Is the number of Photons a Classical Invariant?

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Abstract

We describe a paradox in classical electrodynamics, and its two resolutions. The paradox is concerned with the Lorentz invariance of the classical analog of the number of photons.
1 Introduction

Photon are quantum objects and a-priori have no business in classical electrodynamics. So, what can one possibly mean by: Is the number of Photons a Classical Invariant?

Consider a box filled with monochromatic radiation of frequency $\omega$. If $U$ denotes the total electromagnetic energy in the box, then, the right hand side of

$$\bar{h}N = \frac{U}{\omega}$$

(1)

is a purely classical quantity. The left hand side gives the interpretation and quantization of this quantity, namely, that it counts the number of photons, $N$, in units of $\bar{h}$. Ehrenfest [3], in the early days of quantum mechanics, stressed the significance of classical quantities associated with quantum numbers, i.e. the right hand side of Eq. (1). According to Ehrenfest such quantities have distinguished invariance properties.

Let us recall how this applies to the classical harmonic oscillator. The ratio of energy to frequency of an oscillator is a classical quantity whose importance in quantum mechanics comes from the fact that it is a function of the quantum number:

$$\bar{h} \left( n + \frac{1}{2} \right) = \frac{U}{\omega}$$

(2)

But, as Ehrenfest stressed, the ratio of energy to frequency is the classical adiabatic invariant for the Harmonic oscillator [1]. According to Ehrenfest, therefore, the classical quantity on the right hand side of Eq. (1) should also be distinguished by its invariance properties. The simplest of these is Lorentz invariant. Since neither the energy nor the frequency are Lorentz invariants, the Lorentz invariance of the ratio is not obvious. If, indeed, the ratio is Lorentz invariant then one can discover, and motivate, photons on classical grounds. This approach has its limitations, of course. One still needs quantum mechanics to understand quantization, and $\bar{h}$ to actually count photons.

2 The Paradox

Here is a calculation of how Eq. (1) Lorentz transforms in a simple example. In this example $\frac{U}{\omega}$ turns out not to be Lorentz invariant.
Consider a linearly polarized, plane monochromatic wave of frequency $\omega$ traveling in the $\hat{x}$ direction. The electric and magnetic fields are:

$$\mathbf{E} = E_0 \cos(kx - \omega t) \hat{y}, \quad \mathbf{B} = E_0 \cos(kx - \omega t) \hat{z}. \quad (3)$$

The electromagnetic energy density is

$$\frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) = \frac{E_0^2}{4\pi} \cos^2(kx - \omega t). \quad (4)$$

Consider a fictitious rectangular box of proper length $L$, aligned with the $x$ axis, whose cross section is $A$. Suppose that the length of the box is much larger than the wave length of the radiation. The total energy in the box is

$$U = \frac{AE_0^2}{4\pi} \int_0^L dx \cos^2(kx - \omega t) \approx \frac{AL E_0^2}{8\pi}. \quad (5)$$

The number of photons in this box, according to Eq. (1), appears to be

$$\hbar N = \frac{U}{\omega} = \frac{E_0^2 AL}{8\pi \omega}. \quad (6)$$

Now, let us compute the number of photons, $N'$, in the same box, but as viewed in a frame, $S'$, moving with velocity $v$ along the $x$ axis. In $S'$, the electric field amplitude is [4]:

$$E'_0 = \frac{E_y - (v/c)B_z}{\sqrt{1 - (v/c)^2}} = \frac{E_0 - (v/c)E_0}{\sqrt{1 - (v/c)^2}} = E_0 \sqrt{\frac{1 - v/c}{1 + v/c}}. \quad (7)$$

The box’s length experiences Lorentz contraction and is now:

$$L' = L \sqrt{1 - (v/c)^2} \quad (8)$$

The electromagnetic energy in the box in the moving frame is therefore

$$U' \approx \frac{(E'_0)^2}{8\pi} AL' = \frac{E_0^2}{8\pi} \frac{1 - v/c}{1 + v/c} AL \sqrt{1 - (v/c)^2}. \quad (9)$$

