A Family of Indecomposable Positive Linear Maps based on Entangled Quantum States

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Abstract

We introduce a new family of indecomposable positive linear maps based on entangled quantum states. Central to our construction is the notion of an unextendible product basis. The construction lets us create and conjecture indecomposable positive linear maps in matrix algebras of arbitrary high dimension.
I. INTRODUCTION

One of the central problems in the emergent field of quantum information theory [1] is the classification and characterization of the entanglement (to be defined in section II) of quantum states. Entangled quantum states have been shown to be valuable resources in (quantum) communication and computation protocols. In this context it has been shown [2] that there exists a strong connection between the classification of the entanglement of quantum states and the structure of positive linear maps. Very little is known about the structure of positive linear maps even on low dimensional matrix algebras, in particular the structure of indecomposable positive linear maps. We denote the $n \times n$ matrix algebra as $M_n(C)$. The first example of an indecomposable positive linear map in $M_3(C)$ was found by Choi [3]. There have been only several other examples of indecomposable positive linear maps (see [4] for some recent literature); they seem to be hard to find and no general construction method is available. In this paper we make use of the connection with quantum states to develop a method to create indecomposable positive linear maps on matrix algebras $M_n(C)$ for any $n > 2$. This construction exhibits some of the structure of positive linear maps which is present in almost any dimension. In section II we present the general construction. In section III we present two examples and discuss various open problems.

II. UNEXTENDIBLE PRODUCT BASES AND INDECOMPOSABLE MAPS

A $n$-dimensional complex Hilbert space is denoted as $\mathcal{H}_n$. The set of all linear operators on a Hilbert space $\mathcal{H}_n$ will be denoted as $B(\mathcal{H}_n)$. The subset of Hermitian positive semidefinite operators is denoted as $B(\mathcal{H}_n)^+$. We will use the conventional bra and ket notation in quantum mechanics, i.e. a vector $\vec{\psi}$ in $\mathcal{H}_n$ is written as a ket,

$$|\psi\rangle \in \mathcal{H}_n, \quad (1)$$

and the Hermitian conjugate of $\vec{\psi}$, $\vec{\psi}^*$, is denoted as a bra $\langle \psi |$. The complex innerproduct between vectors $|\psi\rangle$ and $|\phi\rangle$ in $\mathcal{H}_n$ is denoted as
\[\langle \psi | \phi \rangle \equiv \vec{\psi}^* \vec{\phi}. \] (2)

The vectors \(|\psi\rangle \in \mathcal{H}\) are usually normalized, \(\langle \psi | \psi \rangle = 1\). Elements of \(B(\mathcal{H}_n)^+\) are denoted as

\[\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|, \] (3)

where \(|\psi_i\rangle\) are the normalized eigenvectors of \(\rho\) and \(\lambda_i \geq 0\) are the eigenvalues. When \(\rho\) has trace equal to one, \(\rho\) is said to be a density matrix. The physical state of a quantum mechanical system is given by its density matrix. If a density matrix \(\rho\) has rank one, \(\rho\) is called a pure state and can be written as

\[\rho = |\psi\rangle \langle \psi|. \] (4)

Let \(S: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)\) be a linear map. \(S\) is positive when \(S: B(\mathcal{H}_n)^+ \rightarrow B(\mathcal{H}_m)^+\). Let \(id_k\) be the identity map on \(B(\mathcal{H}_k)\). We define the map \(id_k \otimes S: B(\mathcal{H}_k \otimes \mathcal{H}_n) \rightarrow B(\mathcal{H}_k \otimes \mathcal{H}_m)\) for \(k = 1, 2, \ldots\) by

\[(id_k \otimes S) \left( \sum_i \sigma_i \otimes \tau_i \right) = \sum_i \sigma_i \otimes S(\tau_i), \] (5)

where \(\sigma_i \in B(\mathcal{H}_k)\) and \(\tau_i \in B(\mathcal{H}_n)\). The map \(S\) is \(k\)-positive when \(id_k \otimes S\) is positive. The map \(S\) is completely positive when \(S\) is \(k\)-positive for all \(k = 1, 2, \ldots\). Following Lindblad [5], the set of physical operations on a density matrix \(\rho \in B(\mathcal{H}_n)^+\) is given by the set of completely positive trace-preserving maps \(S: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)\). Similarly as \(k\)-positive, one can define a \(k\)-copositive map. Let \(T: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_n)\) be defined as matrix transposition in a chosen basis for \(\mathcal{H}_n\), i.e.

