Exact BPS monopole solution in a self-dual background

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ABSTRACT

An exact one monopole solution in a uniform self-dual background field is obtained in the BPS limit of the $SU(2)$ Yang-Mills-Higgs theory by using the inverse scattering method.

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There has been much theoretical interest concerning magnetic monopole solutions in an $SU(2)$ Yang-Mills-Higgs theory after 't Hooft and Polyakov [1] made the initial discovery of such structure in the seventies. Especially, in the Bogomolny-Prasad-Sommerfend (BPS) limit [2, 3], the ADHMN method [4, 5] can be used to construct exact static multi-monopole solutions satisfying the first-order Bogomolny equations

$$F_{ij} = -\epsilon_{ijk}D_k\Phi,$$

(1)

where $F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \equiv \epsilon_{ijk}B_k$ and $D_k\Phi = \partial_k\Phi + i[A_k, \Phi]$ (with $A_i \equiv A_i^a\tau^a/2, \Phi \equiv \Phi^a\tau^a/2$). BPS monopoles refer to solutions of Eq. (1), with the asymptotic fields approaching the Higgs vacuum (as is necessary for any finite-energy configuration). At large distances, they feature the field $B_i$ characteristic of a system of localized magnetic monopoles and also the (gauge-invariant) magnitude of the Higgs field given as

$$|\Phi(\vec{r})| \approx v - \frac{g}{4\pi r}, \text{ for large } r \quad (2)$$

where $g = 4\pi n$ ($n = 1, 2, ...$) is the strength of the magnetic charge. Note that studies of BPS monopoles are directly relevant in nonperturbative investigations of certain supersymmetric gauge theories.

In this letter, we shall discuss a new solution of Eq. (1) which becomes possible if we assume a more general asymptotic configuration than the Higgs vacuum. As a particular solution of Eq. (1), we have the uniform self-dual field described by (up to arbitrary gauge transformation)

$$A_i = -\frac{1}{2}(\vec{r} \times \vec{B}_0)i\tau^3/2, \quad \phi = -(v + \vec{B}_0 \cdot \vec{r})r^3/2. \quad (3)$$

If the magnetic field strength $\vec{B}_0$ were zero, this would reduce to the usual Higgs vacuum. In this work, we will look for a solution of Eq. (1) which describes a (static) monopole in the asymptotic uniform field background of the form (3) with $\vec{B}_0 \neq 0$. For sufficiently weak $\vec{B}_0$, the corresponding, everywhere regular, solution was first discussed in Ref. [6] (see Eqs. (3.35)-(3.37) of this article). From the latter, we know that the Higgs field in an appropriate gauge takes the form

$$\Phi(\vec{r}) = -\left[v(coth vr - \frac{1}{vr}) + \frac{1}{2}\vec{B}_0 \cdot \vec{r}(2coth vr - \frac{vr}{\sinh^2 vr})\right]\hat{r} \cdot \hat{r}/2 - \frac{1}{2}\frac{r}{\sinh vr}\left[\vec{B}_0 \cdot \hat{r}/2 - (\vec{B}_0 \cdot \hat{r})\hat{r} \cdot \hat{r}/2\right], \quad (4)$$

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where \( \hat{r} = \vec{r}/r \). (Note that, with \( \vec{B}_0 = 0 \), this reduces to the well-known Prasad-Sommerfield expression [2]). This is a perturbative solution, i.e., valid only to the first order in \( \vec{B}_0 \), and therefore we still have no guarantee for the existence of the corresponding, globally well-defined, exact solution (with a finite background field \( \vec{B}_0 \)) to the full nonlinear system (1). The full solution (see Eq. (28)), which reduces to the perturbative result (4) for small \( \vec{B}_0 \), will be found below with the help of the inverse scattering method. However, as we shall see, there arises some unusual feature when one tries to extend the solution to the whole 3-dimensional space.

