Optical depth evaluation in pixel microlensing

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ABSTRACT

We propose an estimator of the microlensing optical depth from pixel lensing data that involves only measurable quantities. In comparison to the only previously proposed estimator, it has the advantage of not being limited to events with large magnification at maximum, and it applies equally well to satellite and ground-based observations.

Subject headings: gravitational lensing
1. INTRODUCTION

An important physical quantity that we want to determine from gravitational microlensing data is the microlensing optical depth. This is the number of gravitational lenses lying within one Einstein radius around the line of sight to a given source.

When monitoring individual stars as proposed by Paczyński (1986), the optical depth is estimated from observational data as a weighted sum of the Einstein times of the individual events (see eq. [7] below; Griest 1991). The Einstein times are determined by fitting the theoretical microlensing lightcurve to the measured lightcurve. The fit is almost degenerate in the three fit parameters Einstein time, maximum magnification, and star flux in absence of lensing (Gould 1996; Woźniak & Paczyński 1997). In particular, the fitted value of the Einstein time depends sensitively on the value of the star flux in absence of lensing, especially at large maximum magnifications.

In pixel microlensing, which is gravitational microlensing of unresolved stars, the star flux in absence of lensing is not generally accessible, since the combined light of all the stars contributing to a pixel is monitored simultaneously. Thus, because of the above–mentioned degeneracy, the Einstein time cannot be reliably determined from a fit to the lightcurve. In this case, the optical depth cannot be estimated by means of the usual expression, and this seems to hinder the possibility of obtaining it from pixel lensing data.

A partial solution to this problem has been proposed by Gould (1995 & 1996) under conditions that may be possible in future satellite observations but are definitely not typical of present microlensing searches towards the Andromeda galaxy M31 and the Large Magellanic Cloud (Crotts 1992, Bailon et al. 1993, Crotts and Tomaney 1996, Tomaney and Crotts 1996, Ansari et al. 1997 for M31; Melchior et al. 1997 & 1998 for the LMC). He assumed regular observations with constant exposure times, a constant level of noise dominated by photon counting, and large magnifications at maximum so that the microlensing lightcurve assumes a completely degenerate form (eq. [5] below). The present pixel–lensing searches instead have irregular time gaps between observations, show a variable level of noise depending on factors other than photon counting, and expect relatively low magnifications for which the microlensing curve is not well approximated by the completely degenerate form.

In this letter, we present an estimator of the optical depth in pixel microlensing that applies to all magnifications, involves measurable quantities only, and is of direct applicability to the data. It is based on the substitution of the Einstein time with the product of the flux increase at maximum and of the full event duration at half-maximum, defined as the total time during which the flux increase exceeds half of its maximum value.
2. USUAL OPTICAL DEPTH ESTIMATOR

The flux increase in a microlensing event is given by (for pointlike lens and pointlike source; Einstein 1936)

\[ \Delta F(t) = F f(u^2(t)), \]

with

\[ f(x) = \frac{2 + x}{\sqrt{x(4 + x)}} - 1. \]

Here \( F \) is the source flux in absence of lensing, \( u = \theta/\theta_e \), \( \theta \) is the angular separation of lens and source, and \( \theta_e \) is the angular Einstein radius

\[ \theta_e = \left[ \frac{4Gm}{c^2} \cdot \frac{s - l}{sl} \right]^{1/2}, \]

where \( m \) is the lens mass, and \( s \) and \( l \) are the distances of the source and the lens from the observer.

The relative motion of lens and source during the event is usually well approximated by a uniform motion, for which

\[ u^2(t) = \beta^2 + \left( \frac{t - t_{\text{max}}}{t_e} \right)^2. \]

\( t_{\text{max}} \) is the time of maximum magnification, \( \beta \) is the impact parameter in units of the Einstein radius, and \( t_e \) is the Einstein time. This we define as the time the lens would take to cross the radius of the Einstein ring, \( t_e = \theta_e/\mu \), where \( \mu \) is the relative proper motion of lens and source.