\[T(A_{ij}) = A_{ji}, \] (6)

on a matrix \(A_{ij} \in B(\mathcal{H}_n)\). The map \(S\) is \(k\)-copositive when \(id_k \otimes ST\) is positive. A positive linear map \(S: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)\) is decomposable if it can be written as

\[S = S_1 + S_2 T, \] (7)
where $S_1: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ and $S_2: B(\mathcal{H}_n) \rightarrow B(\mathcal{H}_m)$ are completely positive maps. It has been shown by Woronowicz [6] that all positive linear maps $S: B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_2)$ and $S: B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_3)$ are decomposable.

**Definition 1** Let $\rho$ be a density matrix on a finite dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A state $|\psi\rangle$ of the form $|\psi^A\rangle \otimes |\psi^B\rangle$ is a (pure) product state in $\mathcal{H}_A \otimes \mathcal{H}_B$. The density matrix $\rho$ is entangled iff $\rho$ cannot be written as a nonnegative combination of pure product states, i.e. there does not exist an ensemble $\{p_i \geq 0, |\psi_i^A \otimes \psi_i^B\rangle\}$ such that

$$\rho = \sum_i p_i |\psi_i^A\rangle\langle \psi_i^A| \otimes |\psi_i^B\rangle\langle \psi_i^B|. \quad (8)$$

When $\rho$ is not entangled $\rho$ is called separable.

The problem of deciding whether a bipartite density matrix $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled can be quite hard. The following theorem by the Horodeckis [2] formulates a necessary and sufficient condition for a density matrix $\rho$ to be entangled:

**Theorem 1 (Horodecki)** A density matrix $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff there exists a positive linear map $S: \mathcal{H}_B \rightarrow \mathcal{H}_A$ such that

$$(\text{id}_A \otimes S) \rho \quad (9)$$

is not positive semidefinite. Here $\text{id}_A$ denotes the identity map on $B(\mathcal{H}_A)$.

**Remark** An equivalent statement as theorem 1 holds for positive linear maps $S: \mathcal{H}_A \rightarrow \mathcal{H}_B$ and the positivity of $S \otimes \text{id}_B$.

The consequences of Theorem 1 and Woronowicz’ result is that a bipartite density matrix $\rho$ on $\mathcal{H}_2 \otimes \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_3$ is entangled iff $(\text{id}_A \otimes S_1 + S_2 T) \rho$ is not positive semidefinite. As $S_1$ and $S_2$ are completely positive maps this is equivalent to saying that $(\text{id}_A \otimes T) \rho$ is not positive semidefinite.

The more complicated structure of the positive linear maps in higher dimensional matrix algebras, namely the existence of indecomposable positive maps, is reflected in the existence of entangled density matrices $\rho$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ for which $(\text{id}_A \otimes T) \rho$ is positive semidefinite.
The first example of such a density matrix on $\mathcal{H}_2 \otimes \mathcal{H}_4$ and $\mathcal{H}_3 \otimes \mathcal{H}_3$ was given by P. Horodecki [7]. In [8] a method was discovered to construct entangled density matrices $\rho$ with positive semidefinite $(\text{id}_A \otimes T)(\rho)$ in various dimensions $\dim \mathcal{H}_A > 2$ and $\dim \mathcal{H}_B > 2$. The construction was based on the notion of an unextendible product basis. Let us give the definition.

**Definition 2** Let $\mathcal{H}$ be a finite dimensional Hilbert space of the form $\mathcal{H}_A \otimes \mathcal{H}_B$. A partial product basis is an orthonormal set $S$ of pure product states spanning a proper subspace $\mathcal{H}_S$ of $\mathcal{H}$. An unextendible product basis is a partial product basis whose complementary subspace $\mathcal{H}_S^\perp$ contains no product state.