As we make the choice \( \vec{B}_0 = B_0 \hat{z} \) (with \( B_0 > 0 \)), an obvious starting point for the solution, suggested by the symmetry consideration, will be the following cylindrical ansatz [7]:

\[
A^a_i = -\hat{\varphi}i \left[ \eta_2(\rho, z) \frac{1}{\rho} \hat{z}^a + \eta_1(\rho, z) \hat{\rho}^a \right] + \left[ \hat{z}^i W_1(\rho, z) + \hat{\rho}^i W_2(\rho, z) \right] \hat{\varphi}^a, \\
\Phi^a = \eta_1(\rho, z) \hat{\rho}^a + \eta_2(\rho, z) \hat{z}^a, 
\]

(5)

where \((\rho, \varphi, z)\) refer to cylindrical coordinates, and we have introduced normalized basis vectors (in ordinary 3-space and isospin space) \( \hat{\rho} = (\cos \varphi, \sin \varphi, 0) \), \( \hat{\varphi} = (-\sin \varphi, \cos \varphi, 0) \) and \( \hat{z} = (0, 0, 1) \). Performing a judicious (singular) gauge transformation with Eq. (5), it is also possible to write the ansatz in an alternative form [8] (here note that \( A_i \equiv A^a_i \tau^a_i/2 \)):

\[
A_{\rho} = \cos \varphi A_1 + \sin \varphi A_2 = -W_2 \frac{\tau^1}{2} = \frac{1}{2} \left( \begin{array}{cc} 0 & -W_2 \\ -W_2 & 0 \end{array} \right), \\
A_{\varphi} = -\sin \varphi A_1 + \cos \varphi A_2 = -\frac{\eta_1 \tau^2}{\rho} - \frac{\eta_2 \tau^3}{\rho} = \frac{1}{2\rho} \left( \begin{array}{cc} -\eta_2 & i\eta_1 \\ -i\eta_1 & \eta_2 \end{array} \right), \\
A_3 = -W_1 \frac{\tau^1}{2} = \frac{1}{2} \left( \begin{array}{cc} 0 & -W_1 \\ -W_1 & 0 \end{array} \right), \quad \Phi = \frac{1}{2} \left( \begin{array}{cc} \phi_2 & -i\phi_1 \\ i\phi_1 & -\phi_2 \end{array} \right). 
\]

(6)

Using either form, one finds from the Bogomolny equation in Eq. (1) that the functions \( \phi_1, \phi_2, \eta_1, \eta_2, W_1 \) and \( W_2 \) should satisfy the coupled equations

\[
\partial_{\rho} \phi_1 - W_2 \phi_2 = -\frac{1}{\rho} (\partial_{\rho} \eta_1 - W_1 \eta_1), \quad \partial_{\rho} \phi_1 - W_1 \phi_2 = \frac{1}{\rho} (\partial_{\rho} \eta_1 - W_2 \eta_2), \\
\partial_{\rho} \phi_2 + W_2 \phi_1 = -\frac{1}{\rho} (\partial_{\rho} \eta_2 + W_1 \eta_1), \quad \partial_{\rho} \phi_2 + W_1 \phi_1 = \frac{1}{\rho} (\partial_{\rho} \eta_2 + W_2 \eta_1), \\
\partial_{\rho} W_1 - \partial_{z} W_2 = -\frac{1}{\rho} (\eta_1 \phi_2 - \eta_2 \phi_1). 
\]

(7)

