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Spatial coherence of bending magnet radiation and application limit of the van Cittert Zernike theorem

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In this paper we discuss how the first order spatial coherence of bending magnet radiation is determined, and present some numerical calculations based on first principles. It is shown that if the electron beam size is so large that some conditions are satisfied, the van Cittert Zernike theorem can be used without any modification in the vertical and horizontal directions. Especially, in the horizontal direction, the condition is determined only by the bending radius and the electron beam parameters, and does not depend on the wavelength of the light.

The presented formalism will be useful to judge whether the electron beam size in the storage ring can be estimated directly from the van Cittert Zernike theorem using the synchrotron radiation interferometer (SR interferometer).

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I. INTRODUCTION

A measurement of the electron beam emittance in an electron storage ring is one of the most important themes for accelerator physics. In order to obtain a high brightness or luminosity, low emittance rings have been constructed. For an extremely low emittance ring, less than 1 nm-rad in the horizontal direction, an accurate measurement of the emittance is very difficult. The emittance can be determined by measuring the electron beam size σ_e , since the beam size and the emittance ε are simply related by the equation

$$\sigma_e = \sqrt{\beta\varepsilon + \left(\eta \frac{\Delta E}{E_0}\right)^2}, \quad (1)$$

where β and η are the beta function and the dispersion function at a point where the electron beam size is measured, respectively. E_0 and ΔE are the average beam energy and the standard deviation of the beam energy, respectively. β , η , E_0 and ΔE are calculated and measured within some accuracy.

Recently, a novel method to estimate the electron beam size using a SR interferometer, which measures the spatial coherence of the bending magnet radiation in the visible light region, has been under development [1]. To estimate the spatial coherence, the van Cittert Zernike theorem is used for an ordinal incoherent light source. In the far field limit, the spatial coherence can be represented by the formula [2] [3] [4]

$$\gamma(D) = \frac{\int dS(\mathbf{x}) I(\mathbf{x}) \exp\left(\frac{i\mathbf{k}\cdot\mathbf{x}D}{L}\right)}{\int dS(\mathbf{x}) I(\mathbf{x})}, \quad (2)$$

where $\mathbf{k}$, D , L are the wave number vector of light, the separation of two points and the distance between the light source to the plane where the coherence is estimated, respectively. $I(\mathbf{x})$ is the intensity distribution of the light source and the integration is performed on the surface of the light source. From (2) the intensity distribution of the light source is estimated by measuring the spatial coherence $\gamma(D)$. For synchrotron radiation, special cases must be assumed in order to use this theorem, because the electrons in the storage ring move at nearly speed of light. Moreover, the radiation is emitted in a narrow cone due to the relativistic effect. For this reason it is needed to build an extended formula to calculate the spatial coherence of the bending magnet radiation.

This paper consists of the followings. In section II we examine how the bending magnet radiation by an electron can be described by the wave form of the radiation field and the arrival time of the radiation. The light is emitted homogeneously in the horizontal plane on which an electron moves. Therefore the wave forms are same at any observer point on this plane, if the distance between the observer and the emitting point is same. An only difference between the fields is a relative time delay. This corresponds to the phase difference of the fields in the frequency domain, that is a key issue for the spatial coherence. The phase difference of the fields at two observer positions is determined by the distance between the effective emitting points on the electron trajectory and the difference of the distance from the effective emitting point to the observer. The effective emitting point is defined as a point where the scalar

product of two vectors, the velocity vector of the electron and the unit vector from the electron to the observer, has the maximum value. Therefore a position of the effective emitting point depends on the electron trajectory and the observer position. In the vertical plane, the field is not homogeneous because the radiation is concentrated in the observation angle of $1/\gamma$, where γ is the electron energy normalized by the electron rest mass. Outside of this angle, the intensity of the radiation decreases rapidly. Therefore, we can not treat the field as a spherical wave in the vertical plane and need to take into account not only the phase but also the wave form, both of which depend on the electron trajectory and the observer position.

In section III, the spatial coherence is calculated with the approximate field. The condition that the van Cittert Zernike theorem is available is given individually for each of the vertical and horizontal directions. In the horizontal direction the condition is completely given by the bending radius and the electron beam parameters. In the vertical direction the condition depends on the wavelength of light additionally, because the vertical divergence of the radiation is a function of the wavelength.

In order to justify our discussions made in sections II and III, some numerical calculations are conducted without any approximation and presented in section IV. It shows that the analytical considerations of sections II and III are quite reasonable.

II. APPROXIMATION OF BENDING MAGNET RADIATION

In this section we give the trajectory of an electron in a bending magnet and the electric field emitted by it.

A single electron moves on an arc with an angular frequency ω_ρ , as shown in Fig. 1. The field of the bending magnet is assumed to have only a y component. We refer to the x and y directions as the horizontal and vertical directions, respectively. We suppose at time $t' = 0$ the electron passes the position $\mathbf{P}(\theta = 0)$ with velocity $c\beta(\theta = 0)$, where c is the light velocity and θ is defined as $\theta = \omega_\rho t'$. Then, the position and velocity of the electron at arbitrary time t' are written as

$$P_x(t') = \rho_0\beta_x(0)\sin\theta + \rho_0\beta_z(0)(1 - \cos\theta) + P_x(0), \quad (3)$$

$$P_y(t') = \rho_0\beta_y(0)\theta + P_y(0), \quad (4)$$

$$P_z(t') = \rho_0\beta_z(0)\sin\theta - \rho_0\beta_x(0)(1 - \cos\theta) + P_z(0), \quad (5)$$

and

$$\beta_x(t') = \beta_x(0)\cos\theta + \beta_z(0)\sin\theta, \quad (6)$$

$$\beta_y(t') = \beta_y(0), \quad (7)$$

$$\beta_z(t') = \beta_z(0)\cos\theta - \beta_x(0)\sin\theta, \quad (8)$$

where we put $\rho_0 = c/\omega_\rho$. We use the initial conditions

$$\mathbf{P}(0) = (x_e, y_e, 0), \quad (9)$$

$$\beta(0) = \sqrt{\frac{1 - \frac{1}{\gamma^2}}{1 + \tan^2\psi_x + \tan^2\psi_y}} (\tan\psi_x, \tan\psi_y, 1), \quad (10)$$

The electron beam has divergence ψ_x and ψ_y in the horizontal and vertical directions at $\theta = 0$, respectively. We set the observer coordinate $\mathbf{Q} = (x_o, y_o, L)$, where L is a large constant value compared with x_o, y_o, x_e and y_e . The bending radius ρ is given by

$$\rho = \rho_0 \sqrt{\beta_x^2(0) + \beta_z^2(0)} \simeq \rho_0, \quad (11)$$

where we have assumed that $\psi_x, \psi_y \ll 1$.

