Do naked singularities generically occur in generalized theories of gravity?

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Abstract

A new mechanism for causing naked singularities is found in an effective superstring theory. We investigate the gravitational collapse in a spherically symmetric Einstein-Maxwell-dilaton system in the presence of a pure cosmological constant “potential”, where the system has no static black hole solution. We show that once gravitational collapse occurs in the system, naked singularities necessarily appear in the sense that the field equations break down in the domain of outer communications. This suggests that in generalized theories of gravity, the non-minimally coupled fields generically cause naked singularities in the process of gravitational collapse if the system has no static or stationary black hole solution.

The singularity theorem [1] states that the occurrence of singularities is inevitable under some physical conditions in general relativity. There are two notable scenarios where singularities may appear in our universe. One is the initial singularity at the birth of the universe and the other is the final stage of gravitational collapse. In the latter case we believe that an event horizon is formed which encloses all occurring singularities as the collapse proceeds, following the cosmic censorship hypothesis (CCH) [2]. CCH is classified into two types, the weak cosmic censorship hypothesis (WCCCH) and the strong one (SCCH). WCCCH says that observers at an infinity should not see singularities while SCCH says that no observer should see them and the whole region of a space-time can be uniquely determined by initial regular data. Mathematically, SCCH is equivalent to the statement that no Cauchy horizon can be formed in a physical gravitational collapse.

Recently many elegant results [3] suggest that SCCH holds for charged and/or rotating black holes due to the destruction of the Cauchy horizon by the mass inflation phenomenon. The general proof of CCH is, however, far from complete and many counter examples have been found in the framework of general relativity [4]. A new approach has been considered in the context of generalized theories of gravity. Gibbons and Maeda [5] discovered the static black hole solutions in the Einstein-Maxwell-dilaton system, which comes from an effective superstring theory, and later Garfinkel, Horowitz and Strominger [6] showed that the inner (Cauchy) horizon in the Reissner-Nordström solution is replaced by a space-like singularity. This suggests that the occurrence of the inner horizon is not generic and hence SCCH holds if we take the effect of string theory into account. On the other hand Horne and Horowitz [7] obtained the opposite result that extremal electrically charged non-static black hole solutions in the presence of a central charge have timelike singularities. It seems, however, not to be a counter example of SCCH in the sense that such solutions have no regular initial spacelike hypersurface because of the central singularities. Thus, the following question naturally arises. Does CCH really hold in generalized theories of gravity? There is not much evidence deciding the matter yet because the above results do not take any physical process of the gravitational collapse from initial regular data into account.

Although a large number of studies have been made on finding static/stationary black hole solution and investigating their thermodynamical properties and geometrical stability and so on in generalized theories of gravity, only few studies have so far been made on investigating gravitational collapse. We should not overlook that the gravitational collapse in generalized theories of gravity will be much different from that in general relativity. In general relativity, the black hole no-hair conjecture states that the matter fields are swallowed by a black hole and a space-time asymptotically approaches the electrovacuum Kerr-Newman solution after gravitational collapse. On the other hand, the above aspects are not necessarily satisfied in generalized theories of gravity because the dilaton field couples to the curvature and/or other matter fields.

In this letter we investigate the gravitational collapse in the spherically symmetric Einstein-Maxwell-dilaton system analytically in the presence of a pure positive cosmological constant, which corresponds to $g_1 = 0$ in the Liouville-type potential [8]. This system is interesting because it has been proved that no static black hole solution exists for spherically symmetric space-times [9]. Such property seems to be general in the sense that there exists no asymptotically flat, asymptotically de Sitter, or asymptotically anti-de Sitter solution in the arbitrary exponential dilaton potential [8]. Furthermore, it is known that there is a critical mass below which no asymptotically flat black hole solution exists in the system which