By making a judicious gauge choice, it was shown in Refs. [8, 9] that the solution to
Eq. (7) can always be written as
\[ \phi_1 = \frac{\partial_\rho \psi}{f}, \quad \phi_2 = -\frac{\partial_z \psi}{f}, \quad \eta_1 = -\frac{\partial_\rho \psi}{f}, \]
\[ \eta_2 = \frac{\partial_\rho \psi}{f}, \quad W_1 = -\phi_1, \quad W_2 = \frac{1}{\rho} \eta_1 \]  
with the two real functions \( f = f(\rho, z) \) and \( \psi = \psi(\rho, z) \) which must satisfy the Ernst equations [10] (here, \( \nabla^2 \equiv \partial_\rho^2 + \partial_z^2 + \frac{1}{\rho} \partial_\rho \))
\[ f \nabla^2 \psi - 2 \nabla f \cdot \nabla \psi = 0, \quad f \nabla^2 f - |\nabla f|^2 + |\nabla \psi|^2 = 0. \]  
If we here define the real symmetric, \( 2 \times 2 \) unimodular matrix \( g \) by
\[ g = \frac{1}{f} \begin{pmatrix} 1 & \psi \\ \psi & \psi^2 + f^2 \end{pmatrix}, \]
Eq. (9) can further be changed into the chiral equation (or Yang’s equation [11] for axially symmetric monopoles)
\[ \partial_\rho [\rho(\partial_\rho g)g^{-1}] + \partial_z [\rho(\partial_z g)g^{-1}] = 0. \]  
Note that, for the Prasad-Sommerfield one-monopole solution, we have [12]
\[ f = \frac{\rho}{F}, \quad \psi = \frac{1}{F}(z \cosh \nu - r \sinh \nu \coth \nu) \]  
where \( F \equiv r \sinh \nu + r \cosh \nu \coth \nu \).

In order to incorporate the effect of the background field on the result (12), we may use the inverse scattering method with the above chiral equation [9, 13]. It is based on the fact that Eq. (11) can be viewed as the compatibility conditions of the linear system
\[ D_1 \Psi \equiv (\partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda) \Psi = \frac{\rho[\rho(\partial_z g)g^{-1} - \lambda(\partial_\rho g)g^{-1}]}{\lambda^2 + \rho^2} \Psi, \]
\[ D_2 \Psi \equiv (\partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2} \partial_\lambda) \Psi = \frac{\rho[\rho(\partial_\rho g)g^{-1} + \lambda(\partial_z g)g^{-1}]}{\lambda^2 + \rho^2} \Psi \]  
for a \( 2 \times 2 \) matrix \( \Psi = \Psi(\rho, z; \lambda) \). Now, for some initial solution \( g = g_0(\rho, z) \) of Eq. (11), suppose that we know a corresponding solution \( \Psi_0(\rho, z; \lambda) \) of Eq. (13), with the boundary condition \( \Psi_0(\rho, z; \lambda = 0) = g_0(\rho, z) \) satisfied. Then, the dressed functions, \( \Psi(\rho, z; \lambda) = \chi(\rho, z; \lambda) \Psi_0(\rho, z; \lambda) \) and \( g(\rho, z) = \chi(\rho, z; \lambda = 0)g_0(\rho, z) \), give new solutions of Eqs. (11) and (13), provided that \( \chi(\rho, z; \lambda) \) satisfies
\[ D_1 \chi = \frac{\rho[\rho(\partial_z g)g^{-1} - \lambda(\partial_\rho g)g^{-1}]}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho[\rho(\partial_\rho g_0)g^{-1} - \lambda(\partial_\rho g_0)g^{-1}]}{\lambda^2 + \rho^2}, \]
\[ D_2 \chi = \frac{\rho[\rho(\partial_\rho g)g^{-1} + \lambda(\partial_z g)g^{-1}]}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho[\rho(\partial_\rho g_0)g^{-1} + \lambda(\partial_z g_0)g^{-1}]}{\lambda^2 + \rho^2}, \]  
(14)
and also the condition (originating from the hermiticity of $g$ and $g_0$)

$$\chi(\rho, z; \lambda) = g(\rho, z)[\chi(\rho, z; -\rho^2 / \bar{\lambda})]^\dagger g_0(\rho, z)^{-1}. \quad (15)$$