**Remark** This definition can be extended to product bases in $\mathcal{H} = \bigotimes_{i=1}^m \mathcal{H}_i$ with arbitrary $m$. Note we restrict ourselves to sets orthonormal sets $S$.

With this notion we can construct the following density matrix:

**Theorem 2** [8] Let $S$ be a bipartite unextendible product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a density matrix $\rho$ as

$$\rho = \frac{1}{\dim \mathcal{H} - |S|} \left( \text{id}_{AB} - \sum_i |\alpha_i\rangle \langle \alpha_i| \otimes |\beta_i\rangle \langle \beta_i| \right),$$

where $\text{id}_{AB}$ is the identity on $\mathcal{H}$. The density matrix $\rho$ is entangled. Furthermore, the state $(\text{id}_A \otimes [S_1 + TS_2])\rho$ for all completely positive maps $S_1$ and $S_2$, is positive semidefinite.

**Proof** The density matrix $\rho$ is proportional to the projector on the complementary subspace $\mathcal{H}_S^\perp$. As $S$ is unextendible $\mathcal{H}_S^\perp$ contains no product states. Therefore the density matrix is entangled. It is not hard to see that $(\text{id}_A \otimes T)\rho$ is positive semidefinite. It has been proved in [9] that when $(\text{id}_A \otimes T)\rho$ is positive semidefinite that $(\text{id}_A \otimes TS_2)\rho$ where $S_2$ is any completely positive map, is also positive semidefinite. Therefore $(\text{id}_A \otimes [S_1 + TS_2])\rho$ is also positive semidefinite. $\Box$

We are now ready to present our results relating these density matrices obtained from the construction in Theorem 2 to indecomposable positive linear maps. We will need the definition of a maximally entangled pure state:
Definition 3 Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $|\psi\rangle$ be a normalized state in $\mathcal{H}$ and
\[\rho_{A,\psi} = \text{Tr}_B |\psi\rangle \langle \psi|,\]
where $\text{Tr}_B$ indicates that the trace is taken with respect to Hilbert space $\mathcal{H}_B$ only. The state $|\psi\rangle \in \mathcal{H}$ is maximally entangled when
\[S(\rho_{A,\psi}) = -\text{Tr} \rho_{A,\psi} \log_2 \rho_{A,\psi} = \log_2 \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)\] (12)
The function $S(\rho_{A,\psi})$ is the von Neumann entropy of the density matrix $\rho_{A,\psi}$.

Remarks For pure states $|\psi\rangle$ the von Neumann entropy of $\rho_{A,\psi}$ is always less than or equal to $d \equiv \log_2 \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. For maximally entangled states we will have $\rho_{A,\psi} = \text{diag}(1/d, \ldots, 1/d, 0, \ldots, 0)$ so that the maximum von Neumann entropy, Eq. (12), is achieved. When $\dim \mathcal{H}_A = \dim \mathcal{H}_B$ one can always make an orthonormal basis for $\mathcal{H}$ with maximally entangled states [10].

The following lemma bounds the innerproduct between a maximally entangled state and any product state.

Lemma 1 Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $|\Psi\rangle \in \mathcal{H}$ be a maximally entangled state. Let $d = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. For all (normalized) product states $|\phi_A\rangle \otimes |\phi_B\rangle$,
\[|\langle \Psi | \phi_A \rangle \otimes | \phi_B \rangle|^2 \leq \frac{1}{d}.\] (13)

Proof We write the maximally entangled state $|\Psi\rangle$ in the Schmidt polar form [11] as
\[|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |a_i\rangle \otimes |b_i\rangle,\] (14)
where $\langle a_i | a_j \rangle = \delta_{ij}$ and $\langle b_i | b_j \rangle = \delta_{ij}$. Thus we can write
\[|\langle \Psi | \phi_A \rangle \otimes | \phi_B \rangle|^2 = \frac{1}{d^2} \left| \sum_{i=1}^{d} \langle a_i | a_i \rangle \langle \phi_B | b_i \rangle \right|^2 \leq \frac{1}{d},\] (15)
using the Schwarz inequality and $\sum_{i=1}^{d} |\langle a_i | a_i \rangle|^2 \leq 1$ and $\sum_{i=1}^{d} |\langle \phi_B | b_i \rangle|^2 \leq 1$. ◻