$\omega$ is transformed according to the Doppler formula [2]:

$$\omega' = \omega \sqrt{\frac{1 - v/c}{1 + v/c}}. \quad (10)$$

Hence the number of photons in the moving box appears to be:

$$\hbar N' = \frac{U'}{\omega'} \approx \frac{E_0^2 AL}{8\pi \omega} (1 - v/c) \approx \hbar N(1 - v/c), \quad (11)$$

which is manifestly not Lorentz invariant.
Figure 1. Space time diagram. Each photon is represented by a dotted line (denoted by p1-p3). The solid lines (a) and (b) represent the box as viewed at $t = 0$, $t' = 0$ from the two frames $S$ and $S'$. The number of intersections between the photon world lines and the box gives the total photons inside the box. It is seen that p2 and p3 are not counted in $S'$, and therefore there will be more photons counted in $S$.

Figure 1 gives a geometric description of this result and illustrates in a direct way why different photon numbers seem to appear in different frames.

3 What Went Wrong?

What, if anything, went wrong? One easy way out is to refuse to admit that there is a problem. A way to do that is to say that Photons can only be correctly discussed in a quantum context, and the classical point of view of Ehrenfest is, anyway, of only historical interest. To correctly compute the number of photons one has to construct the quantum fields and compute the photon number in the framework of quantum field theory. This is, of course, nothing but a copout.

The origin of the paradox is not computational or quantum mechanical but conceptual. It all has to do with what is the correct energy $U$ to put in the Eq. (1). Let us analyze this in some detail.

Eq. (1) must be viewed as a formula that gives the number of photons in a field configuration in a given time. A field configuration is, of course, extended in space. The field configuration associated with a plane wave is problematic because the total electromagnetic energy is infinite, and so is the total number of photons. The energy in a box is finite, however. But, the box we picked is a virtual box: A box that lets light escape and enter. So what we learn is that one can not take a part of a field configuration and chop it more or less arbitrarily and still hope that Eq. (1) will correctly count the number of photons. The equation comes with the proviso that the energy
is the total electromagnetic energy of a field configuration. To make a field configuration with finite energy \(^1\) one can confine the electromagnetic field to an ideal, but still real box. This means a box with reflecting (that is the real part) and lossless (that is the ideal part) walls. The field configuration we have picked does not have this properties.

A second way to resolve the paradox is to think about Eq. (1) differently, namely, to think of \(U\) as the energy absorbed by a photo-detector. In this case, the energy \(U\) is associated with the energy flux swept by a photo-detector while it is operating, see fig.2. The relevant box is now not a box in space but a box in time.

![Figure 2. The square plate represents the photodetector, and the dots represent photons.](image)

Advantage of a detector is that one can apply Eq. (1) also to field configurations, like plane waves, with infinite energy.

Since simultaneity is not a Lorentz invariant concept extended objects are a pain in special relativity and a source of many paradoxes. Therefore, a good photodetector must be a small, and ideally, point-like object.

### 4 Photons in a Box

Photons confined to a box correspond classically to a standing wave. A standing wave is a superposition of two monochromatic waves of equal frequency and amplitude, traveling in opposite directions.

Let \(N_\rightarrow\) and \(N_\leftarrow\) denote the number of right and left traveling photons, respectively. In the box’s rest frame, these numbers are equal, and we will denote them by \(N/2\). In the moving frame, the numbers transform according to (11):

\[
N'_\rightarrow = \frac{N}{2} \left( 1 - \frac{v}{c} \right)
\]

\[
N'_\leftarrow = \frac{N}{2} \left( 1 + \frac{v}{c} \right)
\]

And cheerfully, we find \(N = N'\) and therefore invariant. So, although the number of right and left movers are not Lorentz invariant, their sum is.