In the limit of large magnifications at maximum, the microlensing curve assumes the completely degenerate form

\[ \Delta F(t) \simeq \frac{F}{u(t)} = \frac{F}{\beta} \left[ 1 + \left( \frac{t - t_{\text{max}}}{\beta t_e} \right)^2 \right]^{-1/2} \quad (F, \beta \to 0; F/\beta = \text{const}). \]

The three parameters \( F, \beta \) and \( t_e \) have reduced to two: \( F/\beta \) and \( \beta t_e \). A fit to the data can determine neither the star flux \( F \) nor the Einstein time \( t_e \) separately from the unknown impact parameter \( \beta \) (see e.g. Woźniak & Paczyński 1997).

The microlensing optical depth is the number of gravitational lenses lying within one Einstein radius \( \theta_e \) around the line of sight to a given source,

\[ \tau = \int_0^s \pi \theta_e^2 l^2 n(l) \, dl, \]
where the integral is along the line of sight to the source at distance $s$ from the observer, and $n(l)$ is the lens number density at distance $l$ along the same line of sight.

When monitoring $N_s$ sources for a time $T$, the optical depth is usually estimated as

$$\tau = \frac{\pi}{2N_sT} \sum_{\text{events}} \frac{t_e}{\epsilon(t_e)}.$$  \hspace{1cm} (7)

Here $\tau$ is the optical depth averaged over the monitored sources, the sum is over the observed events, and $\epsilon(t_e)$ is the detection efficiency (normalized to impact parameters $< \theta_e$) as a function of Einstein time $t_e$. The Einstein time of each event is determined experimentally by a fit of the theoretical microlensing lightcurve to the measured lightcurve.

This usual estimator is based on the relation

$$\int_{\beta<1} t_e dN_s d\Gamma = \frac{\pi}{2} N_s \tau.$$  \hspace{1cm} (8)

In the right hand side, $\tau \equiv N_s^{-1} \int \tau dN_s$; in the left-hand side, $dN_s$ is the differential number of monitored stars, $d\Gamma$ is the event rate per source, $d\Gamma = (2\tau/\pi t_e) d\beta$, and the integration is limited to impact parameters $\beta<1$. Eq. (7) follows from solving eq. (8) for $\tau$ and rescaling the observed number of events $dN_{\text{obs}}$ to the expected number of events $dN = TdN_s d\Gamma$ by means of the detection efficiency $\epsilon(t_e)$ normalized to $\beta<1$. Explicitly

$$\sum_{\text{events}} \frac{t_e}{\epsilon(t_e)} = \int_{\beta<1} t_e \frac{\epsilon(t_e)}{\epsilon(t_e)} dN_{\text{obs}} = \int_{\beta<1} t_e dN = \frac{\pi}{2} TN_s \tau.$$  \hspace{1cm} (9)

3. NEW OPTICAL DEPTH ESTIMATOR

We now prove a relation similar to eq. (8) but not involving the Einstein time $t_e$, which is very poorly measurable in pixel lensing. Let $t_{\text{fwhm}}$ be the event duration at half-maximum and $\Delta_{\text{max}}$ be the flux increase at maximum. Both are easily determined from a fit to the lightcurve. Then

$$\int t_{\text{fwhm}} \Delta_{\text{max}} dN_s d\Gamma = \frac{\pi}{2} IF_s \tau_F.$$  \hspace{1cm} (10)

Here $F_s$ is the total flux of the monitored sources, $I$ is 2.1465, and $\tau_F$ is the optical depth averaged over the monitored sources, each source weighted by its flux, $\tau_F = F_s^{-1} \int F \tau dN_s$. This flux-weighted optical depth $\tau_F$ equals the averaged optical depth $\tau$ whenever the source luminosity function is, or can be considered as, independent of source position, because in this case the averages over source positions and fluxes are independent. This is
the case for a single source population.\footnote{The flux-weighted optical depth may actually find an interesting application for individually monitored sources, for example in the microlensing data towards the galactic bulge: comparing it to the unweighted optical depth may give information on a spatial segregation of the star populations.}