We will also need the following lemma:
Lemma 2 Let \( S \) be an unextendible product basis \( \{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|} \) in \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). Let

\[
f(|\phi_A\rangle, |\phi_B\rangle) = \sum_{i=1}^{|S|} |\langle \alpha_i | \phi_A \rangle|^2 |\langle \beta_i | \phi_B \rangle|^2.
\]

(16)

The minimum of \( f \) over all pure states \( |\phi_A\rangle \in \mathcal{H}_A \) and \( |\phi_B\rangle \in \mathcal{H}_B \) exists and is strictly larger than 0.

Proof The set of all pure product states \( |\phi_A\rangle \otimes |\phi_B\rangle \) on \( \mathcal{H} \) is a compact set. The function \( f \) is a continuous function on this set. Therefore, if there exists a set of states \( |\phi_A\rangle \otimes |\phi_B\rangle \) for which \( f \) is arbitrary small then there would also exist a pair \( |\phi'_A\rangle \otimes |\phi'_B\rangle \) for which \( f = 0 \). This contradicts the fact that \( S \) is an unextendible product basis.

The following two theorems contain the main result of this paper.

Theorem 3 Let \( S \) an unextendible product basis \( \{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|} \) in \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). Let \( \rho \) be the density matrix

\[
\rho = \frac{1}{\dim \mathcal{H} - |S|} \left( \text{id}_{AB} - \sum_{i=1}^{|S|} |\alpha_i\rangle \langle \alpha_i | \otimes |\beta_i\rangle \langle \beta_i | \right),
\]

(17)

Let \( d = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B) \). Let \( H \) be a Hermitian operator given by

\[
H = \sum_{i=1}^{|S|} |\alpha_i\rangle \langle \alpha_i | \otimes |\beta_i\rangle \langle \beta_i | - d |\Psi\rangle \langle \Psi |,
\]

(18)

where \( |\Psi\rangle \) is a maximally entangled state such that

\[
\langle \Psi | \rho | \Psi \rangle > 0,
\]

(19)

and

\[
\epsilon = \min_{|\phi_A\rangle \otimes |\phi_B\rangle} \sum_{i=1}^{|S|} |\langle \alpha_i | \phi_A \rangle|^2 |\langle \beta_i | \phi_B \rangle|^2,
\]

(20)

where the minimum is taken over all pure states \( |\phi_A\rangle \in \mathcal{H}_A \) and \( |\phi_B\rangle \in \mathcal{H}_B \). For any unextendible product basis \( S \) it is possible to find a maximally entangled state \( |\Psi\rangle \) such that Eq.(19) holds. \( H \) has the following properties:

\[
\text{Tr} H \rho < 0,
\]

(21)
and for all product states $|\phi_A\rangle \otimes |\phi_B\rangle \in \mathcal{H}$,

$$\text{Tr} \mathcal{H} |\phi_A\rangle \langle \phi_A| \otimes |\phi_B\rangle \langle \phi_B| \geq 0.$$  \hfill (22)

Proof Eq. (22) follows from the definition of $\epsilon$, Eq. (20), and Lemma 1. Consider Eq. (21). As the density matrix $\rho$ is proportional to the projector on $\mathcal{H}_S^\perp$, one has

$$\text{Tr} \mathcal{H} \rho = -d\epsilon \langle \Psi | \rho | \Psi \rangle,$$  \hfill (23)

which is strictly smaller than zero by Lemma 2 and the choice of the maximally entangled state, Eq. (19). When $\dim \mathcal{H}_A = \dim \mathcal{H}_B$ there exist bases of maximally entangled states and thus there will be a basis vector $|\Psi\rangle$ for which $\langle \Psi | \rho | \Psi \rangle$ is nonzero. In case, say, $\dim \mathcal{H}_A > \dim \mathcal{H}_B$, the maximally entangled states form bases of a subspace $\mathcal{H}' = \mathcal{H}'_A \otimes \mathcal{H}_B$ with $\mathcal{H}'_A \subset \mathcal{H}_A$. If $\rho$ has a nonzero projection onto $\mathcal{H}'$ we can find a maximally entangled state $|\Psi\rangle$ such that $\langle \Psi | \rho | \Psi \rangle > 0$. On the other hand if $\rho \in B(\mathcal{H}'^\perp)$, then $\rho$ will be a separable density matrix as $\mathcal{H}'$ has a tensorproduct structure. This is in contradiction with the fact that $\rho$ is entangled. This completes the proof. \hfill \Box