The electric field emitted by the electron is written as [5]

$$\mathbf{E}(\mathbf{Q}, t) = \frac{e}{4\pi\epsilon_0} \left[\frac{1}{\kappa} \frac{\mathbf{n}}{R^2} + \frac{1}{c\kappa} \frac{d}{dt'} \left(\frac{\mathbf{n} - \beta}{\kappa R} \right) \right]_{t=t'+R(t')/c}, \quad (12)$$

$$\mathbf{R} = \overrightarrow{PQ}, \quad (13)$$

$$R = |\mathbf{R}|, \quad (14)$$

$$\mathbf{n} = \mathbf{R}/R, \quad (15)$$

$$\kappa = 1 - \mathbf{n} \cdot \beta, \quad (16)$$

where e and ε_0 are the electric charge and the dielectric constant, respectively. All of the variables in the integration (12) must be evaluated at emitter time t' . An electric field with the angular frequency ω can be written as

$$\mathbf{E}(\mathbf{Q}, \omega) = \frac{1}{\sqrt{2\pi}} \int dt \mathbf{E}(\mathbf{x}, t) e^{i\omega t}. \quad (17)$$

FIG. 1. Electron motion in the bending magnet. It is supposed that the electron's position and the divergence at $\theta = 0$ are $(x_e, y_e, 0)$ and $1/\sqrt{(1 + \tan^2 \psi_x + \tan^2 \psi_y)(\tan \psi_x, \tan \psi_y, 1)}$, respectively.

From (12) the main contribution to the electric field comes from the short arc, where $\kappa(t')$ has the minimum value, for the high energy electron beam. For example, if β and n are parallel each other, κ becomes extremely small value, $1/(2\gamma^2)$. The position where κ has the minimum value can be obtained by solving the equation

$$\frac{d\kappa(t')}{dt'} = 0. \quad (18)$$

The solution $t' = t'_e$ depends on the initial conditions of the electron trajectory as well as an observer position.

The observed field should have the maximum intensity at $t = t'_e + R(t'_e)/c$. Hence, we write the electric field in (12) as

$$\mathbf{E}(\mathbf{Q}, t) = \mathbf{f}\left(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma; t - t'_e - \frac{R(t'_e)}{c}\right). \quad (19)$$

Using (19), (17) is written as

$$\mathbf{E}(\mathbf{Q}, \omega) = \mathbf{F}(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma) e^{i\Phi(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma)}, \quad (20)$$

$$\Phi(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma) = kR(t'_e) + \omega t'_e, \quad (21)$$

$$\mathbf{F}(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma) = \frac{1}{\sqrt{2\pi}} \int dt \mathbf{f}(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma; t) e^{i\omega t}, \quad (22)$$

where $k = \omega/c$ is the wave number. Although $\mathbf{F}$ and Φ depends on ω , we omit it for convenience.

Here, we make an important assumption that the phase of $\mathbf{F}$ does not depend on the electron trajectory, namely x_e, y_e, ψ_x and ψ_y , for any angular frequency ω . Moreover, we assume that the energy spread of the electron beam is so small that $\mathbf{F}$ is independent of the electron energy. In this case, we can put

$$F_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma) \equiv G_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y) e^{i\chi_i(\mathbf{Q})}, \quad (23)$$

where F_i and G_i is the i component of vector $\mathbf{F}$ and $\mathbf{G}$, respectively. $\mathbf{G}$ and χ_i are real functions. We omit to write the dependence of $\mathbf{G}$ and χ_i on the average electron beam energy γ_0 . We can regard $\mathbf{G}$ and Φ as the wave form and phase of the field, respectively. The phase χ_i does not affect the coherence as shown in the following section.

Since the phase χ_i is only a function of observer point $\mathbf{Q}$, we have

$$\frac{F_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma)}{F_i(\mathbf{Q}, 0, 0, 0, 0, \gamma_0)} = \frac{G_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y)}{G_i(\mathbf{Q}, 0, 0, 0, 0)}, \quad (24)$$

Substituting (22) into (24), we have

$$\int dt \left\{ f_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma; t) - \frac{G_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y)}{G_i(\mathbf{Q}, 0, 0, 0, 0)} f_i(\mathbf{Q}, 0, 0, 0, 0, \gamma_0; t) \right\} e^{i\omega t} = 0. \quad (25)$$

Since this equation is always satisfied for any angular frequency ω , we have

$$f_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma; t) = \frac{G_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y)}{G_i(\mathbf{Q}, 0, 0, 0, 0)} f_i(\mathbf{Q}, 0, 0, 0, 0, \gamma_0; t). \quad (26)$$

According to (26), the above assumption is equivalent to that the electric field can be factored with the two functions, one of which depends only on the time and observer position, and the other of which depends only on the electron trajectory and the observer position.

Next, we calculate the phase term in (21) using the following approximations:

$$\begin{aligned} x_o, y_o, x_e, y_e &\ll \rho_0, L, \\ \psi_x, \psi_y &\ll 1, \\ \frac{1}{\gamma_0} &\ll 1, \\ \frac{\Delta\gamma}{\gamma_0} &\ll 1, \\ \theta_e &\ll 1. \end{aligned} \quad (27)$$

The average electron beam energy and its spread are defined as E_0 and ΔE , respectively. γ_0 and $\Delta\gamma$ are defined as $\gamma_0 = E_0/(m_e c^2)$ and $\Delta\gamma = \Delta E/(m_e c^2)$, respectively, where m_e is the electron rest mass. By using (3) - (10) and the conditions in (27), we have

$$\Phi(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma) = kL + \frac{k}{2L} ((x_o - x_e)^2 + (y_o - y_e)^2) + h(x_o - x_e - L\psi_x, y_o - y_e - L\psi_y, \gamma), \quad (28)$$

and

$$h(x, y, \gamma) = k\rho_0 \left\{ \frac{\theta_e}{2\gamma^2} + \frac{\theta_e^3}{6} - \frac{\theta_e^2 x}{2L} + \frac{\theta_e(x^2 + y^2)}{2L^2} \right\}, \quad (29)$$

where we put $\theta_e = \omega_\rho t'_e$. Since we consider up to third order for (29), we just solve t'_e up to first order in (18), because second order term for θ_e gives higher order terms than fourth order in (29). This can be easily performed; the result is

$$\theta_e = \frac{(x_o - x_e - L\psi_x)}{L}. \quad (30)$$

Consequently, we have

$$h(x, y, \gamma_0) = \frac{k\rho_0 x}{2L} \left(\frac{1}{\gamma_0^2} + \frac{x^2 + 3y^2}{3L^2} \right). \quad (31)$$

In (31), γ is replaced with γ_0 because $\Delta\gamma$ contribute to only higher terms. Therefore, γ in (28) is also replaced with γ_0 . Hereafter, we simply use $h(x, y)$ and $\Phi(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y)$ in place of $h(x, y, \gamma_0)$ and $\Phi(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y, \gamma_0)$, respectively.

The first two terms on the RHS in (28) come from a par-axial approximation, which always appears in the far field calculation. On the other hand, the third term is the characteristic of the bending magnet radiation because this term depends on the bending radius. Since this term is third order, it should be much smaller than unity and can be neglected for some conditions. It is noted that the third term is not symmetric with respect to x and y .

III. SPATIAL COHERENCE OF BENDING MAGNET RADIATION

Since we obtained the approximate field of the bending magnet radiation, we calculate the (first order) spatial coherence.