The proof of eq. (10) is as follows. We express the half-maximum duration $t_{\text{fwhm}}$ as a function of $t_e$ and $\beta$ by finding the value of the impact parameter $\beta_w$ at which the flux increase equals half of its maximum value. We also express the maximum flux increase $\Delta_{\text{max}}$ in terms of the source flux $F$ in absence of lensing and of the impact parameter $\beta$. We find $t_{\text{fwhm}} = t_e w(\beta)$ and $\Delta_{\text{max}} = F \delta(\beta)$, with $w(\beta) = 2 \left( \beta_w^2 - \beta^2 \right)^{1/2}$, $\beta_w^2 = 2 f(\delta(\beta))$, $\delta(\beta) = f(\beta^2)$, and $f(x)$ given in eq. (2). The integral over $\beta$ in eq. (10) is then

$$I \equiv \int_0^\infty w(\beta) \delta(\beta) d\beta = 2.1465.$$ \hfill (11)

The remaining integral over the sources gives the total source flux $F_s = \int F dN_s$ times the flux-weighted optical depth $\tau_F$. Putting the factors together gives eq. (10).

If all events could be detected, then eq. (10) would immediately give an estimator for the optical depth, namely

$$\tau_F = \frac{\pi}{2ITF_s} \sum_{\text{all events}} t_{\text{fwhm}} \Delta_{\text{max}}$$ \hspace{1cm} (no noise, 100% efficiency). \hfill (12)

Notice that apart from the numerical factor $\pi/2I \simeq 0.73$, $\tau_F$ equals the sum over events of the products of the fractional flux increase at maximum $\Delta_{\text{max}}/F_s$ and of the fraction of observing time spent above half-maximum $t_{\text{fwhm}}/T$. The estimator in eq. (12) does not depend on the source luminosity function.

In practice, there is always a minimum detectable $\Delta_{\text{max}}$, if only because of Poissonian fluctuations in photon counting. Moreover, the irregularity and the discreteness of the time coverage limit the range of event time scales a microlensing search can access.

We can restrict the sum in eq. (12) – and the integral in eq. (10) – to events with $\Delta_{\text{max}} > \Delta$, a convenient detection threshold. We obtain

$$\bar{\tau} = \frac{\pi}{2ITF_{\text{eff}}(\Delta)} \sum_{\Delta_{\text{max}} > \Delta} t_{\text{fwhm}} \Delta_{\text{max}},$$ \hfill (13)

where we have defined an effective flux

$$F_{\text{eff}}(\Delta) = I^{-1} \int dF \phi(F) \frac{1}{\int_0^{\beta_{\Delta/F}} d\beta w(\beta) \delta(\beta)}, \hfill (14)$$
\( \phi(F) \) being the source luminosity function and \( \beta_{\Delta/F} = 2f(2\Delta/F) \). This is the basic form of the new optical depth estimator that we advocate for pixel lensing. Eq. (13) is written for a single source population; for several populations, similar equations apply to each population separately.

In the next section we discuss the dependence of the effective flux \( F_{\text{eff}}(\Delta) \) and of the new optical depth estimator on the source luminosity function.

