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ON THE PROPERTY OF BEING A BASIS
FOR A DENUMERABLE SET OF SOLUTIONS
OF A NONLINEAR SCHRÖDINGER-TYPE
BOUNDARY-VALUE PROBLEM

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1 Introduction. Definitions. Results

The boundary-value problem we consider is the following

$$u'' = f(u^2)u, \quad u = u(x), \quad x \in (0, 1), \quad (1)$$

$$u(0) = u(1) = 0. \quad (2)$$

Here all quantities are real, f is a given sufficiently smooth function and $u(x)$ is an unknown function of the argument $x \in [0, 1]$. We consider solutions of the problem (1)-(2) continuous on the segment $[0, 1]$ and twice continuously differentiable in the interval $(0, 1)$. The function $u(x) \equiv 0$ obviously satisfies the problem (1)-(2) and we look for nontrivial solutions of this problem. Among others, we assume that

$$f(0) \geq 0 \quad \text{and} \quad \lim_{|u| \rightarrow \infty} f(u^2) = -\infty. \quad (3)$$

It is known (see, for example, [1]) that the condition (3) provides the existence of a denumerable set $\{u_n\}_{n=0,1,2,\dots}$ of solutions of the problem (1)-(2) where for each integer $n \geq 0$ the corresponding solution $u_n(x)$ has precisely n roots in the interval $(0, 1)$. In the present paper, we are interested in the natural question, which is originated from the similarity in the qualitative behavior of functions from the system $\{u_n\}_{n=0,1,2,\dots}$ and the functions $e_n(x) = \sqrt{2} \sin \pi(n+1)x$ ($n = 0, 1, 2, \dots$) (the latter functions obviously form bases in standard spaces like $L_2(0, 1)$), if the system of functions $\{u_n\}_{n=0,1,2,\dots}$ is a basis of a functional space containing "arbitrary functions".

The author knows several papers devoted to investigations of the completeness and related properties for systems of solutions (or eigenfunctions) of nonlinear differential equations. In [2] results in this direction are announced (without proofs) for a nonlinear problem which arises from a linear one under small nonlinear perturbations. In our paper [3], the property of being a basis in the space L_2 for the system of eigenfunctions of a

nonlinear Sturm-Liouville-type (or Schrödinger-type) eigenvalue problem (considered in a finite interval) is proved. In fact, this paper contains some mistakes which are corrected in the note [4]. In [5] we present an independent and shorter proof of the above-indicated result from [3,4]. Also, in [6] the results of paper [3] are reestablished without proofs. Our approach in [5] is based on a theorem of N.K. Bary [7-9] stating that a system of functions from the space L_2 minimal (or linearly independent) and quadratically close to a Riesz basis is a Riesz basis. In [3] we exploit other methods the main idea of which consists in a reduction of the nonlinear eigenvalue problem under consideration to a linear Sturm-Liouville eigenvalue problem with a potential depending on the spectral parameter.

Let us introduce some notation. Let $L_2(a, b)$, where $a < b$, be the usual Lebesgue space consisting of real-valued functions of the argument $x \in (a, b)$, square integrable over the interval (a, b) , with the scalar product $(g, h)_{L_2(a, b)} = \int_a^b g(x)h(x)dx$ and the norm $\|g\|_{L_2(a, b)} = (g, g)_{L_2(a, b)}^{\frac{1}{2}}$. By Δ we denote the closure in the space $L_2(a, b)$ of the operator $-\frac{d^2}{dx^2}$ with the domain $C_0^\infty(a, b)$ which consists of real-valued infinitely differentiable functions of the argument $x \in [a, b]$ becoming zero at the ends of this segment. Then, it is well-known that Δ is a self-adjoint positive operator in the space $L_2(a, b)$ which also has a bounded inverse operator Δ^{-1} . For an arbitrary $s \leq 0$, let $H^s(a, b)$ be the completion of the space $L_2(a, b)$ equipped with the scalar product $(g, h)_{H^s(a, b)} = (\Delta^{\frac{s}{2}}g, \Delta^{\frac{s}{2}}h)_{L_2(a, b)}$ and the norm $\|g\|_{H^s(a, b)} = (g, g)_{H^s(a, b)}^{\frac{1}{2}}$. Then, it is clear that $H^s(a, b)$ is a Hilbert space for each $s \leq 0$ and that $H^0(a, b) = L_2(a, b)$. Also, for a Banach space B with a norm $\|\cdot\|_B$, by $\mathcal{L}(B; B)$ we denote the space of linear bounded operators acting from B into B , with the norm

$$\|A\|_{\mathcal{L}(B;B)} = \sup_{\substack{x \in B \\ \|x\|_B = 1}} \|Ax\|_B.$$

Before formulating our results, we introduce some definitions for the convenience of readers. Let B be a real Banach space.

Definition 1. A system $\{g_n\}_{n=0,1,2,\dots} \subset B$ is called *complete* in the space B if and only if the set of all linear combinations $\sum_{n=0}^N a_n g_n$ taken for all integer $N > 0$ and for all real coefficients a_n is dense in the space B . A system $\{g_n\}_{n=0,1,2,\dots} \subset B$ is called *incomplete* in the space B if and only if it is not complete in this space.

Definition 2. A system $\{g_n\}_{n=0,1,2,\dots} \subset B$ is called *linearly independent* in the space B if and only if the equality $\sum_{n=0}^{\infty} a_n g_n = 0$, where a_n are real coefficients and the convergence of the infinite sum is understood in the sense of the space B , is possible only in the case $a_n = 0$ for all numbers n .