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includes the Gauss-Bonnet term which was neglected in
the former works [10,11]. In these systems the following
three interpretations are available: (i) the matter fields
do not cause the gravitational collapse but escape to
infinity without forming a black hole event horizon (BEH);
(ii) the space-time does not approach a stationary space-
time and the matter fields oscillate forever or, (iii) naked
singularities are necessarily formed. If the case (i) is true
for all initial data, we should find that the strong cosmic
no-hair conjecture holds in the present system. By in-
vestigating numerically, however, we confirmed that this
is not the case and some initial data lead to the grav-
itational collapse, namely, a trapped surface forms [12].
This result is consistent with the fact that the dilaton
field satisfies the dominant energy condition in the Ein-
tstein frame. Then it is enough to consider only the cases
(ii) and (iii). The case (ii) seems incompatible with our
naive expectation that after gravitational collapse, the
space-time settles down to an asymptotically stationary
space-time with a black hole. If the case (iii) is true,
we must say that CCH is violated even in generalized
theories of gravity.

Using the double null coordinates, we will show that
the field equations necessarily break down in the domain
of outer communications or at the BEH, once the grav-
itational collapse has occurred. It is worth to note that
"break down" does not imply a coordinate singularity
because our coordinates are, probably, the most extended
"break down" does not imply a coordinate singularity be-
cause our coordinates are, probably, the most extended
in spherically symmetric space-times. It is also supported by numerical calculations which show the
formation of naked singularities in the system [12]. Thus,
our result suggests that naked singularities generically
occur in the gravitational collapse of spherically symmet-
ic systems.

The low-energy effective action (Einstein frame) of
string theory as a dilaton model is

\[ S = \int d^4x \sqrt{-g} [-R + 2(\nabla \phi)^2 + e^{-2\phi} F^2 + 2\Lambda], \]

(1)

where \( R \) is the Ricci scalar, \( \phi \) is a massless dilaton field,
\( \Lambda \) is a positive constant, and \( F_{\mu\nu} \) is the field strength
of the Maxwell field. The double-null coordinates for a
spherically symmetric space-time are

\[ ds^2 = -2e^{-\lambda}(U, V) dU dV + R(U, V)^2 d\Omega, \]

(2)

where \( \partial_U \) and \( \partial_V \) are future ingoing and outgoing null
gedesics, respectively. The Maxwell equation is auto-
matically satisfied for a purely magnetic Maxwell field
\( F = Q \sin \theta d\theta \wedge d\phi (F^2 = 2Q^2/R^4) \), where \( Q \) is the
magnetic charge. An electrically charged solution is
obtained by a duality rotation from the magnetically
charged one [5,6]. Therefore we shall consider only a
purely magnetic case in this letter. The dynamical field
equations are

\[ \lambda_{,UV} - \frac{2R_{,UV}}{R} = 2\phi_{,U} \phi_{,V} + e^{-\lambda} \left( \frac{Q^2 e^{-2\phi}}{R^4} - \Lambda \right), \]

(3)

\[ R_{,UV} + \frac{R_{,U} R_{,V}}{R} = -\frac{e^{-\lambda}}{2R} \left( 1 - \frac{Q^2 e^{-2\phi}}{R^2} - \Lambda R^2 \right), \]

(4)

\[ 2R^3(R_{,U} \phi_{,V} + R_{,V} \phi_{,U} + R_{,V} \phi_{,U}) = Q^2 e^{-2\phi - \lambda}, \]

(5)

where \( A_{a} \) is a partial derivative of \( A \) with respect to \( a \).
The constraint equations are

\[ R_{,UV} + \lambda_{,U} R_{,V} = -(\phi_{,U})^2 R, \]

(6)

\[ R_{,VV} + \lambda_{,V} R_{,U} = -(\phi_{,V})^2 R. \]

(7)

We consider the evolution of the field equations with ini-
tial regular data on a null characteristic hypersurface
whose boundary is a closed future-trapped surface (see
Fig. 1) because we are interested in the gravitational col-
lose in the gravitational collapse as shown in Theorem 1 of Ref. [13], the trapped
surface causes the formation of a BEH \((U = U_B \) null hy-
persurface) if there are no singularities observed from \( I^+ \).
For this initial value problem, we will show the following
theorem.