### 4. EFFECTIVE FLUX AND SENSITIVITY

The effective flux \( F_{\text{eff}}(\Delta) \) is smaller than the total flux \( F_s \) and is equal to \( F_s \) when \( \Delta = 0 \). It depends in principle on the details of the source luminosity function, but in practice the ratio \( F_{\text{eff}}(\Delta)/F_s \) is an almost universal function of the fluctuation magnitude \( m \) (defined in Tonry & Schneider 1988). This is exemplified in fig. 1 for four luminosity functions: (1) the local neighborhood in the V-band (Jahreiß & Wielen 1998); (2) 47 Tucanae in the V-band (Hesser et al. 1987); (3) the M31 bulge in the K-band (Rich, Mould, and Graham 1993); (4) the galactic bulge in the I-band (Terndrup, Frogel and Whitford 1990; Holtzman et al. 1998). The upper panel shows \( \mu_{\text{eff}} - \mu \) as a function of \( m_{\text{thr}} - m \), where \( \mu_{\text{eff}} \) is the surface magnitude corresponding to the effective flux, \( \mu \) is the actual surface magnitude, and \( m_{\text{thr}} \) is the magnitude corresponding to the threshold flux increase \( \Delta \). The lower panel shows the fractional differences in effective flux with respect to the mean over the luminosity functions considered, which is also the relative error in the estimated optical depth. At very low thresholds \( m_{\text{thr}} \gg \bar{m} \), the effective surface magnitude tends to the actual surface magnitude, and this for all luminosity functions. At very high thresholds \( m_{\text{thr}} \ll m_{\text{tip}} \), the magnitude of the brightest star), the difference between effective and actual surface magnitudes is simply \( \mu_{\text{eff}} - \mu = \bar{m} - m_{\text{thr}} - 0.52 \), and this again for all luminosity functions. The largest deviations from a universal behavior occur at detection thresholds close to the fluctuation magnitude, and for the luminosity functions considered here they are never larger than 30%. If the statistical errors in the determination of the optical depth will be smaller than these systematic errors (we recall that the present error on the optical depth towards the LMC is \( \approx 50\% \) [Alcock et al. 1997]), the systematic errors can of course be reduced by measuring the source luminosity function in deep exposures, even independently of the microlensing search itself.

A given microlensing search is only sensitive to a range of Einstein times and radii, and can only measure the contributions to the optical depth coming from this range. Indeed, with a finite time coverage, there is a minimum event duration set by the finite exposure time and the interval between successive images, and a maximum event duration set by the
total time span of the observations. Moreover, for sources with finite angular extensions, there is a reduction in flux increase with respect to the pointlike case when the lens transits the source disk. As a consequence, the fraction of observable events is different according to the Einstein time and radius of each event.

We write the optical depth observable in a given microlensing search as

\[ \tau_{\text{obs}} = \int \pi \theta_e^2 l^2 n(v_T, l) s(t_e, \theta_e) dv_T dl \]  \hspace{1cm} (15)

(compare with eq. [6], where \( n(l) = \int n(v_T, l) dv_T \)). Here \( v_T = l\mu \) is the relative transverse velocity of lens and source.

The function \( s(t_e, \theta_e) \) represents the sensitivity of a given experiment to the different contributions to the optical depth, and depends on the observing conditions and data analysis and on the choice of the optical depth estimator. We call \( s(t_e, \theta_e) \) the optical depth sensitivity. For the usual estimator, \( s(t_e, \theta_e) \) equals the detection efficiency as a function of \( t_e \) and \( \theta_e \). For the new estimator in eq. (13), \( s(t_e, \theta_e) \) is

\[ s(t_e, \theta_e) = \frac{1}{ITF_{\text{eff}}(\Delta)} \int F\phi(F) w(\beta) \delta(\beta) \epsilon_{\text{obs}}(F, \beta, t_e, \theta_e, t_{\text{max}}) dF d\beta dt_{\text{max}}, \]  \hspace{1cm} (16)

where \( \epsilon_{\text{obs}}(F, \beta, t_e, \theta_e, t_{\text{max}}) \) is the event detection efficiency (counting efficiency).

In an actual survey, the optical depth sensitivity, like the efficiency, has to be evaluated in a Monte-Carlo simulation. Events are generated, and two histograms in the binned \((t_e, \theta_e)\) plane are constructed: one counting the events detected by the selection algorithm in the given observing conditions, the other including all events above the reference threshold \( \Delta \). Weighting each event by \( F\phi(F) w(\beta) \delta(\beta) \) and summing over events, the two histograms evaluate the numerator and denominator of eq. (16). The sensitivity is then their bin-by-bin ratio.