Remark 1. Our definition of linearly independent systems in a Banach space is not standard and is taken in view of its convenience for our purposes. Sometimes the linear independence from Definition 2 is called the ω -linear independence.

Definition 3. A system $\{g_n\}_{n=0,1,2,\dots} \subset B$ is called a *basis* of the space B if and only if for an arbitrary point $g \in B$ there exists a unique sequence of real coefficients a_n ($n = 0, 1, 2, \dots$) such that $g = \sum_{n=0}^{\infty} a_n g_n$ in the sense of the space B .

In accordance with papers [7,8] we introduce the following

Definition 4. A basis $\{g_n\}_{n=0,1,2,\dots}$ of a Hilbert space H is

called the *Riesz basis* of this space iff the series $\sum_{n=0}^{\infty} a_n g_n$ with real coefficients a_n converges in the space H if and only if $\sum_{n=0}^{\infty} a_n^2 < \infty$. In fact, in [7,8] this definition is given for $H = L_2(a, b)$.

Now, we establish our results. First, we need the following

Theorem 1. (a) Let $f(u^2)u$ be a real-valued continuously differentiable function of the argument $u \in \mathbb{R}$ and let the function $f(r)$ of the argument $r \in [0, +\infty)$ be continuous and satisfy the condition (3). Then, there exists a denumerable set $\{u_n\}_{n=0,1,2,\dots}$ of solutions of the problem (1)-(2) such that for any integer $n \geq 0$ the solution $u_n(x)$ has precisely n roots in the interval $(0, 1)$;

(b) if in addition to the assumptions from the statement (a) the function $f(r)$ is nonincreasing on the half-line $r \geq 0$, then for any integer $n \geq 0$ the solution of the problem (1)-(2) possessing precisely n roots in the interval $(0, 1)$ is unique up to the coefficient ± 1 .

Let us consider the function $\xi(s) = \sum_{l=1}^{\infty} (2l+1)^{s-1}$ of the argument $s < 0$. Obviously, since

$$\xi(-1) \leq \int_{\frac{1}{2}}^{\infty} (2x+1)^{-2} dx = \frac{1}{4}$$

(this estimate takes place because $(2x+1)^{-2}$ is a convex function of the argument $x > 0$), the equation

$$2\sqrt{6}\pi^{-1}[1 - 24\pi^{-2}\xi(-1)]^{-\frac{1}{2}}\xi(s) = 1$$

with an unknown s has a unique negative root. We denote this root by s_0 .

Our main result is the following.

Theorem 2. *Under the assumptions of Theorem 1(a) for any $s < s_0$ and for an arbitrary system of solutions $\{u_n\}_{n=0,1,2,\dots}$ of the problem (1)-(2) given by Theorem 1(a) the system of functions $\{k_n u_n\}_{n=0,1,2,\dots}$ where $k_n = \|u_n\|_{H^s(0,1)}^{-1}$, is a Riesz basis of the space $H^s(0,1)$.*

2 Proof of Theorem 1

Of course, Eq. (1) can be solved by quadratures. However, we believe that the qualitative analysis we use for proving Theorem 1 is simpler. The statement (a) of Theorem 1 is generally well-known (for example, it follows from theorem 1 of paper [1]). Let us prove the statement (b).

Let us consider the following Cauchy problem

$$u'' = f(u^2)u, \quad u = u(x), \quad x \in R, \quad (4)$$

$$u(0) = 0, \quad u'(0) = p \quad (5)$$

where p is a real parameter. Due to our assumptions, for any fixed value of the parameter p the usual local existence, uniqueness and continuous dependence theorems are valid for the problem (4)-(5). Further, one can easily verify that any solution $u(x)$ of the problem (1)-(2) continuous on the segment $[0, 1]$ and twice continuously differentiable in the interval $(0, 1)$ in view of Eq. (1) has first and second derivatives on the right and on the left, respectively, at the points $x = 0$ and $x = 1$. Therefore, an arbitrary solution of the problem (1)-(2) satisfies the problem (4)-(5) with some value of the parameter p . Also, if $u'(0) = 0$ for a solution $u(x)$ of the problem (1)-(2), then $u(x) \equiv 0$ by the uniqueness theorem. So, $u'(0) \neq 0$ for an arbitrary nontrivial solution of the problem (1)-(2). Hence, since Eq. (1) (or (4)) is invariant with respect to the multiplication of the solution $u(x)$ by -1 , we

get, up to the coefficient ± 1 , all solutions of the problem (1)-(2) considering solutions of the problem (4)-(5) when the parameter p runs over the half-line $(0, +\infty)$ and choosing those solutions which become zero at the point $x = 1$.