The spatial coherence at $\mathbf{Q}_1 = (x_{o1}, y_{o2}, L)$ and $\mathbf{Q}_2 = (x_{o2}, y_{o2}, L)$ is defined as [4]

$$\gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega) = \frac{\Gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega)}{\sqrt{\Gamma_{i,i}(\mathbf{Q}_1, \mathbf{Q}_1; \omega)} \sqrt{\Gamma_{j,j}(\mathbf{Q}_2, \mathbf{Q}_2; \omega)}}, \quad (32)$$

$$\Gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega) = \langle E_i^*(\mathbf{Q}_1, \omega) E_j(\mathbf{Q}_2, \omega) \rangle, \quad (33)$$

where $\langle \dots \rangle$ means the ensemble average with respect to the electrons parameters; i, j mean the polarization. If we use the representation of the electric field in (23), we have

$$\begin{aligned} \Gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega) \\ = e^{-i(\chi_i(\mathbf{Q}_1) - \chi_j(\mathbf{Q}_2))} \langle G_i(\mathbf{Q}_1, x_e, y_e, \psi_x, \psi_y) G_j(\mathbf{Q}_2, x_e, y_e, \psi_x, \psi_y) e^{-i\Phi(\mathbf{Q}_1, x_e, y_e, \psi_x, \psi_y) + i\Phi(\mathbf{Q}_2, x_e, y_e, \psi_x, \psi_y)} \rangle. \end{aligned} \quad (34)$$

The ensemble average can be replaced by integration with the electron phase space density, namely

$$\begin{aligned} \Gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega) &= e^{-i(\chi_i(\mathbf{Q}_1) - \chi_j(\mathbf{Q}_2))} \int dx_e dy_e d\psi_x d\psi_y dE I_0(x_e, \psi_x, y_e, \psi_y; E) \\ &\quad G_i(\mathbf{Q}_1, x_e, y_e, \psi_x, \psi_y) G_j(\mathbf{Q}_2, x_e, y_e, \psi_x, \psi_y) e^{i(-\Phi(\mathbf{Q}_1, x_e, y_e, \psi_x, \psi_y) + \Phi(\mathbf{Q}_2, x_e, y_e, \psi_x, \psi_y))} \\ &= e^{i\Psi_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2)} \int dx_e dy_e d\psi_x d\psi_y I(x_e, \psi_x, y_e, \psi_y) e^{\frac{ik(x_e D_x + y_e D_y)}{L}} \\ &\quad G_i(\mathbf{Q}_1, x_e, y_e, \psi_x, \psi_y) G_j(\mathbf{Q}_2, x_e, y_e, \psi_x, \psi_y) e^{-ih(x_{o1} - \bar{x}_e, y_{o1} - \bar{y}_e) + ih(x_{o2} - \bar{x}_e, y_{o2} - \bar{y}_e)}, \end{aligned} \quad (35)$$

where

$$D_x = x_{o1} - x_{o2}, \quad (36)$$

$$D_y = y_{o1} - y_{o2}, \quad (37)$$

$$\bar{x}_e = x_e + L\psi_x, \quad (38)$$

$$\bar{y}_e = y_e + L\psi_y, \quad (39)$$

$$\Psi_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2) = \frac{k(x_{o2}^2 - x_{o1}^2 + y_{o2}^2 - y_{o1}^2)}{2L} - \chi_i(\mathbf{Q}_1) + \chi_j(\mathbf{Q}_2), \quad (40)$$

and (28) and (31) are used. $I_0(x_e, \psi_x, y_e, \psi_y; E)$ is the five dimensional phase space density of the electron beam. We defined

$$I(x_e, \psi_x, y_e, \psi_y) = \int dE I_0(x_e, \psi_x, y_e, \psi_y; E). \quad (41)$$

If two conditions,

$$(I) G_i(\mathbf{Q}, x_e, y_e, \psi_x, \psi_y) = G_i(\mathbf{Q}), \quad (42)$$

$$(II) h(x_{o1} - \bar{x}_e, y_{o1} - \bar{y}_e) - h(x_{o2} - \bar{x}_e, y_{o2} - \bar{y}_e) = h_0(\mathbf{Q}_1, \mathbf{Q}_2), \quad (43)$$

are satisfied, we obtain a simple form for the coherence. Condition (I) means that the wave form is homogeneous for any electron trajectory. Condition (II) means that the third order phase difference at two points does not depend on the electron trajectory. By using these conditions, (35) is written as

$$\Gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega) = G_i(\mathbf{Q}_1) G_j(\mathbf{Q}_2) e^{i\Psi_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2) - ih_0(\mathbf{Q}_1, \mathbf{Q}_2)} \int dx_e dy_e I_e(x_e, y_e) \exp\left(\frac{ik(x_e D_x + y_e D_y)}{L}\right), \quad (44)$$

where

$$I_e(x_e, y_e) = \int d\psi_x d\psi_y I(x_e, \psi_x, y_e, \psi_y), \quad (45)$$

is the intensity distribution of the electron beam on the $z = 0$ plane in Fig. 1. Substituting (44) into (32) we have the van Cittert Zernike theorem. Therefore, the van Cittert Zernike theorem can be applied for the bending magnet radiation, if conditions (I) and (II) are satisfied. For example, if we put

$$I_e(x_e, y_e) = I_e(0, 0) \exp\left(-\frac{x_e^2}{2\sigma_x^2} - \frac{y_e^2}{2\sigma_y^2}\right), \quad (46)$$

where σ_x and σ_y are the horizontal and vertical electron beam size, respectively, the coherence is written as

$$|\gamma_{i,j}(\mathbf{Q}_1, \mathbf{Q}_2; \omega)| = \exp\left(-\frac{D_x^2}{8\sigma_{cx}^2} - \frac{D_y^2}{8\sigma_{cy}^2}\right), \quad (47)$$

$$\sigma_{cx} = \frac{L\lambda}{4\pi\sigma_x}, \quad (48)$$

$$\sigma_{cy} = \frac{L\lambda}{4\pi\sigma_y}, \quad (49)$$

where λ is the wavelength.

To investigate condition (II) in detail, it is convenient to define the phase difference as

$$h(x_{o1} - \bar{x}_e, y_{o1} - \bar{y}_e) - h(x_{o2} - \bar{x}_e, y_{o2} - \bar{y}_e) = h_0(\mathbf{Q}_1, \mathbf{Q}_2) + h_1(\mathbf{Q}_1, \mathbf{Q}_2, \bar{x}_e, \bar{y}_e) + h_2(\mathbf{Q}_1, \mathbf{Q}_2, \bar{x}_e, \bar{y}_e), \quad (50)$$

where

$$h_0(\mathbf{Q}_1, \mathbf{Q}_2) = h(x_{o1}, y_{o1}) - h(x_{o2}, y_{o2}), \quad (51)$$

$$h_1(\mathbf{Q}_1, \mathbf{Q}_2, \bar{x}_e, \bar{y}_e) = \frac{k\rho_0 D_x}{2L^3} \{-(x_{o1} + x_{o2})\bar{x}_e + \bar{x}_e^2\}, \quad (52)$$

$$h_2(\mathbf{Q}_1, \mathbf{Q}_2, \bar{x}_e, \bar{y}_e) = \frac{k\rho_0}{2L^3} \{-2(x_{o1}y_{o1} - x_{o2}y_{o2})\bar{y}_e + D_x\bar{y}_e^2 - D_y(y_{o1} + y_{o2})\bar{x}_e + 2D_y\bar{x}_e\bar{y}_e\}. \quad (53)$$

