LEC’s of Gasser and Leutwyler [4]. This sum rule was evaluated in Ref. [54] using as input EM data for the isoscalar spectral function and both EM and \( \tau \) decay data for the isovector spectral function. The corrections above, required for the EM data, were not considered in this analysis. It is not clear, from our reading of the discussion of Ref. [54], exactly what the relative weightings of \( \tau \) decay and EM data in the determination of the \( \rho^0 \) contribution to the LHS above actually were. Since, however, \( \tau \) decay data is considerably more precise than electroproduction data, we have assumed in what follows that the determination is dominated by \( \tau \) decay data. To the extent that this is true, we need only make corrections to the (nominally) isoscalar \( \omega \) and \( \phi \) contributions. The result of this exercise is a shift of \( Q(m_{\rho}^2) \) from \((3.7 \pm 2.0) \times 10^{-5}\) to

\[
Q(m_{\rho}^2) = (2.4 \pm 2.0) \times 10^{-5}.
\]

(For reference, making, instead, the somewhat perverse assumption that the determination of the \( \rho^0 \) contribution was dominated by EM data, one would find \( Q(m_{\rho}^2) = (2.0 \pm 2.0) \times 10^{-5} \). The full correction is dominated by that to the \( \omega \) contribution. The reason this correction is so much larger than the others has been discussed above.)

One should bear in mind, in interpreting these results, that (1) there are, in principle, additional corrections to be made to the nominal isovector and isoscalar contributions at higher \( s \), and (2) the \( \bar{K}K2\pi \) contributions were taken to be purely isoscalar in the analysis of Ref. [54]. Because the separations within the higher isovector and isoscalar resonance pairs, \( \rho^0 - \omega \) and \( \rho^0 - \omega' \), are much smaller than the resonance widths, it is not possible to use sum rule methods to extract the individual isospin-breaking decay constants of these resonances. As such, we are unable to estimate the size of the former corrections. Were the \( \bar{K}K2\pi \) states to have a significant isovector component, the effect would be to raise \( Q(m_{\rho}^2) \).

An alternate method for determining \( Q \) is based on the observation that \( Q \) occurs not only in the inverse chiral moment sum rule above, but also in the 2-loop ChPT expression for \( \Pi^{38}(0) \) [33]. It is, thus, possible to make an independent estimate by using the fitted spectral ansatz for \( \Pi^{38} \) to compute \( \Pi^{38}(0) \), assuming negligible contribution from the portion of the spectrum above 2.8 GeV\(^2\). One obtains, from this exercise,

\[
Q(m_{\rho}^2) = (3.3 \pm 0.4) \times 10^{-5}.
\]

Since there is no positivity constraint on \( \rho_{38}(s) \) one does not know in which direction this result would be changed by corrections due to the small higher-\( s \) part of the spectral integral. The results of a study of the effect of including two combined spectral contributions, one for the \( \rho^0 - \omega' \) and one for the \( \rho^0 - \omega'' \), however, shows negligible change in \( Q(m_{\rho}^2) \), suggesting that such corrections are unlikely to be numerically significant. Since the two independent determinations of \( Q \) are completely consistent, within errors, the conclusions that \( Q(m_{\rho}^2) \simeq 3 \times 10^{-5} \) is considerably strengthened.

The last phenomenological application of our results concerns the effect on Narison’s \( \tau \)-decay-like sum rule for \( m_{s} \). Since a detailed discussion of the way in which one implements the isospin-breaking corrections is given in Ref. [47], we report here only the results of employing the improved determinations of the correction factors determined above. In doing so we will also take the opportunity to update the input parameters to the analysis of Ref. [47], employing the newer (1998) ALEPH value of \( \alpha_s(m_{\pi}^2) \) [37]. One finds, for example, using \( \tau \) decay data for the isovector input, that the average over the values of \( m_s \) (1 GeV\(^2\)) extracted using the set of scales \( s_0 = 1.4, 1.5 \) and 1.6 GeV\(^2\) in the sum rule analysis is shifted from 138 MeV to 146 MeV (147 MeV if one retains the 1997 ALEPH value of \( \alpha_s \) as input). Unfortunately, the errors on this value associated with uncertainties in the experimental input are still very large, \( \sim \pm 50 \) MeV at least, and this uncertainty cannot be appreciably reduced without a significant improvement in the accuracy of the determination of the experimental \( \omega \to e^+e^- \) and \( \phi \to e^+e^- \) widths. As such, although the central value is brought into better agreement with that discussed above, little more can learned from the Narison sum rule, at the present time.

ACKNOWLEDGMENTS

KM would like to acknowledge the ongoing support of the Natural Sciences and Engineering Research Council of Canada, and to thank Terry Goldman, Andreas Höcker, Heath O’Connell, Derek Leinweber, Tony Thomas, and Tony
APPENDIX: THE OPE FOR $\Pi^{38}$

The explicit form of the OPE for $\Pi^{38}$, keeping terms only up to dimension six, and to $\mathcal{O}(\alpha_s^2, m_q^2)$, can be obtained, as explained in the text, from the relevant expressions for the flavor-non-diagonal case given in the literature (see Ref. [34] and the paper by Braaten, Narison and Pich (BNP) in Ref. [1]). We list the results by operator dimension.

Dimension 0:

The only isospin-breaking contribution at dimension 0 is that due to EM, and is given by [29]

$$[\Pi^{38}_{\gamma E}]_{D=0} = -\frac{\alpha}{16\pi^3} \frac{1}{4\sqrt{3}} m(Q^2)$$  \hspace{1cm} (A1)

where $\alpha$ is the usual EM coupling.