Let us take an arbitrary $p > 0$. One can easily verify that for the solution $u(x)$ of the problem (4)-(5) the following identity takes place

$$\frac{d}{dx} \{ [u'(x)]^2 - U(u^2(x)) \} = 0$$

where $U(z) = \int_0^z f(t)dt$. Hence, one has

$$p^2 = [u'(x)]^2 - U(u^2(x)) \quad (6)$$

for all values x for which the solution $u(x)$ exists. Since in view of the condition (3) the function $U(r^2)$ of the argument r is bounded from above and $\lim_{r \rightarrow \infty} U(r^2) = -\infty$, the identity (6) implies the boundedness of the functions $u(x)$ and $u'(x)$ in the whole interval of existence of the solution $u(x)$. Also, in view of Eq. (4), the second derivative $u''(x)$ is bounded, too. These facts immediately yield the global solvability of the problem (4)-(5). Indeed, let us suppose that there exists $a > 0$ such that the solution $u(x)$ of the problem (4)-(5) can be continued onto the half-interval $[0, a)$ and cannot be continued on an arbitrary right half-neighborhood of the point $x = a$. Then, we set

$$q = \int_0^a u'(x)dx \quad \text{and} \quad q' = p + \int_0^a u''(x)dx$$

and, considering the Cauchy problem for Eq. (4) with the initial data $u(a) = q$, $u'(a) = q'$, immediately get that our solution $u(x)$ can be continued onto a right half-neighborhood of the point $x = a$, i. e. we arrive at the contradiction. By analogy, the solution $u(x)$ of the problem (4)-(5) can be continued onto the whole half-line $x < 0$.

So, for any $p > 0$ the corresponding solution $u(x)$ of the problem (4)-(5) is global (can be continued onto the whole real line). In what follows, under solutions of the problem (1)-(2) (or (4)-(5)) we mean maximal solutions defined for all $x \in R$ and satisfying boundary conditions (2) (resp., the initial condition (5)) and Eq. (1) (resp., Eq. (4)) for all $x \in R$.

We need some properties of solutions $u_n(x)$ of the problem (1)-(2) given by Theorem 1(a). We establish them with the following

Proposition. *Let the hypotheses of Theorem 1(a) be valid and let n be an arbitrary nonnegative integer. Then, for an arbitrary solution $u_n(x)$ of the problem (1)-(2) possessing precisely n roots in the interval $(0, 1)$ the following properties take place:*

- 1) *the roots of the solution $u_n(x)$ are precisely the points $\frac{k}{n+1}$ where k runs over all integers;*
- 2) *between any two nearest roots $\frac{k}{n+1}$ and $\frac{k+1}{n+1}$ of the solution $u_n(x)$ this function has the unique point of extremum $\frac{2k+1}{2(n+1)}$, $u'_n(x) \neq 0$ in the interval $\left(\frac{2k-1}{2(n+1)}, \frac{2k+1}{2(n+1)}\right)$ and $u_n\left(\frac{2k+1}{2(n+1)} + x\right) = u_n\left(\frac{2k+1}{2(n+1)} - x\right)$ for any $x \in R$ (here $k = 0, \pm 1, \pm 2, \dots$);*
- 3) *$u_n\left(x + \frac{1}{n+1}\right) = -u_n(x)$ for any $x \in R$.*

Proof of Proposition. Let us fix an arbitrary integer $n \geq 0$ and consider a solution $u_n(x)$ of the problem (1)-(2) possessing precisely n roots in the interval $(0, 1)$. As earlier, without loss of generality we can accept that $u'_n(0) > 0$. Then, the function $u_n(x)$ satisfies the problem (4)-(5) taken with $p = u'_n(0) > 0$. Further, since $p > 0$, there exists a neighborhood of the point $x = 0$ in which the function $u_n(x)$ strictly increases. Then, since $u_n(1) = 0$, there exists a point $d \in (0, 1)$ such that $u'_n(d) = 0$ and $u'_n(x) > 0$ for any $x \in [0, d)$. Due to the uniqueness theorem, the autonomy of Eq. (4) and its invariance with respect to the changes of variables $x + a \rightarrow a - x$ and $u(x) \rightarrow -u(x)$ (where a

is an arbitrary real constant), we get that $u_n(d+x) = u_n(d-x)$, $u_n(2d) = 0$, $u_n(2d+x) = -u_n(x)$ and $u_n(2d-x) = -u_n(x)$ for any $x \in R$. Hence, $d = \frac{1}{2(n+1)}$, and Proposition is proved. $\square$

Clearly, to prove the statement (b) of Theorem 1, it suffices to prove that there are no an integer $n \geq 0$ and two real values $p_v, p_w : 0 < p_v < p_w$ of the parameter p such that each of the corresponding solutions $v(x)$ and $w(x)$ of the problem (4)-(5) taken respectively with $p = p_v$ and $p = p_w$ has precisely n roots in the interval $(0, 1)$ and becomes zero at the point $x = 1$. Let us suppose that this is not the case, and such numbers n, p_v, p_w exist. By Proposition, $v'(x) > 0$ and $w'(x) > 0$ in the half-interval $\left[0, \frac{1}{2(n+1)}\right)$ and $v'\left(\frac{1}{2(n+1)}\right) = w'\left(\frac{1}{2(n+1)}\right) = 0$. Further, by the identity (6), we have $0 < v'(x_1) < w'(x_2)$ for any $x_1, x_2 \in \left(0, \frac{1}{2(n+1)}\right)$ such that $v(x_1) = w(x_2)$. Hence, $0 < v(x) < w(x)$ and, consequently, $f(v^2(x)) \geq f(w^2(x))$ for any $x \in \left(0, \frac{1}{2(n+1)}\right)$ (because otherwise there exists $d \in \left(0, \frac{1}{2(n+1)}\right)$ such that $0 < v(x) < w(x)$ for $x \in (0, d)$ and $v(d) = w(d)$, therefore $v'(d) \geq w'(d)$). Also, one can easily observe that there exists an interval $(c, d) \subset \left(0, \frac{1}{2(n+1)}\right)$ such that $f(v^2(x)) > f(w^2(x))$ for all $x \in (c, d)$. Indeed, this follows from the continuity of the function $f(r^2)$ for $r > 0$ and the fact that $\frac{d}{dr}f(r^2) < 0$ for some $r \in \left(0, f\left(w^2\left(\frac{1}{2(n+1)}\right)\right)\right)$ (otherwise $f\left(w^2\left(\frac{1}{2(n+1)}\right)\right) = f(0) \geq 0$ in the contradiction with the maximum principle). Let us multiply Eq. (4) written for $u(x) = v(x)$ by $w(x)$, the same equation written for $u(x) = w(x)$ by $v(x)$, subtract these identities from each other and integrate the obtained equality between 0 and $\frac{1}{2(n+1)}$. Then, we get