Usually $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are chosen to be symmetric. Hence, we set

$$\begin{aligned} x_{o1} &= D_x/2, \\ x_{o2} &= -D_x/2, \\ y_{o1} &= D_y/2, \\ y_{o2} &= -D_y/2. \end{aligned} \quad (54)$$

Then, h_1 and h_2 in (52) and (53) are simply written as

$$h_1(D_x, \bar{x}_e, \bar{y}_e) \equiv \frac{k\rho_0 D_x \bar{x}_e^2}{2L^3}, \quad (55)$$

$$h_2(D_x, D_y, \bar{x}_e, \bar{y}_e) \equiv \frac{k\rho_0}{2L^3} (D_x \bar{y}_e^2 + 2D_y \bar{x}_e \bar{y}_e), \quad (56)$$

where we used the variables D_x and D_y instead of $\mathbf{Q}_1$ and $\mathbf{Q}_2$. By using these terms condition (II) is rewritten as

$$(II) \quad h_1(D_x, \bar{x}_e, \bar{y}_e) = 0, \quad h_2(D_x, D_y, \bar{x}_e, \bar{y}_e) = 0. \quad (57)$$

Since the preparation has been completed, in the followings derived will be the individual conditions for each of the horizontal and vertical directions under which the van Cittert Zernike theorem is valid.

A. Horizontal direction ($D_x = D \geq 0, D_y = 0$)

Since the field has a strong σ -polarization component, we take only this component to calculate the coherence and omit suffixes for the polarization.

If the electron beam does not distribute in the vertical direction, the radiation in the horizontal direction is almost homogeneous for any electron trajectory, since there is no specific direction in the horizontal plane. Therefore condition (I) is satisfied. Moreover the wave form $\mathbf{G}$ does not depend on the observer position $\mathbf{Q}$, if the conditions in (27) are satisfied. On the other hand, if the electron beam distributes in the vertical direction, condition (I) is broken because the intensity distribution of the field is not homogeneous in the vertical direction and the wave form depends on the electron trajectory as well as the observer position.

However, we neglect it because of the following consideration. If the vertical distribution is not zero, the coherence decreases because the phase of the field changes rapidly for large y_e or ψ_y according to (56). This effect, which spoils

the van Cittert Zernike theorem, is suppressed by the rapid decrease of the radiation intensity with large y_e or ψ_y . Hence, the coherence calculated by first principles must be better than the coherence calculated with the assumption that condition (I) is satisfied. Therefore, we assume condition (I) satisfied in order to derive the sufficient condition for the van Cittert Zernike theorem to be applied. The obtained condition in this way should be stricter than the reality.

As for condition (II), h_1 and h_2 in (55) and (56) are finite and written as

$$h_1(D, \bar{x}_e, \bar{y}_e) = \frac{k\rho D \bar{x}_e^2}{2L^3}, \quad (58)$$

$$h_2(D, 0, \bar{x}_e, \bar{y}_e) = \frac{k\rho D \bar{y}_e^2}{2L^3}. \quad (59)$$

From (47) we know that the coherence decreases to $e^{-\frac{1}{2}}$ for $D = 2\sigma_{cx}$, if the above two terms do not exist. Therefore, the h_1 and h_2 terms hardly affect the coherence if they are much smaller than unity for $D = 2\sigma_{cx}$ and the typical electron beam parameters. By substituting $D = 2\sigma_{cx}$ into (58) and (59) together with $\bar{x}_e = \sqrt{\sigma_x^2 + L^2\sigma_x'^2}$ and $\bar{y}_e = \sqrt{\sigma_y^2 + L^2\sigma_y'^2}$, the conditions that the h_1 and h_2 terms can be neglected are written as

$$h_1(2\sigma_{cx}, \sqrt{\sigma_x^2 + L^2\sigma_x'^2}, \sqrt{\sigma_y^2 + L^2\sigma_y'^2}) = \frac{k\rho(2\sigma_{cx})(\sigma_x^2 + L^2\sigma_x'^2)}{2L^3} \ll \frac{1}{2}, \quad (60)$$

$$h_2(2\sigma_{cx}, 0, \sqrt{\sigma_x^2 + L^2\sigma_x'^2}, \sqrt{\sigma_y^2 + L^2\sigma_y'^2}) = \frac{k\rho(2\sigma_{cx})(\sigma_y^2 + L^2\sigma_y'^2)}{2L^3} \ll \frac{1}{2}, \quad (61)$$

where σ_x, σ_y are the standard deviation of the electron beam size in the horizontal and vertical directions, respectively and σ_x', σ_y' are those of the electron beam divergence. The factor 1/2 of RHS is introduced so as to make the latter calculation simple. We evaluate σ_{cx} by substituting σ_x into (48) and the conditions (60) and (61) become

$$\sigma_x \gg \rho_0 \left(\sigma_x'^2 + \frac{\sigma_x^2}{L^2} \right) \simeq \rho_0 \sigma_x'^2, \quad (62)$$

$$\sigma_x \gg \rho_0 \left(\sigma_y'^2 + \frac{\sigma_y^2}{L^2} \right) \simeq \rho_0 \sigma_y'^2, \quad (63)$$

where we used $L\sigma_x' \gg \sigma_x, L\sigma_y' \gg \sigma_y$. As an example of that the van Cittert Zernike theorem is available, we have $\sigma_x \gg 8.66 \mu\text{m}$ for $\rho_0 = 8.66 \text{ m}$ and $\sigma_x' = \sigma_y' = 1 \text{ mrad}$. If these conditions are not satisfied, we must take into account the h_1 and h_2 terms in evaluating (35).

When the electron beam size is very small,

$$\sigma_x \ll \rho_0 \left(\sigma_x'^2 + \frac{\sigma_x^2}{L^2} \right) \simeq \rho_0 \sigma_x'^2, \quad (64)$$

$$\sigma_x \ll \rho_0 \left(\sigma_y'^2 + \frac{\sigma_y^2}{L^2} \right) \simeq \rho_0 \sigma_y'^2, \quad (65)$$

h_1 and h_2 give the dominant contribution in (35). Then, (35) is written as

$$\Gamma(\mathbf{Q}_1, \mathbf{Q}_2; \omega) = G(\mathbf{Q}_1)G(\mathbf{Q}_2) e^{i\Psi(\mathbf{Q}_1, \mathbf{Q}_2)} \int d\psi_x d\psi_y I_\psi(\psi_x, \psi_y) e^{-ih_1(D, -L\psi_x, -L\psi_y) - ih_2(D, 0, -L\psi_x, -L\psi_y)}, \quad (66)$$

where

$$I_\psi(\psi_x, \psi_y) = \int dx_e dy_e I(x_e, \psi_x, y_e, \psi_y). \quad (67)$$

This is the opposite of the van Cittert Zernike theorem and the coherence is determined by the angular divergence of the electron beam. Supposing that there is no electron beam divergence in the vertical direction, we put

$$I_\psi(\psi_x, \psi_y) \propto \exp\left(-\frac{\psi_x^2}{2\sigma_x'^2}\right) \delta(\psi_y). \quad (68)$$

In this case the h_2 term gives no contribution and the integration of (66) is easily performed to give the coherence as

$$|\gamma(\mathbf{Q}_1, \mathbf{Q}_2)| = \frac{1}{\sqrt[4]{1 + \left(\frac{k\rho_0\sigma'^2 D}{L}\right)^2}}. \quad (69)$$

This curve is not Gaussian because of a contribution by the third order term. If the electron beam size is extremely small and the conditions (64) and (65) are satisfied, we can estimate the electron beam divergence by measuring the coherence and applying (69).