To illustrate the dependence of the optical depth sensitivity on the luminosity function, we consider a simple expression for the detection efficiency. We count all events for which the flux increase stays above the threshold \( \Delta \) for a duration longer than \( t_1 \) but shorter than \( t_2 \). In addition, in this example we neglect photometric errors and finite-source effects (so to have a dependency on \( t_e \) only). This translates into \( \epsilon_{\text{obs}} = 1 \) for \( t_1 < 2t_e \sqrt{\beta^2/\Delta_F} - \beta^2 < t_2 \) and \( \epsilon_{\text{obs}} = 0 \) otherwise. We calculate the optical depth sensitivity for the luminosity functions mentioned previously, and plot the results in fig. 2. Each panel corresponds to a value of the threshold magnitude \( m_{\text{thr}} \), and \( t_2/t_1 \) is fixed at 100. Although the details differ from luminosity function to luminosity function, the sensitivity curves are similar, especially considering that with a more realistic expression for the detection efficiency these curves would be smoothed out and that in a comparison with a model they would be folded
with the model $t_e$ distribution and then integrated. Again, a measurement of the luminosity function would reduce even these small systematic uncertainties.

We finally comment on finite-source effects. With sources of finite angular diameter, the flux increase in a microlensing event does not reach the value of the pointlike case whenever the lens is over the disk of the source. The exact light curve depends on the luminosity distribution on the disk of the star, and in general it is a complicated function. A simple account of finite-size effects consists in stopping the flux increase when $u(t) < \theta_s/2\theta_e$, where $\theta_s$ is the angular radius of the source. Given the minimum detectable flux increase $\Delta$, events with $F\delta(\theta_s/\theta_e) < \Delta$ but $F\delta(\beta) > \Delta$ are undetected. The efficiency becomes a function of the Einstein radius $\theta_e$, as does the sensitivity. The optical depth sensitivity as a function of $t_e$ and $\theta_e$ can be calculated from eq. (16) once a relation between stellar radius and luminosity is specified. We have checked that also in this case the behavior of the sensitivity is almost universal.

5. CONCLUSIONS

We have proposed an estimator of the optical depth in pixel microlensing experiments that involves only measurable quantities. Our new estimator is not limited to events with large magnifications at maximum and applies equally well to satellite and ground-based observations.

Our estimator is based on the idea of characterizing the event not by the Einstein time and the maximum magnification but by the full event duration at half-maximum and the flux increase at maximum. The latter are easily determined from a fit to the event lightcurve.

The new estimator of the microlensing optical depth is an efficiency-weighted sum over events of the products of the fractional flux increase at maximum and of the fraction of observing time spent above half-maximum.

In the ideal case of no noise and complete time coverage, our proposed optical depth estimator does not depend on the source luminosity function. In a real situation, an experiment is only sensitive to events above a threshold and to a range of Einstein times and radii. We have shown that the sensitivity and the effective flux depend on the luminosity function in an almost universal way through the surface brightness fluctuation magnitude. Residual uncertainties can be removed, if needed, by measuring the luminosity function in deep exposures.
Our new estimator applies not only to pixel microlensing data but also to blended and individually monitored sources. In the latter case, it could be interestingly compared with the standard estimator based on Einstein times. If agreement is found, the optical depth estimator we propose may indeed open the way to the determination of physical parameters from pixel microlensing data.

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REFERENCES

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Fig. 1.— Dependence of the effective surface magnitude $\mu_{\text{eff}}$ on the threshold magnitude $m_{\text{thr}}$ for various luminosity functions. $\mu$ is the actual surface magnitude and $\bar{m}$ is the fluctuation magnitude. The lower panel shows the relative systematic error in the effective flux with respect to the mean of the luminosity functions considered.

Fig. 2.— Optical depth sensitivity as a function of Einstein time $t_e$ for a simple detection efficiency and various luminosity functions. Each panel corresponds to a different threshold magnitude $m_{\text{thr}}$. 
LUMINOSITY FUNCTIONS

- local [V band]
- 47 Tuc [V band]
- M31 bulge [K band]
- galactic bulge [I band]
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