$$\left[v'(x)w(x) - v(x)w'(x)\right]\Bigg|_0^{\frac{1}{2(n+1)}} =$$

$$= \int_0^{\frac{1}{2(n+1)}} v(x)w(x)[f(v^2(x)) - f(w^2(x))]dx.$$

But here the left-hand side is equal to zero and the right-hand side is positive. Therefore, we get the contradiction. Thus, Theorem 1 is proved. $\square$

3 Proof of Theorem 2

Let $\{u_n\}_{n=0,1,2,\dots}$ be a system of solutions of the problem (1)-(2) given by Theorem 1(a). As in Section 2, without loss of generality we accept that $u'_n(0) > 0$ for all $n = 0, 1, 2, \dots$. Further, let $v_n(x) = \frac{u_n(x)}{\|u_n\|_{L_2(0,1)}}$ ($n = 0, 1, 2, \dots$). For an arbitrary integer $n \geq 0$ let us consider the function $w_n(x) = v_n\left(\frac{x}{n+1}\right)$. Due to Proposition, w_n is a continuous function positive in the interval $(0, 1)$. Therefore,

$$w_n(x) = \sum_{k=0}^{\infty} a_k^n e_k(x)$$

in the space $L_2(0, 1)$ where a_k^n are real numbers and $a_0^n = (w_n, c_0)_{L_2(0,1)} > 0$. Hence,

$$v_n(x) = \sum_{k=0}^{\infty} b_k^n e_k(x) \tag{7}$$

where $b_{(k+1)(n+1)-1}^n = a_k^n$ and $b_m^n = 0$ if $m \neq (k+1)(n+1) - 1$ for all integer $k \geq 0$ (here, of course, the Fourier series converges in the sense of the space $L_2\left(0, \frac{1}{n+1}\right)$). Then, since by Proposition $v_n\left(\frac{m}{n+1} + x\right) = -v_n\left(\frac{m}{n+1} - x\right)$ and since the direct verification shows that $c_{(k+1)(n+1)-1}\left(\frac{m}{n+1} + x\right) = -c_{(k+1)(n+1)-1}\left(\frac{m}{n+1} - x\right)$ for any integer m and for any $x \in R$, the equality (7) also holds in the sense of each of the spaces $L_2\left(\frac{m-1}{n+1}, \frac{m}{n+1}\right)$ where $m = 2, n+1$.

Hence, the equality (7) is valid in the sense of the space $L_2(0, 1)$.

Remark 2. We obviously have $b_n^n = a_0^n > 0$ for each number n . Therefore, taking into account the facts that the matrix $(b_k^n)_{n,k=0,1,2,\dots}$ is upper triangular and that all elements of its principal diagonal are nonzero, one may think that the system of functions $\{v_n\}_{n=0,1,2,\dots}$ is always complete (for example, in the space $L_2(0, 1)$). Here, we demonstrate with the following simple example that it is not so and generally the system of functions $\{v_n\}_{n=0,1,2,\dots}$ can be incomplete.

Example. Let H be a separable real Hilbert space with a scalar product $(\cdot, \cdot)_H$ and the corresponding norm $\|\cdot\|_H$ and let $\{\bar{e}_n\}_{n=0,1,2,\dots}$ be an orthonormal basis in this space. Let $\{\bar{v}_n\}_{n=0,1,2,\dots}$ be a sequence of elements of the space H normalized to 1 and such that the expansions (7) take place with $c_n = \bar{e}_n$, $v_n = \bar{v}_n$ ($n = 0, 1, 2, \dots$), $b_n^n = b \in (0, 1)$, $b_{n+1}^n = -\sqrt{1-b^2}$ and $b_k^n = 0$ for all other values of n and k (here b is a constant independent of n and k). Let also $B = \frac{\sqrt{1-b^2}}{b}$. We choose $b \in (0, 1)$ to satisfy the condition $B \geq 3$ and want to show that the system of functions $\{\bar{v}_n\}_{n=0,1,2,\dots}$ is incomplete in the space H . For this aim, it suffices to prove that there exists $c > 0$ such that $\|\alpha_m\|_H \geq c$ for all $m = 1, 2, 3, \dots$, all positive integers N_m and all real coefficients c_n^m ($n = 0, 1, \dots, N_m$), where $\alpha_m = \sum_{n=0}^{N_m} c_n^m \bar{v}_n - \bar{e}_0$. Let us suppose that this is not right and there exist sequences of positive integers N_m and of real numbers c_n^m , where $m = 1, 2, 3, \dots$ and $n = \overline{0, N_m}$, such that $\|\alpha_m\|_H \rightarrow 0$ as $m \rightarrow \infty$. Setting $\gamma_{m,n} = (\alpha_m, \bar{e}_n)_H$, we easily derive (by multiplication the expressions for α_m by elements $\bar{e}_n$ in H) that

$$c_0^m = b^{-1}(1 + \gamma_{m,0}) \quad \text{and} \quad c_n^m = Bc_{n-1}^m + b^{-1}\gamma_{m,n} \quad \text{for } n = \overline{1, N_m}.$$