More generally, the integration in (35) can be performed in the horizontal direction if we suppose that the electron phase space density is Gaussian, which is

$$\begin{aligned} I_0(x_e, \psi_x, y_e, \psi_y; E) \\ = I_0(0, 0, 0, 0; E_0) \exp \left(-\frac{\gamma_{0x} \left(x_e - \eta \frac{E-E_0}{E_0}\right)^2 + 2\alpha_{0x} \left(x_e - \eta \frac{E-E_0}{E_0}\right) \left(\psi_x - \eta' \frac{E-E_0}{E_0}\right) + \beta_{0x} \left(\psi_x - \eta' \frac{E-E_0}{E_0}\right)^2}{2\varepsilon_{0x}} \right) \\ \exp \left(-\frac{\gamma_y y_e^2 + 2\alpha_y y_e \psi_y + \beta_y \psi_y^2}{2\varepsilon_y} - \frac{(E-E_0)^2}{2(\Delta E)^2} \right), \end{aligned} \quad (70)$$

where $\alpha_{x0}, \beta_{x0}, \gamma_{x0}$ and ε_{0x} are the Twiss parameters and the emittance in the horizontal direction, respectively and $\alpha_y, \beta_y, \gamma_y$ and ε_y are those of the vertical direction. η and η' are the horizontal dispersion and its derivative, respectively, and the vertical dispersion is assumed to be zero. After the integration in (41), we have

$$I(x_e, \psi_x, y_e, \psi_y) = I(0, 0, 0, 0) \exp \left(-\frac{\gamma_x x_e^2 + 2\alpha_x x_e \psi_x + \beta_x \psi_x^2}{2\varepsilon_x} - \frac{\gamma_y y_e^2 + 2\alpha_y y_e \psi_y + \beta_y \psi_y^2}{2\varepsilon_y} \right), \quad (71)$$

where

$$\alpha_x = \frac{\left(\alpha_{0x} - \frac{1}{\varepsilon_{0x}} \eta \eta' \left(\frac{\Delta E}{E_0}\right)^2\right)}{\sqrt{1+\delta}}, \quad (72)$$

$$\beta_x = \frac{\left(\beta_{0x} + \frac{1}{\varepsilon_{0x}} \left(\eta \frac{\Delta E}{E_0}\right)^2\right)}{\sqrt{1+\delta}}, \quad (73)$$

$$\gamma_x = \frac{\left(\gamma_{0x} + \frac{1}{\varepsilon_{0x}} \left(\eta' \frac{\Delta E}{E_0}\right)^2\right)}{\sqrt{1+\delta}}, \quad (74)$$

$$\varepsilon_x = \varepsilon_{0x} \sqrt{1+\delta}, \quad (75)$$

$$\delta = \left(\frac{\Delta E}{E_0}\right)^2 \frac{(\gamma_{0x} \eta^2 + 2\alpha_{0x} \eta \eta' + \beta_{0x} \eta'^2)}{\varepsilon_{0x}}. \quad (76)$$

The beam size and divergence are given by

$$\sigma_x = \sqrt{\beta_x \varepsilon_x} = \sqrt{\beta_{0x} \varepsilon_{0x} + \left(\eta \frac{\Delta E}{E_0}\right)^2}, \quad (77)$$

$$\sigma'_x = \sqrt{\gamma_x \varepsilon_x} = \sqrt{\gamma_{0x} \varepsilon_{0x} + \left(\eta' \frac{\Delta E}{E_0}\right)^2}, \quad (78)$$

in the horizontal direction, respectively and

$$\sigma_y = \sqrt{\beta_y \varepsilon_y}, \quad (79)$$

$$\sigma'_y = \sqrt{\gamma_y \varepsilon_y}, \quad (80)$$

in the vertical direction, respectively. After performing the integration in (35) with (71), we have

$$|\gamma(\mathbf{Q}_1, \mathbf{Q}_2)| = \frac{\exp\left(-\frac{k^2 D^2 \beta_x \varepsilon_x}{2L^2} (1 - \xi_h)\right)}{\sqrt[4]{\left(1 + \left(\frac{k\rho_0 D \varepsilon_x \bar{\beta}_x}{L^3}\right)^2\right) \left(1 + \left(\frac{k\rho_0 D \varepsilon_y \bar{\beta}_y}{L^3}\right)^2\right)}}, \quad (81)$$

where

$$\xi_h = \frac{\frac{\bar{\beta}_x}{\beta_x} \left(\frac{(\beta_x - L\alpha_x)k\rho_0 D \varepsilon_x}{L^3}\right)^2}{1 + \left(\frac{k\rho_0 D \varepsilon_x \bar{\beta}_x}{L^3}\right)^2}, \quad (82)$$

$$\bar{\beta}_x = \beta_x - 2L\alpha_x + L^2\gamma_x, \quad (83)$$

$$\bar{\beta}_y = \beta_y - 2L\alpha_y + L^2\gamma_y. \quad (84)$$

$\bar{\beta}_x$ and $\bar{\beta}_y$ can be obtained by transforming β_x and β_y in the free space with distance L , respectively. We must be careful that (81) gives just the lower limit if the vertical electron beam size or beam divergence has a finite value, as discussed in the beginning of this section.

By comparing (81) and (47), the van Cittert Zernike theorem is available, if the followings are satisfied for $D = \frac{L}{k\sigma_x}$.

$$\frac{k\rho_0 D \varepsilon_x \bar{\beta}_x}{L^3} \ll 1, \quad (85)$$

$$\frac{k\rho_0 D \varepsilon_y \bar{\beta}_y}{L^3} \ll 1, \quad (86)$$

$$\xi_h \ll 1. \quad (87)$$

As a result, the following three conditions are derived to make the van Cittert Zernike theorem valid:

$$\sigma_x \gg \frac{\rho_0 \bar{\beta}_x \varepsilon_x}{L^2}, \quad (88)$$

$$\sigma_y \gg \frac{\rho_0 \bar{\beta}_y \varepsilon_y}{L^2}, \quad (89)$$

and

$$\sigma_x \gg \frac{\rho_0 \varepsilon_x |\beta_x - L\alpha_x|}{L^2} \sqrt{\frac{\bar{\beta}_x}{\beta_x}}. \quad (90)$$