The function $\chi(\rho, z; \lambda)$, needed in generating $N$-monopole solutions, may have only simple poles in the complex $\lambda$-plane (see Refs. [9, 12]), viz.,

$$\chi(\rho, z; \lambda) = 1 + \sum_{k=1}^N \frac{R_k(\rho, z)}{\lambda - \mu_k(\rho, z)} \quad (16)$$

with the poles $\mu_k(\rho, z)$ explicitly given by

$$\mu_k(\rho, z) = w_k - z + \sqrt{(w_k - z)^2 + \rho^2}, \quad (17)$$

where $w_k$ are arbitrary constants. The residues $R_k(\rho, z)$ are also found readily and then the resulting expression for $\chi(\rho, z; \lambda)$ may be used to secure the following formula for the new solution $g = g_{ph}(\rho, z)$ of Eq. (11):

$$g_{ph} = g/\sqrt{\det g}, \quad g_{ab} = (g_0)_{ab} - \sum_{i=1}^N \sum_{j=1}^N (\mu_i \bar{\mu}_j)^{-1} (\Gamma_{ij})_{ab} (g_0)_{ac} \bar{m}_c^i m_d^j (g_0)_{db}, (a, b = 1, 2) \quad (18)$$

where $m^k_c = M^k_c [\Psi_0(\rho, z; \mu_k)^{-1}]_{ab} (M^k_c \text{ are constants})$ and $\Gamma_{ij} = m^i_a (g_0)_{ab} \bar{m}_b^j / (\rho^2 + \mu_i \bar{\mu}_j)$.

For our problem, we may apply the above dressing method on the initial solutions which correspond to uniform self-dual fields. By a direct integration of the Ernst equation (9), we have a particular solution

$$g_0 = \begin{pmatrix} 1/f_0 & 0 \\ 0 & -f_0 \end{pmatrix}; \quad f_0 = \exp \left[ vz + \frac{B_0}{2} \left( z^2 - \frac{\rho^2}{2} \right) \right] \equiv \exp(vZ), \quad (19)$$

and the corresponding fields, if used in Eq. (6), yield precisely the uniform field configuration given in Eq. (3). The minus sign in the component of $g_0$ is introduced in order to make $\det g$ in Eq. (18) to be positive definite [9]. Given the matrix $g_0$ as in (19), we may then solve the linear equations (13) for $\Psi_0 = \begin{pmatrix} \Psi_0^1 \\ 0 \end{pmatrix}$. All together, we have here four equations for $\Psi_0^1$ and $\Psi_0^2$, which may be integrated by noticing that two equations from the four in fact imply

$$[\partial_z + (v + B_0 z) + \frac{\lambda}{\rho} \partial_{\rho}] \Psi_0^1 = 0,$$

$$[\partial_z - (v + B_0 z) + \frac{\lambda}{\rho} \partial_{\rho}] \Psi_0^2 = 0. \quad (20)$$
For a solution \( \Psi_0(\rho, z; \lambda) \) which satisfies the boundary condition \( \Psi_0(\rho, z; \lambda = 0) = g_0(\rho, z) \), we have found through this analysis the following expression:

\[
\Psi_1(\rho, z; \lambda) = \exp \left[ -v(z + \frac{\lambda}{2}) - \frac{B_0}{2} (z^2 - \frac{1}{2} \rho^2 + \lambda z + \frac{1}{4} \lambda^2) \right]^{-1} \equiv K. \tag{21}
\]

Then, for the one monopole case (i.e., \( N = 1 \) in Eq. (16)), the dressing method yields the \( 2 \times 2 \) matrix \( g(\rho, z) \) with

\[
\begin{align*}
g_{11} &= -\frac{\mu^2 f_0^2 K M_2^2 + \rho^2 M_1^2}{\mu^2 f_0^2 (M_1^2 - K^4 M_2^2 f_0^2)} \\
g_{12} &= g_{21} = \frac{(\rho^2 + \mu^2) f_0 K^2 M_1 M_2}{\mu^2 (M_1^2 - K^4 M_2^2 f_0^2)} \\
g_{22} &= -\frac{\mu^2 f_0 M_1^2 + \rho^2 K^4 M_2^2 f_0^2}{\mu^2 (M_1^2 - K^4 M_2^2 f_0^2)}, \tag{22}
\end{align*}
\]

where \( \mu = -z + r \), \( r \equiv \sqrt{z^2 + \rho^2} \), and \( f_0 \) is given in Eq. (19). Finally, a new solution can be constructed directly from Eq. (18). However, in order to compare with previously known results in the limiting case, we make a gauge transformation of \( g_{ph} \) through \( g_{ph} \rightarrow hg_{ph} h^{-1} \) where

\[
h = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.
\]