\(^1\)and well defined frequency
This is good news, because there are no additional quantum numbers in this problem besides the total number of photons.

Although this calculation gives the desired result, it is a cheat: Generally, electromagnetic energies do not add linearly. However, in this case the total energy can be decomposed into two contributions due to the left and right traveling radiation. Let \( E_\rightarrow = \hat{y}E_\rightarrow(x, t) \) and \( B_\rightarrow = \hat{z}E_\rightarrow(x, t) \) denote the electric and magnetic fields of the right going wave, respectively. Analogously, the fields of the left going wave are \( E_\leftarrow = \hat{y}E_\leftarrow(x, t) \) and \( B_\leftarrow = -\hat{z}E_\leftarrow(x, t) \). The sign of \( B_\leftarrow \) is negative because the direction of motion is reversed. The energy density is:

\[
\frac{\left(E_\rightarrow + E_\leftarrow\right)^2 + \left(B_\rightarrow + B_\leftarrow\right)^2}{8\pi} = \frac{\left(E_\rightarrow(x, t) + E_\leftarrow(x, t)\right)^2 + \left(E_\rightarrow(x, t) - E_\leftarrow(x, t)\right)^2}{8\pi} = \frac{2E_\rightarrow^2(x, t) + 2E_\leftarrow^2(x, t)}{8\pi} \tag{13}
\]

We see that the cross terms cancel, and the energies of the two waves indeed add linearly. Note that this result is true regardless of the reference frame, since we did not assume any relation between \( E_\rightarrow(x, t) \) and \( E_\leftarrow(x, t) \).

Another way of solving the problem is shown in fig. 3.

![Figure 3. Photons in a closed optical fiber. Here, unlike in the box, photons going in opposite directions don’t interfere, and the energies of the right and left movers are clearly add.](image)

### 5 Photo-Detector

A different approach to counting photons in a Lorentz invariant way is to replace the box by a photodetector. Consider a monochromatic plane wave passing through a thin photon detector whose surface is perpendicular to the \( x \) axis, as can be seen in fig. 2. We will find the number of photons passing through the detector during a given proper time \( t \), assuming that the photons are point particles.
In the rest frame of the detector, the total energy received by the detector during the time $\tau$ is

$$U = \frac{E_0^2}{8\pi} A \tau$$  \hspace{1cm} (14)

Where $A$ is the detector’s surface area. This yields:

$$\hbar N = \frac{E_0^2 A \tau}{8\pi \omega}$$  \hspace{1cm} (15)

for the number of detected photons.

In a moving frame the field intensity and frequency transform according to (2) and (4) respectively. The measurement time experiences time dilation:

$$t' = \frac{\tau}{\sqrt{1 - (v/c)^2}}$$  \hspace{1cm} (16)

What volume will the detector sweep during $t'$? The detector moves towards the photons a distance of $vt'$, while each photon, treated as a point particle, travels towards the detector a distance of $ct'$. Therefore, the last photon to meet the detector at time $t'$ is exactly $vt' + ct'$ far from the detector at $t = 0$. The volume swept by the detector is $A(v + c)t'$. Now we can find $N'$:

$$\hbar N' = \frac{(E_0')^2}{8\pi} A(c + v) \frac{t'}{\omega'} =$$

$$= \frac{E_0^2}{8\pi} \frac{1 - v/c}{1 + v/c} A(c + v) \frac{\tau}{\sqrt{1 - (v/c)^2}} \frac{1}{\omega} \sqrt{1 + v/c} \frac{1}{1 - v/c}$$

$$= A \frac{E_0^2 c \tau}{8\pi \omega'} = \hbar N$$  \hspace{1cm} (17)

The number of photons seen by the two detectors is Lorentz invariant.

6 Epilogue

This is an account of a paradox and its resolution. It grew out of teacher-students interaction in the spring semester class of classical electrodynamics at the Technion. Puzzles and paradoxes are effective means to teach and learn especially when the teacher does not already know the resolution.

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References