The conditions in (88) and (89) are almost equivalent to the conditions in (62) and (63). **Consequently, the van Cittert Zernike theorem is available in the horizontal direction if the three conditions in (88), (89) and (90) are satisfied.**

B. Vertical direction ($D_x = 0$, $D_y = D \geq 0$)

Since the radiation is not homogeneous in the vertical plane, condition (I) is not always satisfied. Namely if the electron beam has a large divergence, an intensity imbalance of the radiation by a single electron occurs and the coherence decreases. The polarization depends on the observer point also, and the beam profile for the π -polarization component has two peaks in the vertical direction, and the coherence can not be easily treated [6]. For this reason we consider only the σ -component again. If we suppose that condition (II) is satisfied and approximately regard the field as a Gaussian beam, the spatial coherence is written as [6]

$$\gamma(D) = \exp\left(-\frac{D^2}{8\sigma_c^2}\right), \quad (91)$$

$$\sigma_c = \frac{L\lambda\sqrt{\sigma_p'^2 + \sigma_y'^2}}{4\pi\sqrt{\sigma_y^2\sigma_y'^2 + \sigma_y^2\sigma_p'^2 + \sigma_y'^2\sigma_p^2 - \alpha_y^2\varepsilon_y}}, \quad (92)$$

$$\sigma_p\sigma_p' = \frac{\lambda}{4\pi}, \quad (93)$$

where σ_p, σ'_p are the beam size and the beam divergence of the radiation by a single electron at the waist, respectively, and the electron distribution in the phase space is given by (71). If the condition

$$\sigma_y^2 \sigma_y'^2 + \sigma_y^2 \sigma_p'^2 \gg |\sigma_y'^2 \sigma_p^2 - \alpha_y^2 \varepsilon_y|, \quad (94)$$

namely

$$\sigma_y \gg \sqrt{\frac{|\sigma_y'^2 \sigma_p^2 - \alpha_y^2 \varepsilon_y|}{\sigma_y'^2 + \sigma_p'^2}}, \quad (95)$$

is satisfied, the coherent size in (92) agrees with that in (49) and the van Cittert Zernike theorem can be used. As a result (95) is equivalent to condition (I). For example, the beam divergence of the σ -polarization component in the vertical direction σ'_p , which is obtained by fitting the far field beam profile with the Gaussian shape, is 1.86 mrad for $\rho = 8.66$ m, $\gamma = 4900$ (electron beam energy = 2.5 GeV), $\lambda = 500$ nm and $\lambda_c = 0.31$ nm, where λ_c is the critical wavelength defined as

$$\lambda_c = \frac{4\pi\rho}{3\gamma^3}. \quad (96)$$

If we assume $\sigma'_y = 1$ mrad and $\alpha_y = 0$, the condition in (95) is reduced to $\sigma_y \gg 10.1 \mu\text{m}$. It is noted that this condition is just a rough standard, because to regard the bending magnet radiation as a Gaussian beam is a poor approximation.

As for conditions (II) in (55) and (56), h_1 and h_2 are written as

$$h_1(0, \bar{x}_e, \bar{y}_e) = 0, \quad (97)$$

$$h_2(0, D, \bar{x}_e, \bar{y}_e) = \frac{k\rho_0 D \bar{x}_e \bar{y}_e}{L^3}, \quad (98)$$

and we have the following condition through the same procedure as in the horizontal direction.

$$\sigma_y \gg \rho_0 \sqrt{\left(\sigma_x'^2 + \frac{\sigma_x^2}{L^2}\right) \left(\sigma_y'^2 + \frac{\sigma_y^2}{L^2}\right)} \simeq \rho_0 \sigma'_x \sigma'_y. \quad (99)$$

For example, we have $\sigma_x \gg 8.66 \mu\text{m}$, if we put $\rho_0 = 8.66$ m, $\sigma'_x = \sigma'_y = 1$ mrad.

We have a more accurate condition for (99) assuming that condition (I) is satisfied and using the electron phase space density defined in (71). We obtain an analytical form of the coherence similar to the case of the horizontal direction;

$$|\gamma(\mathbf{Q}_1, \mathbf{Q}_2)| = \frac{\exp\left(-\frac{k^2 D^2 \beta_y \varepsilon_y}{2L^2} (1 - \xi_v)\right)}{\sqrt{1 + \varepsilon_x \varepsilon_y \bar{\beta}_x \bar{\beta}_y \left(\frac{k\rho_0 D}{L^3}\right)^2}}, \quad (100)$$

where

$$\xi_v = \frac{\frac{\varepsilon_x \varepsilon_y \bar{\beta}_x}{\bar{\beta}_y} \left(\frac{(\beta_y - L\alpha_y) k\rho_0 D}{L^3}\right)^2}{1 + \varepsilon_x \varepsilon_y \bar{\beta}_x \bar{\beta}_y \left(\frac{k\rho_0 D}{L^3}\right)^2}. \quad (101)$$

Hence, condition (II) can be satisfied for

$$\sigma_y \gg \frac{\rho_0 \sqrt{\bar{\beta}_x \bar{\beta}_y \varepsilon_x \varepsilon_y}}{L^2}, \quad (102)$$

and

$$\sigma_y \gg \frac{\rho_0 |\beta_y - L\alpha_y| \sqrt{\varepsilon_x \varepsilon_y}}{L^2} \sqrt{\frac{\bar{\beta}_x}{\bar{\beta}_y}}. \quad (103)$$

As a result, the van Cittert Zernike theorem is available in the vertical direction if conditions (95), (102) and (103) are satisfied.

In this section we calculated the spatial coherence of the bending magnet radiation and derived some conditions under which the van Cittert Zernike theorem can be used.

The conditions are given individually for each of the horizontal and vertical directions. It is important that in the horizontal direction we don't need to care about the wave form, and just need to investigate the conditions in (88), (89) and (90). In the vertical direction, the wave form may affect the coherence if the condition in (95), which depends on the wavelength, is not satisfied. Since the Gaussian approximation is not satisfactory for the bending magnet radiation, the condition in (95) which based on the Gaussian approximation is not complete one. On the other hand the conditions in (88), (89), (90), (102) and (103) are derived by the consideration of the geometry. Taking into account these circumstances, it is more reliable to use the van Cittert Zernike theorem in the horizontal direction for estimation of the electron beam size than in the vertical direction.

The discussion in this section is based on the approximation in (23) and (31). To confirm that the approximation is reasonable, we will calculate the spatial coherence from first principles and compare with the results derived in this section.

IV. NUMERICAL CALCULATIONS

In this section we calculate the spatial coherence, $|\gamma(\mathbf{Q}_1, \mathbf{Q}_2, \omega)|$, using (12), (17) and (32). Since we consider only the σ -polarization component, suffixes for the polarization are not explicitly written in this section.

At first we calculate the coherence when the electron beam has some size and no divergence in the vertical and horizontal directions. The coherence can be calculated by the van Cittert Zernike theorem in this case.

Next, we calculate the coherence when the electron beam has a small divergence and no size at the origin ($z = 0$), and the separation of two observer points, $\mathbf{Q}_1$ and $\mathbf{Q}_2$, is small. In this case conditions (I) and (II) are nearly satisfied. If the electron beam has a finite divergence, the electron beam changes size as it moves on the arc. We show that the van Cittert Zernike theorem is also available in the horizontal and vertical directions.