Dimension 2:

The $D = 2$ term consists of the leading mass-dependent part of the perturbative contribution to the OPE, and follows from the expression given in Ref. [34]. One finds

$$Q^2 [\Pi^{38}(Q^2)]_{D=2} = \frac{3}{2\pi^2} \frac{1}{4\sqrt{3}} \left[ (m_u^2 - m_d^2)(Q^2) \right] \left[ 1 + \frac{8}{3} a(Q^2) + \left( \frac{17981}{432} + \frac{62}{27}(3) - \frac{1045}{54} \zeta(5) \right) a^2(Q^2) \right]$$  \hspace{1cm} (A2)

where $a(Q^2) = \alpha_s(Q^2)/\pi$, and $\zeta(n)$ is the Riemann zeta function. Further details on how the running of the coupling and the masses is handled can be found in Section IV.

Dimension 4:

Our expression for the $D = 4$ contribution also follows from that given in Ref. [34]. Only the $m_q^4$ and quark condensate terms survive once one takes the relevant isospin-breaking difference. The former are numerically tiny compared to the latter, and hence have not been written down explicitly. We then find

$$Q^4 [\Pi^{38}(Q^2)]_{D=4} = \frac{2}{3\pi^2} \frac{1}{4\sqrt{3}} \left[ \frac{1}{3} a(Q^2) + \frac{11}{2} a^2(Q^2) \right] .$$  \hspace{1cm} (A3)

The scale-invariant $<mqq>$ difference can be written in terms of the $m_d - m_u$, $<\bar{d}d - \bar{u}u>$, and the averages of the $u$ and $d$ quark masses and condensates. Since isospin-breaking in the condensates is much smaller than in the masses, the term proportional to $m_d - m_u$ dominates numerically. It can be recast in terms of the isospin-breaking quark mass ratio, $r$, and $f_\pi$, $m_\pi$, as explained in the text.

Dimension 6:

The 4-quark operators are the dominant operators at dimension 6. Their contribution to $\Pi^{38}$ can be obtained from the expressions given in the Appendix of BNP [1]. Since lack of phenomenological information on the various condensates forces one to work with the re-scaled version of the VSA, one must, for consistency, drop the terms of $\mathcal{O}(\alpha_s^2)$ contained there. (See the discussion of this point contained in BNP.) One then finds

$$4\sqrt{3} Q^6 [\Pi^{38}(Q^2)]_{D=6} = -8\pi^2 a(Q^2) \left( \langle \bar{u}\gamma_\mu\gamma_5 T^a u \bar{u}\gamma^\mu\gamma_5 T^a u \rangle - \langle \bar{d}\gamma_\mu\gamma_5 T^a d \bar{d}\gamma^\mu\gamma_5 T^a d \rangle \right)$$  \hspace{1cm} (A4)

$$-\frac{16\pi^2}{9} a(Q^2) \sum_k \left( \langle \bar{u}\gamma_\mu T^a u \bar{k}\gamma^\mu T^a k \rangle - \langle \bar{d}\gamma_\mu T^a d \bar{k}\gamma^\mu T^a k \rangle \right) ,$$  \hspace{1cm} (A5)

where $T^a$ is an $SU(3)$ generator. Implementing the re-scaled VSA, this expression reduces to

$$4\sqrt{3} Q^6 [\Pi^{38}(Q^2)]_{D=6} = \frac{448\pi}{81} \rho_{red}\gamma(\rho\alpha_s\langle q\bar{q}\rangle^2) ,$$  \hspace{1cm} (A6)

Williams for useful discussions.
where $\gamma$, $\rho$ and $\rho_{red}$ are as defined in the text.

TABLE I. Results for the $D = 6$ VSA-violation parameter, $\rho_{red}$, and the spectral strength parameters, $f_\rho$, $f_\omega$, $f_\phi$ and $f_{\rho'\omega'}$ as a function of the isospin-breaking mass ratio, $r$. The first line, for each value of $r$, corresponds to the results obtained using the single-pinch weight family, the second line to those obtained using the double-pinch family.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\rho_{red}$</th>
<th>$f_\rho (\times 10^3)$</th>
<th>$f_\omega (\times 10^3)$</th>
<th>$f_\phi (\times 10^3)$</th>
<th>$f_{\rho'\omega'} (\times 10^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.251</td>
<td>1.02</td>
<td>2.3</td>
<td>1.7</td>
<td>-0.28</td>
<td>-0.020</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.3</td>
<td>1.7</td>
<td>-0.28</td>
<td>-0.020</td>
</tr>
<tr>
<td>0.288</td>
<td>1.15</td>
<td>2.6</td>
<td>2.0</td>
<td>-0.32</td>
<td>-0.026</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.6</td>
<td>2.0</td>
<td>-0.32</td>
<td>-0.026</td>
</tr>
<tr>
<td>0.325</td>
<td>1.28</td>
<td>2.9</td>
<td>2.2</td>
<td>-0.36</td>
<td>-0.032</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.9</td>
<td>2.2</td>
<td>-0.36</td>
<td>-0.032</td>
</tr>
</tbody>
</table>
FIG. 1. The FESR “Pac-man” contour
FIG. 2. The variation of $f_\rho$ with $\rho_{red}$ for the single- and double-pinch weight families. Results are displayed here for the central value $r = 0.288$. The solid line corresponds to the single-pinch weight analysis, the dashed line to the double-pinch analysis. The intersection point determines the value of $\rho_{red}$ quoted in the table.

49 K. Maltman, R. Gupta and T. Bhattacharya, work in progress.