Theorem

Let us consider the dynamical evolution of the model (1)
in a spherically symmetric space-time or equivalently for
the equations (3)-(7) with initial data on the character-
istic null hypersurface \( N \). Then, there is \( U_1(\leq U_B) \) such
that the system of equations breaks down at \( U = U_1 \).

We shall show the above theorem by contradiction
below. First, we consider the asymptotic behavior of
field functions near the cosmological event hori-
zon (CEH), which is defined as a past Cauchy horizon
\( H^{-}(I^+) \) when a BEH exists (see Ref. [14]). It is con-
venient to rescale the coordinate \( U \) such that \( U \) is an
affine parameter of a null geodesic of the CEH, i.e., \( \lambda \)
is constant along the CEH. Hereafter we use a charac-
ter \( u (= f(U)) \) instead of \( U \) for parameterization to avoid
confusion. Under such coordinates, Eq. (6) on the CEH
is

\[ -R_{,uu} = (\phi_{,u})^2 R. \]

(8)

If there are no singularities in \( J^-(I^+) \cap J^+(N) \), the null
geodesic generators of the BEH and the CEH are future
complete. Then, by the non-decreasing area law for the
CEH and by existence of an upper bound of the area [14],
\( \lim_{u \to \infty} R = C_1 \) (hereafter, \( C_i (i = 1, 2, \ldots) \) means a posi-
tive constant), and hence future asymptotic behavior of
\( R_{,u} \) on the CEH is represented as follows [15],

\[ R_{,u} \sim C_2 u^{-\alpha - 1} \quad (\alpha > 0). \]

(9)

Then, by Eq. (8)
\[ \phi_{,u} \sim u^{-\alpha/2-1}, \]  
(10)

and hence \( \lim_{u \to \infty} \phi(u) = \text{const} \). We shall obtain the asymptotic value of \( R_V \) on the CEH by solving Eq. (4) as

\[ R_V = \left( \int_{u_i}^{u} \frac{KR}{R_i} du + R_{,V}|_i \right) \frac{R_i}{R}, \]  
(11)

where \( R_{,V}|_i, R_i \) are initial values of \( R_V, R \) on \( u = u_i, \) respectively and \( K = -(1 - Q^2 e^{-2\phi}/R^2 - \Lambda R^2)/2Re^\lambda. \) \( K \) must approach a positive constant \( K_\infty \) as \( u \to \infty \) because \( R, \phi \to \text{const.} \) and the expansion \( \theta_i \equiv R_{,V}/R \) of each outgoing null geodesic is positive (if it were negative, the area element \( d\Omega \) would become 0 along the outgoing null geodesics). Then, the asymptotic value of \( R_{,V} \) is

\[ R_{,V} \sim K_\infty u > 0. \]  
(12)

We shall also obtain the asymptotic value of \( \phi_{,V} \) on the CEH by solving Eq. (5) with the following solution

\[ \phi_{,V} = \left( \int_{u_i}^{u} \frac{HR}{R_i} du + \phi_{,V}|_i \right) \frac{R_i}{R}, \]  
(13)

where \( \phi_{,V}|_i \) is an initial value of \( \phi_{,V} \) on \( u = u_i \), and \( H = (Q^2 e^{-2\phi}/R^2 - 2R^3 R_{,V} \phi_{,u})/2R^4. \) \( H \) approaches a positive constant \( H_\infty \) as \( u \to \infty \) because \( R_{,V} \phi_{,u} \sim u^{-\alpha/2-2} \) to 0 by Eqs. (10) and (12). The asymptotic value of \( \phi_{,V} \) is

\[ \phi_{,V} \sim H_\infty u > 0. \]  
(14)