Last, we calculate the coherence when the electron beam has a large divergence and no size. If the van Cittert Zernike theorem were to be applied, the coherence would be unity for any divergence of the electron beam. However, since the condition (I) or (II) are not satisfied, the coherence must decrease. We show that the coherence can be calculated using (69) in the horizontal direction (condition (II) is not satisfied) and the coherence decreases in the vertical direction because the field is not uniform (condition (I) is not satisfied).

A. Finite beam size

Here, we calculate the coherence when the electron beam has some size and no divergence.

We consider the arrangement as shown in Fig. 2. Two vectors, $\mathbf{n}$ and $\boldsymbol{\beta}$, become parallel at the origin when the observer coordinate $\mathbf{Q}_c$ is at the center of two points, $\mathbf{Q}_1$ and $\mathbf{Q}_2$. For all numerical calculations in this section, we choose the distance L between the origin and $\mathbf{Q}_c$, the bending radius ρ_0 , the wavelength of the light λ and the electron beam energy to be 10 m, 8.66 m, 500 nm and 2.5 GeV, respectively. We use three kinds of electron beam sizes, which are 50, 100 and 200 μm . The electron density $I_e(x_e, y_e)$ is chosen to be

$$I_e(x_e, y_e) = I_e(0, 0) \exp\left(-\frac{x_e^2}{2\sigma_x^2} - \frac{y_e^2}{2\sigma_y^2}\right). \quad (104)$$

For consideration of the horizontal direction, we put $\mathbf{Q}_1 = (D/2, 0, L)$, $\mathbf{Q}_2 = (-D/2, 0, L)$ and $\sigma_y = 0$, and for the vertical direction we put $\mathbf{Q}_1 = (0, D/2, L)$, $\mathbf{Q}_2 = (0, -D/2, L)$ and $\sigma_x = 0$. If the van Cittert Zernike theorem can be applied, the spatial coherence should be given by (47) - (49).

FIG. 2. Arrangement to calculate the spatial coherence for a finite beam size with no divergence.

The results of numerical calculations are shown in Figs. 3, 4 and Table I. In Fig. 3 and Fig. 4, the absolute values of the spatial coherence are plotted as a function of the separation of two observer points D in the horizontal and vertical directions, respectively. The curves of the spatial coherence are almost same in both directions. By fitting these curves with the Gaussian shape defined in (47), we obtain the coherent size numerically and compare this with the coherent size given by the van Cittert Zernike theorem in (48) and (49). The van Cittert Zernike theorem gives the same results as the numerical calculations, so that this theorem is available in the vertical and horizontal directions.

Theoretically, the van Cittert Zernike theorem can be applied for any beam size. However, it is quite difficult to measure very small size in the vertical direction. If the vertical size σ_y is very small, the separation D , which must be larger than the coherent size $2\sigma_{cy}$ defined in (49), can be very large. Since the intensity of light decreases rapidly for $D/2 > L\sigma'_p$ in the vertical direction, where $L\sigma'_p$ is the beam size of the light at the observer point, the light itself is too weak to be measured accurately. Therefore, in order to obtain enough light intensity, the coherent size σ_{cy} must be smaller than $L\sigma'_p$. Using (49) and (93), we have a condition

$$\sigma_y > \sigma_p. \quad (105)$$

FIG. 3. Spatial coherence in the horizontal direction with a horizontal beam size.

FIG. 4. Spatial coherence in the vertical direction with a vertical beam size.

electron beam size (μm)	van Cittert Zernike theorem (mm)	numerical calculation horizontal (mm)	numerical calculation vertical (mm)
50	7.958	7.981	7.980
100	3.979	3.987	3.987
200	1.989	1.994	1.993

TABLE I. Comparison of the coherent size for a finite electron beam size. The coherent size calculated by the van Cittert Zernike theorem and numerical calculation in the horizontal and vertical directions are compared.

B. Small beam divergence

Next, we consider an electron beam with a finite divergence, σ'_x or σ'_y , but no beam size at the origin.

FIG. 5. Arrangement to calculate the spatial coherence for a finite small divergence, but no beam size at $\theta = 0$.

In this case the electron beam changes size as the observer point moves. In Fig. 5, we put $\mathbf{Q}_c = (L \sin \theta, 0, L \cos \theta)$ at the observer point B . Then, two vectors, $\mathbf{n}$ and β , become parallel at B' to which the electron beam moves by angle θ' from the origin. θ and θ' are related by the equation

$$\theta' = \tan^{-1} \left(\theta - \frac{\rho}{L \cos \theta} \right) + \sin^{-1} \left(\frac{\rho}{\sqrt{L^2 + \rho^2 - 2L\rho \sin \theta}} \right). \quad (106)$$

Then, the electron beam size at B' is written as

$$\sigma_x = \rho \sigma'_x \theta', \quad (107)$$

$$\sigma_y = \rho \sigma'_y \theta'. \quad (108)$$

The distance between B and B' is written as

$$R = \sqrt{(L \sin \theta - \rho + \rho \cos \theta')^2 + (L \cos \theta - \rho \sin \theta')^2}. \quad (109)$$

In this case, the coherent size expected by the van Cittert Zernike theorem is written as

$$\sigma_{cx} = \frac{R\lambda}{4\pi\sigma_x}, \quad (110)$$

$$\sigma_{cy} = \frac{R\lambda}{4\pi\sigma_y}. \quad (111)$$

We choose three parameters for θ , which are 0, 1 and 2 degrees. We set the electron beam divergence, σ'_x or σ'_y , to be 0.5 mrad. For these parameters the conditions in (62) and (63) are satisfied, except for the case of $\theta = 0$. The distribution of the electron beam divergence is defined as

$$I_\psi(\psi_x, \psi_y) = I_\psi(0, 0) \exp \left(-\frac{\psi_x^2}{2\sigma_x'^2} - \frac{\psi_y^2}{2\sigma_y'^2} \right). \quad (112)$$

In the same way as the previous case, for consideration of the horizontal direction, we put $\mathbf{Q}_1 = (D/2, 0, L)$, $\mathbf{Q}_2 = (-D/2, 0, L)$ and $\sigma'_y = 0$, and for the vertical direction we put $\mathbf{Q}_1 = (0, D/2, L)$, $\mathbf{Q}_2 = (0, -D/2, L)$ and $\sigma'_x = 0$.

The results of numerical calculations are shown in Figs. 6, 7 and Table II. In Fig. 6 and Fig. 7 the spatial coherence is plotted as a function of the separation of two observer points D in the horizontal and the vertical directions, respectively. For the case of $\theta = 0$, the coherent size calculated numerically is finite, which is not consistent with the expected value, infinity. For the $\theta \neq 0$ cases, the electron beam size satisfies the condition introduced in the previous section, and the van Cittert Zernike theorem and the numerical calculation agree for the coherent size within several % error. These errors are introduced because condition (I) or (II) is not completely satisfied.