Note that this is indeed a gauge transformation which leaves the chiral equation (11) covariant. This gives rise to the identification:

\[
1/f = \frac{\mu}{2\rho} (g_{11} - g_{21} - g_{12} + g_{22})
\]

\[
\psi = \frac{\mu f}{2\rho} (g_{11} - g_{22}). \tag{23}
\]

Explicit evaluation then gives the expressions

\[
\begin{align*}
f &= \frac{\rho}{F}, \quad \hat{F} \equiv \frac{r}{\sinh R} + r \cosh Z \cosh vR - z \sinh vZ, \\
\psi &= \frac{1}{F} (z \cosh Z - r \sinh Z \coth vR), \tag{24}
\end{align*}
\]

where \( R \equiv r(1 + \frac{B_0 z}{2v}) \) and \( Z \equiv z + \frac{B_0}{2v} (z^2 - \frac{1}{2} \rho^2) \) (see Eq. (19)).

Note that, with \( B_0 = 0 \) (i.e., in the zero background field limit), our expressions (24) reduce to the known results (12); in this sense, Eq. (24) provides a deformation of the Prasad-Sommerfield solution by allowing the background magnetic field. If the functions \( (\phi_1, \phi_2, \eta_1, \eta_2, W_1, W_2) \), calculated using Eqs. (8) and (24), are inserted into Eq. (5), we have an exact solution to the Bogomolny equations (1) which are regular at \( r = 0 \) and also on the \( z \)-axis. Explicitly, for the Higgs field, we find

\[
\Phi^a(\vec{r}) = -v \{(1 + \frac{B_0}{v} z) \coth vR - \frac{1}{vR} \} \hat{\rho}^a \cos \Lambda(\rho, z) + \hat{z} \sin \Lambda(\rho, z)
\]

\[
+ \frac{B_0 \rho}{2\sinh R} [\hat{\rho}^a \sin \Lambda(\rho, z) - \hat{z} \cos \Lambda(\rho, z)] \tag{25}
\]

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with the function \( \Lambda(\rho, z) \) defined through
\[
\tan \Lambda(\rho, z) = \frac{z(1 + \cosh \nu Z \cosh \nu R) - r \sinh \nu Z \sinh \nu R}{\rho (\cosh \nu Z + \cosh \nu R)}.
\tag{26}
\]
This leads to the gauge-invariant Higgs field magnitude
\[
|\Phi(\vec{r})|^2 = v^2 \left[ (1 + \frac{B_0 z}{v}) \coth \nu R - \frac{1}{\nu R} \right]^2 + \frac{B_0^2 \rho^2}{4 \sinh^2 \nu R}.
\tag{27}
\]
The small \( B_0 \)-limit of this expression can easily be shown to coincide with the gauge-invariant magnitude obtained using the perturbative solution (4); up to gauge transformation, the solution we have above is what we were after. Also, the appearance of the function \( Z(z, \rho) \) above can be ascribed to a gauge artifact. After performing an appropriate (complicated) gauge transformation with the above solution, we have succeeded in casting our full solution, including \( A_a^i(\vec{r}) \), into the form (with \( R \equiv r[1 + \frac{1}{2v} \vec{B}_0 \cdot \hat{r}] \))
\[
\Phi^a(\hat{r}) = -\hat{r}^a v \left[ (1 + \frac{1}{v} \vec{B}_0 \cdot \hat{r}) \coth \nu R - \frac{1}{\nu R} \right] - \frac{r}{2 \sinh \nu R} \left[ (\vec{B}_0)^a - \vec{B}_0 \cdot \hat{r} \hat{r}^a \right],
\]
\[
A^i_a(\hat{r}) = -\epsilon_{aij} \hat{r}_j \left[ 1 - (1 + \frac{1}{v} \vec{B}_0 \cdot \hat{r}) \frac{vr}{\sinh \nu R} \right] + \epsilon_{aij} \left[ (\vec{B}_0)_j - \vec{B}_0 \cdot \hat{r} \hat{r}_j \right] \frac{r}{2 \sinh \nu R} + \frac{r}{2} \left( \frac{1 - \cosh \nu R}{\sinh \nu R} \right) \hat{r}^a \epsilon_{ilm} \hat{r}_l (\vec{B}_0)_m.
\tag{28}
\]
[We have verified that Eq. (7), and hence Eq. (1), is satisfied by this expression]. From Eq. (27) we note that the Higgs zero or the monopole center, originally at the origin for \( B_0 = 0 \), gets displaced along the \( z \)-axis for nonzero \( B_0 \). Evidently, \(|\Phi|^2 \) may have zeros only along the line \( \rho = 0 \).) In fact, a detailed analysis shows that zeros of the Higgs field occurs in a rather nontrivial way depending on the strength of the background field \( B_0 \) (See below). Nevertheless, at large distances where \( \nu R \gg 1 \), we find
\[
|\Phi(\hat{r})| \approx v + B_0 z - \frac{1}{r}, \quad (\nu R \gg 1)
\tag{29}
\]
which is the expected behavior if an \( n = 1 \) monopole is situated near the origin in the presence of the background field \( \vec{B}_0 = B_0 \hat{z} \). But, at points on the plane \( z = -2v/B_0 \) (which is, for small \( B_0 \), on the far left of our monopole), \( R = r(1 + \frac{B_0}{2v} z) \to 0 \) and \(|\Phi(\hat{r})| \) in Eq. (27) diverges, therefore, our solution possesses a surface singularity.