Also, since $\alpha_m \rightarrow 0$ in H as $m \rightarrow \infty$, we have that $\sum_n \gamma_{m,n}^2 \rightarrow +0$ as $m \rightarrow \infty$. Hence, $c_0^m \rightarrow b^{-1} > 1$ as $m \rightarrow \infty$ and, using the

induction and the facts that $B \geq 3$ and $|\gamma_{m,n}| \leq b$ for all sufficiently large numbers m and for all n , we get that $|c_n^m| \geq 1$ for all sufficiently large numbers m and for all $n = \overline{1, N_m}$. But then $\|\alpha_m\|_H^2 \geq (c_{N_m}^m)^2(1 - b^2) \geq 1 - b^2 > 0$ for all sufficiently large numbers m , and we get the contradiction. $\square$

Let $A_n = \max_x u_n(x)$ and $B_n = \max_x v_n(x)$. In view of Proposition $A_n = u_n\left(\frac{1}{2(n+1)}\right)$ and $B_n = v_n\left(\frac{1}{2(n+1)}\right)$.

Lemma 1. *For any $\epsilon > 0$ there exists a number $N_0 > 0$ such that $B_n \leq \sqrt{3} + \epsilon$ for all numbers $n \geq N_0$.*

Proof. By the usual comparison theorem we get that $A_n \rightarrow +\infty$ as $n \rightarrow \infty$. Let us take an arbitrary $\gamma \in (0, 1)$ and let $D_\gamma^n = \{x \in R : u_n(x) \in [0, \gamma A_n]\}$. Let us prove that

$$A_n^{-1} \min_{x \in D_\gamma^n} |u'_n(x)| \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \quad (8)$$

For this aim, let us consider the identity (6) written for $u(x) = u_n(x)$. Then, we get that $u'_n(0) = [-U(A_n^2)]^{\frac{1}{2}}$. Also, in view of the condition (3) $A_n = o([-U(A_n^2)]^{\frac{1}{2}})$ as $n \rightarrow \infty$. Further, for any $x \in R$ we have $[u'_n(x)]^2 = [u'_n(0)]^2 + U(u_n^2(x))$. Hence, since in view of the condition (3) $U(u_n^2(x)) \geq U(\gamma^2 A_n^2)$ for all $x \in D_\gamma^n$ and for all sufficiently large numbers n and since by (3) $\lim_{n \rightarrow \infty} \frac{-U(A_n^2) + U(\gamma^2 A_n^2)}{A_n^2} = +\infty$, we get (8).

Clearly, $\|u_n\|_{L_2(0,1)} \leq A_n$ for all numbers n . Also, solutions of Eq. (1) become concave in the domain $u > c$ for a constant $c > 0$. For an arbitrary number $n \geq 0$ let us consider the linear function $l_n(x) = \frac{2(n+1)(A_n B_n - c)}{A_n(1 - 2p_n(n+1))}(x - p_n) + \frac{c}{A_n}$ where $p_n \in \left(0, \frac{1}{2(n+1)}\right)$ is a point such that $v_n(p_n) = \frac{c}{A_n}$ (we recall that, according to Proposition, the function $v_n(x)$ monotonously increases in the interval $\left(0, \frac{1}{2(n+1)}\right)$). This function coincides with the function $v_n(x)$ at the points $x = p_n$ and $x = \frac{1}{2(n+1)}$ and $l_n(x) \leq v_n(x)$ for all $x \in \left(p_n, \frac{1}{2(n+1)}\right)$ in view of the concavity of the function $v_n(x)$

in the interval $x \in \left(p_n, \frac{1}{2(n+1)}\right)$. Also, $\lim_{n \rightarrow \infty} p_n = 0$ by (8). Let us take an arbitrary $\epsilon > 0$. Then, in view of the above arguments and Proposition, we get

$$\begin{aligned} 1 &= \|v_n\|_{L_2(0,1)}^2 = 2(n+1) \int_0^{\frac{1}{2(n+1)}} v_n^2(x) dx \geq \\ &\geq 2(n+1) \int_{p_n}^{\frac{1}{2(n+1)}} l_n^2(x) dx = \frac{B_n^2}{3} + \delta_n \end{aligned}$$

where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $0 < B_n \leq \sqrt{3} + \epsilon$ for all sufficiently large numbers n . $\square$

Lemma 2. *For any $\epsilon > 0$ there exists a number $N_0 > 0$ such that for all numbers $n \geq N_0$ and for all positive integers l one has $|b_{(2l+1)(n+1)-1}^n| \leq \frac{2\sqrt{6}+\epsilon}{\pi(2l+1)}$ and $b_{2l(n+1)-1}^n = 0$ (we remind that in the expansions (7) $b_m^n = 0$ if $m \geq 0$ and $m \neq (k+1)(n+1) - 1$ for all $k = 0, 1, 2, \dots$)*