Next, let us consider an infinitesimally small neighborhood \( \mathcal{U}_C \) of the CEH which contains a timelike hypersurface \( T_C \) such that \( T_C \) is in the past of the CEH by \( \epsilon > 0 \), where \( \epsilon \) is a fixed affine parameter distance along the outgoing null geodesic intersecting the CEH [16]. We denote each point of the intersection of \( T_C \) and \( u = \text{const.} \) hypersurface by \( p(u) \). By Eq. (5), the solution of \( h \equiv \phi_{,u} \) along each \( u = \text{const.} \) is

\[ h = \left( -\int_{V_C}^{V} \frac{LR}{R^2} dV + h_C \right) \frac{R_C}{R}, \]  
(15)

where \( R_C \) and \( h_C \) are values of \( R \) and \( h \) at the CEH \( (V = V_C) \), respectively, and \( L = (Q^2 e^{-2\phi}/R^2 - 2R^3 R_{,V} \phi_{,u})/2R^4. \) Since \( (R_{,V} \phi_{,V}/R)|_{V_C} \sim u^{-\alpha} \) to 0 for large values of \( u, L \sim L_\infty > 0 \) asymptotically. Differentiating \( h \) by \( V \) in Eq. (15),

\[ h_{,V} = L + \frac{R_C R_{,V}}{R^2} \left( \int_{V}^{V_C} \frac{LR}{R^2} dV - h_C \right). \]  
(16)

By the relation \( \epsilon = dr \sim udV \) for large \( u \) [17], \( h_{,V} \sim L_\infty + O(\epsilon) + O(u^{-\alpha/2}). \) Since \( \epsilon \) is an arbitrary small value, asymptotically \( h_{,V} > 0 \) on the \( u = \text{const.} \) null segment \([V_{|u(i)}, V_C] \subset \mathcal{U}_C. \)

In the next step we investigate the behavior of \( \phi, \phi_{,V} \) and \( R \) on BEH, just like the CEH case. We rescale \( V \) into \( v \) such that \( v \) is an affine parameter and \( \lambda = \text{const.} \) on the BEH, while we leave \( U \) unchanged. Since the area of the BEH is non-decreasing and also has an upper bound as shown in [13,14], \( \lim_{v \to \infty} R = C_3, \) and \( R_{,V} \sim C_4 \epsilon^2 - \beta - 1 \) (\( \beta > 0 \)). This means that by Eq. (7) \( \phi_{,V} \sim v^{-\beta/2 - 1} \) as obtained in the CEH case. By replacing \( u \) by \( v \) in the argument of the CEH case and solving Eqs. (4) and (5), the asymptotic values of \( \phi_{,V} \) and \( R_{,V} \) become

\[ R_{,V} \sim v \phi_{,V} \sim C_5 v, \]  
respectively.

Hence the first and third terms in the l.h.s. of the dilaton field Eq. (5) are negligible asymptotically and

\[ \lim_{v \to \infty} k_{,U} = \lim_{v \to \infty} \phi_{,U}v = C_6 > 0. \]  
(17)

We define \( k \equiv \phi_{,U}. \) Consider an infinitesimally small neighborhood \( \mathcal{U}_B \) of the BEH. There is a small \( \epsilon \) such that the timelike hypersurface \( T_B \) which is in the past of the BEH by a fixed affine parameter distance \( \epsilon \) of ingoing null geodesics intersecting the BEH is contained in \( \mathcal{U}_B. \) By the relation between the affine parameter and \( dU, \) i.e., \( dU \propto \epsilon/v, k \) on \( T_B \) is asymptotically

\[ k|_{T_B} \sim k|_{\text{BEH}} + k|_{\text{BEH}} (-dU) \sim k|_{\text{BEH}} - C_6 \epsilon v^{-1} \sim (v^{-\beta/2} - C_6 \epsilon)v^{-1}. \]  
(18)