Theorem 4 Let $S$ be an unextendible product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{[S]}$ in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\mathcal{H}$ be defined as in Theorem 3, Eq. (18). Choose an orthonormal basis $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_A}$ for $\mathcal{H}_A$. Let $S$: $B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ be a linear map defined by

$$S (|i\rangle \langle j|) = \langle i| \mathcal{H} |j\rangle.$$  \hfill (24)

Then $S$ is positive but not completely positive. $S$ is indecomposable.

Proof The relation between $S$ and $\mathcal{H}$, Eq. (24), follows from the isomorphism between Hermitian operators on $\mathcal{H}_A \otimes \mathcal{H}_B$ with the property of Eq. (22) and linear positive maps, see [2,12]. In particular, iff a Hermitian $\mathcal{H}$ operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ has the property of Eq. (22) then the linear map $\mathcal{R}$: $B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ defined by

\[ \]
\[ H = \sum_{i,j} (|i\rangle\langle j|)^* \otimes \mathcal{R}(|i\rangle\langle j|), \]  

(25)

is positive for any choice of the orthonormal basis \(\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_A}\). The map \(\mathcal{S} = \mathcal{R}T\) where \(T\) is matrix transposition in the basis \(\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_A}\) of Eq. (24) is then positive as well.

We will show how the density matrix \(\rho\) derived from the unextendible product basis, Eq. (17) shows that \(\mathcal{S}\) is not completely positive. At the same time we prove that the assumption that \(\mathcal{S}\) is decomposable leads to a contradiction. We can rewrite Eq. (24) as

\[ H = (\text{id}_A \otimes \mathcal{S})(|\Psi^+\rangle\langle \Psi^+|) \]  

(26)

where \(|\Psi^+\rangle\) is equal to the (unnormalized) maximally entangled state \(\sum_{i=1}^{\dim \mathcal{H}_A} |i\rangle \otimes |i\rangle\). Let \(\mathcal{S}^* : B(\mathcal{H}_B) \rightarrow B(\mathcal{H}_A)\) be the Hermitian conjugate of \(\mathcal{S}\). We use the definition of \(\mathcal{S}^*\)

\[ \text{Tr} \mathcal{S}^*(A^*) B = \text{Tr} A^* \mathcal{S}(B), \]  

(27)

and Eq. (26) to derive that Eq. (21) can be rewritten as

\[ \text{Tr} H \rho = \langle \Psi^+ | (\text{id}_A \otimes \mathcal{S}^*) (\rho) |\Psi^+\rangle < 0, \]  

(28)

Thus \(\mathcal{S}^*\) cannot be completely positive and therefore \(\mathcal{S}\) itself is not completely positive. If \(\mathcal{S}\) was decomposable, then \(\mathcal{S}^*\) would be of the form \(\mathcal{S}_1 + T\mathcal{S}_2\) where \(\mathcal{S}_1\) and \(\mathcal{S}_2\) are completely positive maps. The density matrix \(\rho\) is positive semidefinite under any linear map of the form \(\mathcal{S}_1 + T\mathcal{S}_2\) by Theorem 2. This is in contradiction with Eq. (28) and therefore \(\mathcal{S}\) cannot be decomposable. \(\square\)

III. EXAMPLES AND DISCUSSION

As we have shown the structure of unextendible product bases carries over to indecomposable positive linear maps. In this section we will list some of the results that have been obtained about unextendible product bases. We will take two examples of unextendible product bases and demonstrate the construction of Theorem 3 and Theorem 4.
1. In [8] it was shown that there exist no unextendible product bases in $\mathcal{H}_2 \otimes \mathcal{H}_n$ for any $n \geq 2$.