Proof. We have $b_{(2l+1)(n+1)-1}^n = (v_n, e_{(2l+1)(n+1)-1})_{L_2(0,1)}$. Let us consider the segment $\left[0, \frac{1}{n+1}\right]$ (here $\frac{1}{n+1}$ is the minimal positive root of the function $v_n(x)$). On this segment the function $e_{(2l+1)(n+1)-1}(x)$ has $2l+2$ roots $x_k = \frac{k}{(2l+1)(n+1)}$ where $k = \overline{0, 2l+1}$. Consider the following integrals: $I_k =$

$\int_{x_{k-1}}^{x_k} v_n(x) e_{(2l+1)(n+1)-1}(x) dx$ where $k = \overline{1, 2l+1}$. Then, due to the properties of the function $v_n(x)$ established in Proposition, we have $|I_{l+1}| > |I_l| = |I_{l+2}| > \dots > |I_1| = |I_{2l+1}|$ and $\text{sign } I_{l+1} = -\text{sign } I_l = -\text{sign } I_{l+2} = \dots = (-1)^l \text{sign } I_1 = (-1)^l \text{sign } I_{2l+1}$. Hence, due to Proposition $|b_{(2l+1)(n+1)-1}^n| \leq (n+1)|I_{l+1}|$.

Let us take an arbitrary $\epsilon > 0$. Then, for all sufficiently large numbers n we have by Lemma 1

$$|b_{(2l+1)(n+1)-1}^n| \leq$$

$$\leq \left(\sqrt{6} + \frac{\epsilon}{2}\right) (n+1) \int_{\frac{l}{(2l+1)(n+1)}}^{\frac{l+1}{(2l+1)(n+1)}} \sin[\pi(2l+1)(n+1)x] dx = \frac{2\sqrt{6} + \epsilon}{\pi(2l+1)}.$$

Finally, we remark that similar arguments show that for any integer $n \geq 0$ and $l = 1, 2, 3, \dots$ one has $b_{2l(n+1)-1}^n = 0$ because according to Proposition the function $v_n(x)$ is even with respect to the point $x = \frac{1}{2(n+1)}$ and, for each $l = 1, 2, 3, \dots$, the function $e_{2l(n+1)-1}(x)$ is odd with respect to this point. $\square$

For an arbitrary $s \leq 0$ and a nonnegative integer n we obviously have $\|e_n\|_{H^s(0,1)}^2 = \pi^{2s}(n+1)^{2s}$. Further, the sequence of real numbers $b_n^n = (v_n, e_n)_{L_2(0,1)}$ where $n = 0, 1, 2, \dots$ is obviously bounded, hence, by Lemma 2 and since the functions $e_n(x)$, $n = 0, 1, 2, \dots$, are pairwise orthogonal in the space $H^s(0, 1)$ where $s \leq 0$ is arbitrary, for any $s \leq 0$ there exist positive constants C_1 and C_2 such that

$$\begin{aligned} \|v_n\|_{H^s(0,1)}^2 &\leq \pi^{2s}(n+1)^{2s}(b_n^n)^2 + C_1 \sum_{l=1}^{\infty} (2l+1)^{2s-2}(n+1)^{2s} \leq \\ &\leq C_2(n+1)^{2s} \end{aligned}$$

for all sufficiently large numbers n . Therefore, for any $s \leq 0$ there exists $C_3 > 0$ such that

$$\|v_n\|_{H^s(0,1)}^2 \leq C_3(n+1)^{2s} \quad (9)$$

for all $n = 0, 1, 2, \dots$

Let us estimate the coefficients b_n^n in the expansions (7). In view of Lemma 2, we have for sufficiently small $\epsilon > 0$ and sufficiently large numbers n

$$b_n^n \geq \left\{ 1 - \sum_{l=1}^{\infty} \frac{24 + \epsilon}{\pi^2(2l+1)^2} \right\}^{\frac{1}{2}} = [1 - (24 + \epsilon)\pi^{-2}\xi(-1)]^{\frac{1}{2}} \quad (10)$$

where $\xi(-1) \leq \frac{1}{4}$ as in Section 1. The above arguments including the estimate (10) imply, in particular, the existence of constants $0 < b_1 < b_2$ such that

$$b_1 \leq b_n^n \leq b_2, \quad n = 0, 1, 2, \dots \quad (11)$$

Due to the estimate (11), we have for any $s \leq 0$

$$\|v_n\|_{H^s(0,1)}^2 \geq \pi^{2s}(n+1)^{2s}(b_n^n)^2 \geq b_1^2 \pi^{2s}(n+1)^{2s}, \quad n = 0, 1, 2, \dots \quad (12)$$