It turns out that Higgs field has a single zero at \( z = 0 \) only for the vanishing background case, \( B_0 = 0 \). For \( B_0 \neq 0 \), Higgs field has a couple of zeros when \( B_0 \) is small and has no zero at all when \( B_0 \) exceeds a critical value, \( B_0 > B_0^c \approx 0.3 v^2 \). This makes it difficult to
address notions such as the monopole center or the monopole number in terms of Higgs zeros as in the case of the vanishing $B_0$. In order to help understand the situation better, it would be useful to consider the background self-dual solution itself as given in Eq. (3). It has the plane $z = -v/B_0$ as the zero of the background Higgs field. Note that, if there exists some extended region where the Higgs field becomes very small, the topological character usually related to a magnetic monopole gets rather murky. Our solution has a monopole deep in the right half-space $z > -v/B_0$ and shows a plausible behavior in the very right half. On the other hand, the plane $z = -v/B_0$ as the zeros of the background Higgs field has disappeared. In our solution (25) or (28), we have instead an isolated zero near the point $\rho = 0, \ z = -v/B_0$ immersed in the region of small, but non-zero, Higgs field and there is no distinctive long-range tail associated with this zero. Even in the other half-space where the Higgs field was aligned in the opposite direction, $|\Phi(\vec{r})|$ is well approximated by (29) if $z$ is not too close to $-2v/B_0$. However, the divergence of $|\Phi(\vec{r})|$ encountered at $z = -2v/B_0$ is nontrivial; above all, it is not an gauge artifact. Thus, our monopole solution cannot be extended beyond this singular plane. If one is concerned with only restricted physical problems (as in Ref. [6]), this ill-behavior of our solution in the ‘wrong’ Higgs vacuum region might not be taken too seriously. But our opinion is that this singularity issue deserves further investigation in the future.

A couple of comments are in order. We note that the well-known trick [14] may be used on our solution to obtain the corresponding dyon solution which solves the generalized Bogomolny equations [15]

$$B_i = - \cos \beta \ D_i \Phi, \quad E_i = - \sin \beta \ D_i \Phi$$

in the background of a uniform magnetic and electric field. Also, our approach can be applied to the problem of finding exact instanton solutions in nonvanishing background fields as well. In this regard, it would be interesting to extend the ADHM construction [4] and the Nahm equation [5] in the presence of background fields.

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