2. In [13] it was shown how to parametrize all possible unextendible product bases in $\mathcal{H}_3 \otimes \mathcal{H}_3$ as a six-parameter family.

3. In [13] a family of unextendible product bases $\mathcal{H}_n \otimes \mathcal{H}_m$ ($m > 2, n > 2$) for arbitrary $m \neq n$ as well as even $n = m$ has been conjectured. The conjecture was proved in $\mathcal{H}_3 \otimes \mathcal{H}_n$ and $\mathcal{H}_4 \otimes \mathcal{H}_4$.

4. Also in [13] a family of unextendible product bases, based on quadratic residues, in $\mathcal{H}_n \otimes \mathcal{H}_n$ where $n$ is any odd number and $2n - 1$ is prime has been conjectured. The conjecture was proved in $\mathcal{H}_3 \otimes \mathcal{H}_3$ and $\mathcal{H}_7 \otimes \mathcal{H}_7$.

5. In [13] it was shown that when $S_1$ and $S_2$ are unextendible product bases on $\mathcal{H}_A^1 \otimes \mathcal{H}_B^1$ and $\mathcal{H}_A^2 \otimes \mathcal{H}_B^2$ respectively, then the tensorproduct of the two sets, $S_1 \otimes S_2$, is again an unextendible product bases on $(\mathcal{H}_A^1 \otimes \mathcal{H}_A^2) \otimes (\mathcal{H}_B^1 \otimes \mathcal{H}_B^2)$.

**Example 1:** One of the first examples of an unextendible product basis on $\mathcal{H}_3 \otimes \mathcal{H}_3$ was the following set of states [8]. Consider five vectors in real three-dimensional space forming the apex of a regular pentagonal pyramid, the height $h$ of the pyramid being chosen such that nonadjacent apex vectors are orthogonal. The vectors are

$$ |v_i\rangle = N(\cos \frac{2\pi i}{5}, \sin \frac{2\pi i}{5}, h), \quad i = 0, \ldots, 4, \quad (29) $$

with $h = \frac{1}{2}\sqrt{1 + \sqrt{5}}$ and $N = 2/\sqrt{5 + \sqrt{5}}$. Then the following five states in $\mathcal{H}_3 \otimes \mathcal{H}_3$ form an unextendible product basis:

$$ |p_i\rangle = |v_i\rangle \otimes |v_{2i \mod 5}\rangle, \quad i = 0, \ldots, 4. \quad (30) $$

Let $\rho$ be the entangled state derived from this unextendible product basis as in Eq. (10). Theorem 2. We choose a maximally entangled state $|\Psi\rangle$, here named $|\Psi^+\rangle$,

$$ |\Psi^+\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle). \quad (31) $$
One can easily compute that
\[
\langle \Psi^+ | \rho | \Psi^+ \rangle = \frac{1}{4} \left( 1 - \frac{7 + \sqrt{5}}{3(3 + \sqrt{5})} \right) > 0. \quad (32)
\]

The map \( \mathcal{S} \) as defined in Eq. (24) Theorem 4, follows directly:
\[
\mathcal{S}(|i\rangle \langle j|) = \sum_{k=0}^{4} \langle i|v_k\rangle \langle v_{2k \mod 5}| \langle v_{2k \mod 5}| - \epsilon |i\rangle \langle j|. \quad (33)
\]

A positive linear map \( \mathcal{S} : B(\mathcal{H}_n) \to B(\mathcal{H}_m) \) is unital if \( \mathcal{S}(\text{id}_n) = \text{id}_m \). We will demonstrate that \( \mathcal{S} \) is not unital. One can write
\[
\mathcal{S}(\text{id}_A) = \text{Tr}_A H = \sum_{k=0}^{4} \langle v_{2k \mod 5}| \langle v_{2k \mod 5}| - 3\epsilon \text{Tr}_A |\Psi^+\rangle \langle \Psi^+|, \quad (34)
\]

which in turn is equal to
\[
\mathcal{S}(\text{id}_A) = \text{diag} \left( \frac{10}{5 + \sqrt{5}}, \frac{10}{5 + \sqrt{5}}, \sqrt{5} \right) - \epsilon \text{id}_B. \quad (35)
\]

A numerical approximation of \( \epsilon \) as defined in Eq. (20) Theorem 3, gives the value
\[
\epsilon \approx 0.037911, \quad (36)
\]

but we don’t know whether this is the minimum of the function in Eq. (20).