We denote $v_n^s = (b_n^n)^{-1}(n+1)^{-s}\pi^{-s}v_n$. Then, in view of the expansions (7) and by Lemma 2

$$v_n^s = \sum_{k=0}^{\infty} b_k^{n,s} e_k^s, \quad n = 0, 1, 2, \dots, \quad (13)$$

in the spaces $L_2(0, 1)$ and $H^t(0, 1)$ ($t \leq 0$) where

$$e_n^s = \pi^{-s}(n+1)^{-s}e_n \quad (n = 0, 1, 2, \dots),$$

$$b_{(2l+1)(n+1)-1}^{n,s} = (b_n^n)^{-1}(2l+1)^s b_{(2l+1)(n+1)-1}^n,$$

$$b_m^{n,s} = 0 \quad \text{if } m \neq (2l+1)(n+1) - 1 \quad \text{for all}$$

$$l = 0, 1, 2, \dots \quad \text{and} \quad b_n^{n,s} = 1. \quad (14)$$

In view of the estimates (9) and (12), for any $s \leq 0$ there exist positive constants C' and C'' such that

$$C' \leq \|v_n^s\|_{H^s(0,1)} \leq C'', \quad n = 0, 1, 2, \dots \quad (15)$$

Due to Lemma 2, (10) and (14), for an arbitrary $s \leq 0$ and a sufficiently small $\epsilon > 0$ there exists a number $N_1 = N_1(s, \epsilon) > 0$ such that for all $n \geq N_1$

$$\begin{aligned} & |b_{(2l+1)(n+1)-1}^{n,s}| \leq \\ & \leq [(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon] 2\sqrt{6}\pi^{-1}(2l+1)^{s-1}, \quad l = 1, 2, 3, \dots \end{aligned}$$

Also, if $s < s_0$, then in view of the definition of the number s_0 there exist $\epsilon_0 = \epsilon_0(s) > 0$ and $\delta = \delta(s) > 0$ such that

$$\begin{aligned} \sum_{l=1}^{\infty} [(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon_0] 2\sqrt{6}\pi^{-1}(2l+1)^{s-1} = \\ = [(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon_0] 2\sqrt{6}\pi^{-1}\xi(s) \leq 1 - \delta. \end{aligned}$$

Therefore, if $s < s_0$, then

$$\sum_{l=1}^{\infty} \sup_{n \geq N_1(s, \epsilon_0)} |b_{(2l+1)(n+1)-1}^{n,s}| \leq 1 - \delta. \quad (16)$$

Let us fix an arbitrary $s < s_0$ and let L be the closure in the space $H^s(0, 1)$ of the linear span of the functions $\{e_n^s\}_{n \geq N_1(s, \epsilon_0)}$ equipped with the topology of the space $H^s(0, 1)$. Let us consider the following linear operators acting in the space L :

A is the identity operator;

G_l is the operator which for any integer $n \geq N_1(s, \epsilon_0)$ transforms the function e_n^s in the function $b_{(2l+1)(n+1)-1}^{n,s} e_{(2l+1)(n+1)-1}^s$ ($l = 1, 2, 3, \dots$);

$$G = \sum_{l=1}^{\infty} G_l;$$

$$A = \Lambda + G$$

(we mean that the operators G_l and G are extended onto all finite linear combinations of functions e_n^s , $n \geq N_1(s, \epsilon_0)$, too).

Clearly, $\|G_l\|_{\mathcal{L}(L;L)} \leq \sup_{n \geq N_1(s, \epsilon_0)} |b_{(2l+1)(n+1)-1}^{n,s}|$. Therefore, due to

the estimate (16)

$$\|G\|_{\mathcal{L}(L;L)} \leq \sum_{l=1}^{\infty} \|G_l\|_{\mathcal{L}(L;L)} \leq 1 - \delta. \quad (17)$$

The inequality (17) shows that A is a bounded linear operator from L into L . We denote $N = N_1(s, \epsilon_0)$ and remark that the

operator A transforms any e_n^s , where $n \geq N$, into v_n^s . Also, due to (17) this operator possesses a bounded inverse one $A^{-1} = \Lambda + \sum_{r=1}^{\infty} (-1)^r G^r$ (for the proof of this fact, see, for example, [10]).

The following Lemmas 3 and 4 are in fact proved in [8,9]. However, since proofs of these statements are short and simple, we present them for the convenience of readers.

Lemma 3. *For our $s < s_0$ the system of functions $\{v_n^s\}_{n=0,1,2,\dots}$ is linearly independent in the space $H^s(0,1)$.*

Proof. Let us suppose that the statement of this lemma is invalid and there exist real coefficients a_n not all equal to zero and such that

$$\sum_{n=0}^{\infty} a_n v_n^s = 0 \quad (18)$$

in the space $H^s(0,1)$. Let $l \geq 0$ be a number such that $a_0 = \dots = a_{l-1} = 0$ and $a_l \neq 0$. Formulas (13) and (14) show that $(v_n^s, e_m^s)_{H^s(0,1)} = 0$ for arbitrary numbers $m < n$. Using this fact and multiplying the equality (18) by e_l^s in the space $H^s(0,1)$, we get $a_l b_l^{l,s} = 0$, i. e. the contradiction. $\square$

Lemma 4. *For our $s < s_0$ $\{v_n^s\}_{n \geq N}$ is a Riesz basis of the space L .*

Proof. Let us take an arbitrary $v \in L$ and let $u = A^{-1}v \in L$. Then, $u = \sum_{n=N}^{\infty} c_n e_n^s$ in the space L for some real coefficients c_n . We have $v = Au = \sum_{n=N}^{\infty} c_n A e_n^s = \sum_{n=N}^{\infty} c_n v_n^s$ where the infinite sums converge again in the space L . Therefore, in view of the linear independence of the system of functions $\{v_n^s\}_{n \geq N}$ in the space L given by Lemma 3, this system of functions is a basis of the space L . Also, obviously, the infinite sum $\sum_{n=N}^{\infty} c_n e_n^s$ with real coefficients

c_n converges in the space L if and only if $\sum_{n=N}^{\infty} c_n^2 < \infty$. Hence, first, if $\sum_{n=N}^{\infty} c_n^2 < \infty$, then the infinite sum $\sum_{n=N}^{\infty} c_n v_n^s$ converges in the space $H^s(0, 1)$ and, second, if the latter series converges in the indicated space, then $A^{-1} \sum_{n=N}^{\infty} c_n v_n^s = \sum_{n=N}^{\infty} c_n A^{-1} v_n^s = \sum_{n=N}^{\infty} c_n e_n^s \in L$, i. e. $\sum_{n=N}^{\infty} c_n^2 < \infty$. Thus, Lemma 4 is proved. $\square$