This indicates that \( \phi_{,V} \) is negative on \( T_B \) for large values \( v(> v_1). \) Now, consider each null segment \( N_u(g \geq u_f): [V_C - \epsilon/u_f, V_C] \) where \( h_{,V} > 0 \) on \( N_u \) (see Fig. 2). If one takes \( u_f \) large enough, \( N_u \) intersects \( T_B(v > v_1) \) at \( u = u_f. \) Let us take a sequence of \( N_{u_J}(J = 1, 2, ..., L+1) (L \) is a natural number large enough), where \( \delta u = (u_f - u_i)/L \) and \( u_f = u_i + (J - 1)\delta u. \) Assume that \( \phi_{,V} > 0 \) on \( N_J(\equiv N_{u_J}) \), for a moment, then \( h_{,V} > 0 \) on \( N_J \) in the same way as \( h_{,V} > 0 \) on \( [V|_{p(i)}, V_C] \) [18]. Hence we can show that \( \phi_{,V}|_{x_{J+1}} \equiv \phi_{,V}|_{x_J} + h_{,V}|_{x_J} \delta u \) > 0. On the other hand, \( \phi_{,V} > 0 \) on \( u = u_f \) by Eq. (14), hence \( \phi_{,V} > 0 \) for each \( u_f \) by induction. This is a contradiction because as we showed \( \phi_{,V} < 0 \) on \( T_B(u = u_f). \)

It is worth to comment that the possibility of the formation of null singularities on BEH, continuing to \( i^+ \) is not excluded from our theorem. In this case the BEH is a singular null hypersurface even if WCCH holds (SCCH is, probably, violated).

Our theorem says that the case (ii) is not true, as we expected. When we consider the dyon solution or a rotating space-time, non-trivial 3-rank anti-symmetric tensor fields inevitably appear. However, we expect that such fields do not change our result drastically. This implies that if there are no static/stationary black hole solutions, naked singularities generically occur in the process of gravitational collapse in generalized theories of gravity.

Since the model under consideration is the low energy limit of string theory, we cannot exclude the possibility that higher curvature terms and higher order \( \alpha' \) corrections, where \( \alpha' \) is the inverse string tension, may prevent
the space-time from causing naked singularities. However, the theory is useful near the Planck scale. Thus, we can at least say that string theory predicts existence of space-time points with very high curvature in the domain of outer communications. This picture is quite different from that of general relativity because it is strongly believed that space-time points with high curvature do not appear generically in the outer region of a black hole.

ACKNOWLEDGMENTS

We would like to thank Akio Hosoya and Hideki Ishihara and Kei-ichi Maeda for useful discussions, and Christian Baraldo for his critical reading of our letter. The work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture (K.M and T.T), by the Grant-in-Aid for JSPS (No. 199605200 (K.M) and No. 199704162 (T.T)).

[15] It is enough to consider the case of the power law function, $R_u = u^{-\alpha -1}$, to show our theorem because it gives the mildest damping.
[16] The neighborhood of CEH can be covered by Gaussian null coordinates, $ds^2 = -2drd\eta + f(d\eta)^2 + R^2d\Omega$, where $r$ is an affine parameter of outgoing null geodesics and $f = 0$ on the CEH. Then, $\epsilon = dr = \text{const}$.
[17] It seems reasonable to assume that the surface gravity of CEH, $\kappa_c \equiv -f_{,r}/2$, is asymptotically constant. In this case, $e^{\epsilon u} \sim u$ and hence $d\eta \sim du/u$. This indicates that $dr \sim ud\eta$.
[18] One can show that $R_u < 0$ on $T_C$ for large $u$ by using Eq. (4). Therefore, $R_u < 0$ inside $T_C$ because each expansion of ingoing null geodesics from $T_C$ must decrease monotonically. Thus, we can easily get $h_{,V} > 0$ by considering each segment; $[V_C - \epsilon/u_I, V_C - \epsilon/u_J] \notin U_C$, $[V_C - \epsilon/u_I, V_C] \in U_C$. The detailed proof can be found in Ref. [12].

FIG. 1. A Penrose diagram in asymptotically de Sitter space-time. $N$ and $\text{AH}$ are the characteristic hypersurface and the apparent horizon of the black hole, respectively.

FIG. 2. Timelike hypersurfaces $T_B$, $T_C$ are displayed in the neighborhoods $U_B$, $U_C$ of BEH and CEH, respectively. A null segment $N_{ij}$ is displayed by a thick line. $\epsilon$ is a fixed affine parameter distance along outgoing and ingoing null geodesics.