The next example is based on a more general unextendible product bases that was presented in [13].

**Example 2:** The states of \( S \) in \( \mathcal{H}_3 \otimes \mathcal{H}_n \) are
\[
|F_0^k\rangle = \frac{1}{\sqrt{n-2}} |0\rangle \otimes (|1\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle), \quad 1 \leq k \leq n-3, \quad (37)
\]
\[
|F_1^k\rangle = \frac{1}{\sqrt{n-2}} |1\rangle \otimes (|2\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle), \quad 1 \leq k \leq n-3, \quad (38)
\]
\[
|F_2^k\rangle = \frac{1}{\sqrt{n-2}} |2\rangle \otimes (|0\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle), \quad 1 \leq k \leq n-3, \quad (39)
\]
\[
|\psi_3\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes |0\rangle, \quad (40)
\]
\[
|\psi_4\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \otimes |1\rangle, \quad (41)
\]
\[
|\psi_5\rangle = \frac{1}{\sqrt{2}} (|2\rangle - |0\rangle) \otimes |2\rangle, \quad (42)
\]
\[
|\psi_6\rangle = \frac{1}{\sqrt{3n}} \sum_{j=0}^{2} \sum_{j=0}^{n-1} |i\rangle \otimes |j\rangle, \quad (43)
\]
and we have $\omega = \exp(2\pi i/(n-2))$. Here the states $\{|k\rangle\}_{k=0}^{n-1}$ form an orthonormal basis. In total there are $3n-5$ states in the basis. We choose a maximally entangled state, again we take $|\Psi^+\rangle$, Eq. (31). One can show that

$$
\langle \Psi^+ | \rho | \Psi^+ \rangle = \frac{1}{5} \left( \frac{1}{2} - \frac{1}{3n} \right) > 0. \tag{44}
$$

The map $S: B(H_3) \to B(H_n)$ is given as

$$
S(|i\rangle\langle j|) = \sum_{k=1}^{n-3} \sum_{p=0}^{2} \langle i | F_p^k \rangle \langle F_p^k | j \rangle + \sum_{p=3}^{6} \langle i | \psi_p \rangle \langle \psi_p | j \rangle - \epsilon |i\rangle\langle j|. \tag{45}
$$

The following questions concerning the positive maps that were introduced in this paper are left open.

1. Is $S$ always non unital? We conjecture it is. As we showed, see Eq. (34), the answer to this question depends on whether

$$
\sum_{i=1}^{\text{|S|}} |\beta_i\rangle\langle \beta_i| \propto \text{id}_B, \tag{46}
$$

where the set of states $\{|\beta_i\rangle\}_{i=1}^{\text{|S|}}$ are one side of the unextendible product basis. The states $|\beta_i\rangle$ will span $H_B$ but they will not be all orthogonal, nor all nonorthogonal.

2. It was shown in Theorem 4 that the new indecomposable positive linear maps $S: B(H_m) \to B(H_n)$ are not $m$-positive, as they are not completely positive. Are these maps $S$ $k$-positive with $1 < k < m$? The answer to this question will rely on a better understanding of the structure of unextendible product bases.

3. In [8] a single example was given of an entangled density matrix on $H_3 \otimes H_4$ which was positive semidefinite under $\text{id}_3 \otimes T$. The density matrix was based not on an unextendible product basis, but an uncompletable product basis $S$. It could be shown that the Hilbert space $H_3^\perp$ had a product state deficit, i.e. the number of product states in $H_3^\perp$ was less than $\text{dim} H_3^\perp$. It is open question how to generalize this example and whether these kinds of density matrices will give rise to more general family of indecomposable positive linear maps.
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REFERENCES