Lemma 5. *For our $s < s_0$ the system of functions $\{v_n^s\}_{n=0,1,2,\dots}$ is a Riesz basis of the space $H^s(0, 1)$.*

Proof. Let again $N = N_1(s, c_0)$ and P be the orthogonal projector in the space $H^s(0, 1)$ onto the subspace L_N spanned over the functions $e_0^s, \dots, e_{N-1}^s$. By (13) and (14)

$$Pv_n^s = v_n^s - \sum_{k=N}^{\infty} b_k^{n,s} v_k^s, \quad n = \overline{0, N-1} \quad (19)$$

in the space $H^s(0, 1)$. Since $\dim L_N = N$ and since in view of the formulas (13) and (14) the functions $Pv_0^s, \dots, Pv_{N-1}^s$ are linearly independent in the space $H^s(0, 1)$, these functions form a basis of the space L_N . Therefore, in view of Lemma 4, the system of functions $\{Pv_n^s\}_{n=\overline{0, N-1}} \cup \{v_n^s\}_{n \geq N}$ is a basis of the space $H^s(0, 1)$.

Let us take an arbitrary $u \in H^s(0, 1)$. Then, due to the above arguments, there exists a unique sequence $\{c_n\}_{n=0,1,2,\dots}$ of real numbers such that

$$u = \sum_{n=0}^{N-1} c_n Pv_n^s + \sum_{n=N}^{\infty} c_n v_n^s \quad (20)$$

in the space $H^s(0, 1)$. Substituting the expansions (19) into (20), we get

$$u = \sum_{n=0}^{N-1} c_n v_n^s + \sum_{n=N}^{\infty} \left(c_n - \sum_{m=0}^{N-1} c_m b_n^{m,s} \right) v_n^s$$

in the space $H^s(0,1)$. Hence, in view of Lemma 3, the system of functions $\{v_n^s\}_{n=0,1,2,\dots}$ is a basis of the space $H^s(0,1)$. Finally, the fact that this system of functions is a Riesz basis of the above space follows from Lemma 4. Thus, Lemma 5 is proved. $\square$

Now Theorem 2 follows from Lemma 5 and the inequalities (15). $\square$

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**ТЕМАТИЧЕСКИЕ КАТЕГОРИИ ПУБЛИКАЦИЙ
ОБЪЕДИНЕННОГО ИНСТИТУТА
ЯДЕРНЫХ ИССЛЕДОВАНИЙ**

Индекс	Тематика
1.	Экспериментальная физика высоких энергий
2.	Теоретическая физика высоких энергий
3.	Экспериментальная нейтронная физика
4.	Теоретическая физика низких энергий
5.	Математика
6.	Ядерная спектроскопия и радиохимия
7.	Физика тяжелых ионов
8.	Криогеника
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10.	Автоматизация обработки экспериментальных данных
11.	Вычислительная математика и техника
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14.	Исследования твердых тел и жидкостей ядерными методами
15.	Экспериментальная физика ядерных реакций при низких энергиях
16.	Дозиметрия и физика защиты
17.	Теория конденсированного состояния
18.	Использование результатов и методов фундаментальных физических исследований в смежных областях науки и техники
19.	Биофизика

О базисности счетного множества решений
некоторой нелинейной граничной задачи типа Шредингера

Рассматривается задача $u'' = f(u^2) u$ ($x \in (0, 1)$), $u(0) = u(1) = 0$, где f — достаточно гладкая функция, $f(0) \geq 0$ и $\lim_{r \rightarrow \infty} f(r^2) = -\infty$. Известно, что эта задача обладает последовательностью решений $\{u_n\}_{n=0,1,2,\dots}$ такой, что n -е решение $u_n(x)$ имеет ровно n корней в интервале $(0, 1)$. В работе доказано существование постоянной $s_0 < 0$ такой, что для любого $s < s_0$ произвольная описанная выше последовательность решений указанной задачи является базисом пространства $H^s(0, 1)$.

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On the Property of Being a Basis for a Denumerable Set of Solutions
of a Nonlinear Schrödinger-Type Boundary-Value Problem

We consider the problem $u'' = f(u^2) u$ ($x \in (0, 1)$), $u(0) = u(1) = 0$ where f is a sufficiently regular function, $f(0) \geq 0$ and $\lim_{r \rightarrow \infty} f(r^2) = -\infty$. It is known that this problem possesses a sequence of solutions $\{u_n\}_{n=0,1,2,\dots}$ such that the n th solution $u_n(x)$ has precisely n roots in the interval $(0, 1)$. We prove the existence of a constant $s_0 < 0$ such that, for any $s < s_0$, an arbitrary above-described sequence of solutions of our problem is a basis of the space $H^s(0, 1)$.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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