In any case the spatial coherence is almost determined by the electron beam size and doesn't depend on the electron beam divergence. Therefore, even if the electron beam has both beam size and beam divergence, the van Cittert Zernike theorem is a good approximation to calculate the coherent size, as far as conditions (I) and (II) are satisfied.

As in the previous case, there is a condition on the divergence of the light beam under which the light beam reaches at the observer points strongly enough. In this case, the size of the light beam at the observer point is $L\sqrt{\sigma_p'^2 + \sigma_y'^2}$ in the vertical direction. Therefore, the minimum size of the electron beam to be measured in the vertical direction is given by the inequality

$$\sigma_y > \frac{\sigma_p}{\sqrt{1 + \left(\frac{\sigma'_y}{\sigma'_p} \right)^2}}, \quad (113)$$

where we used (49) and (93). It is noted that this condition is different from (95), which is

$$\sigma_y \gg \frac{\sigma_p}{\sqrt{1 + \left(\frac{\sigma'_p}{\sigma'_y} \right)^2}}, \quad (114)$$

for $\alpha_y = 0$.

FIG. 6. Spatial coherence in the horizontal direction with a horizontal beam size and divergence.

FIG. 7. Spatial coherence in the vertical direction with a vertical beam size and divergence.

θ (degree)	electron beam size (μm)	van Cittert Zernike theorem (mm)	numerical calculation horizontal (mm)	numerical calculation vertical (mm)
0	0	∞	277.9	130.4
1	76.15	5.145	5.160	5.305
2	153.5	2.513	2.519	2.496

TABLE II. Comparison of the coherent size for a small electron beam divergence. The coherent size calculated by the van Cittert Zernike theorem and numerical calculation in the horizontal and vertical directions are compared.

C. Large beam divergence

Lastly, we calculate the coherence when the electron beam has a large divergence and no size at the origin. The arrangement is shown in Fig. 8.

We calculate four cases. For each case we use four kinds of the electron beam divergences: 0.5, 1, 2 and 5 mrad. We use the divergence distribution of the electron beam defined in (112). It is noted that if the van Cittert Zernike theorem were available, the coherence would be unity for any case.

FIG. 8. Arrangement to calculate the spatial coherence for a finite large divergence, but no beam size at $\theta = 0$.

1. *horizontal divergence $\rightarrow$ horizontal coherence*

We use the parameters

$$Q_1 = \left(\frac{D}{2}, 0, L \right), Q_2 = \left(-\frac{D}{2}, 0, L \right), \sigma'_x \neq 0, \sigma'_y = 0.$$

The coherence curves calculated numerically are shown in Fig. 9. We see that the curves are not Gaussian shape, although the divergence distribution of the electron beam is Gaussian. This comes from the third order contribution of the phase. Actually, these curves are fitted exactly with the curves expected analytically in (69).

FIG. 9. Spatial coherence in the horizontal direction with a horizontal divergence.

2. vertical divergence $\rightarrow$ vertical coherence

We use the parameters

$$\mathbf{Q}_1 = \left(0, \frac{D}{2}, L\right), \mathbf{Q}_2 = \left(0, -\frac{D}{2}, L\right), \sigma'_x = 0, \sigma'_y \neq 0.$$

The results of numerical calculations are shown in Fig. 10. The coherence decreases more rapidly than the first case (horizontal divergence $\rightarrow$ horizontal coherence), even for the 0.5 mrad case. Moreover, the coherence curve is similar to the Gaussian shape.

According to (92), if we regard the field as a Gaussian beam, the coherent size is given by

$$\sigma_{cy} = \frac{L\sigma'_p \sqrt{\sigma_y'^2 + \sigma_p'^2}}{\sigma'_y}, \quad (115)$$

where $\sigma'_p = 1.86$ mrad in this case. As shown in Table III, we compare this with the coherent size calculated numerically, which is obtained by fitting the curves in Fig. 10 by (47). A large discrepancy is seen, especially for the large coherent size, which comes from that the bending magnet radiation is not exactly a Gaussian beam. Numerical calculation suggests that condition (95), which is derived using the Gaussian approximation, is not complete one, since the Gaussian approximation is a poor approximation for the bending magnet radiation. Therefore, to confirm precisely whether the van Cittert Zernike theorem can be used in the vertical direction, numerical calculation for each specific case is necessary.

FIG. 10. Spatial coherence in the vertical direction with a vertical divergence.

electron beam divergence (mrad)	Gaussian approximation (mm)	numerical calculation (mm)
0.5	71.6	30.5
1	39.3	22.7
2	25.4	18.1
5	19.8	15.1

TABLE III. Comparison of the coherent size for a large electron beam divergence in the vertical direction. The coherent size expected by (92) and the numerical calculation are compared.

3. horizontal divergence $\rightarrow$ vertical coherence

We use the parameters

$$\mathbf{Q}_1 = \left(0, \frac{D}{2}, L\right), \mathbf{Q}_2 = \left(0, -\frac{D}{2}, L\right), \sigma'_x \neq 0, \sigma'_y = 0.$$

The results of numerical calculations are shown in Fig. 11. The coherence is always unity in this case, because there is no phase difference between two observer points at all. Moreover, the intensities at two points are always the same. Therefore, the horizontal divergence of the electron beam does not affect the vertical coherence at all.

FIG. 11. Spatial coherence in the vertical direction with a horizontal divergence.

4. vertical divergence $\rightarrow$ horizontal coherence

We use the parameters

$$\mathbf{Q}_1 = \left(\frac{D}{2}, 0, L\right), \mathbf{Q}_2 = \left(-\frac{D}{2}, 0, L\right), \sigma'_x = 0, \sigma'_y \neq 0.$$

The results of numerical calculations are shown in Fig. 12. If we consider only the effect of the phase term, the curves in this case should coincide with those of the first case (horizontal divergence $\rightarrow$ horizontal coherence). For a large divergence of the electron beam, a big discrepancy is seen between Fig. 9 and Fig. 12. This is why condition (I) is broken for a large divergence of the electron beam, as discussed in section III.

FIG. 12. Spatial coherence in the horizontal direction with a vertical divergence.

V. SUMMARY

We calculated the spatial coherence of the bending magnet radiation while supposing that the radiation is represented with the phase Φ and the wave form $\mathbf{G}$, and compared the result with a numerical calculation.

We showed that the van Cittert Zernike theorem can be applied in the horizontal and vertical directions if the electron beam size is much larger than some value σ_e , and that the electron beam size can be estimated by measuring the spatial coherence. The conditions are written in (88), (89) and (90) for the horizontal direction and in (95), (102) and (103) for the vertical direction. For the horizontal direction, σ_e is completely determined by the bending radius and the electron beam parameters.

Oppositely, if the electron beam size is extremely smaller than σ_e , the electron beam divergence can be estimated by measuring the spatial coherence. For the vertical direction, σ_e depends not only on the electron beam parameters, but also on the wavelength of light, which makes it complicated to justify the condition. To make the argument clear, a numerical calculation based on first principles is necessary for each specific case. Therefore, to investigate the very small size of the electron beam, an estimation of the vertical size is more difficult than that of the horizontal size. However, we can overcome this difficulty by using a vertical bending magnet. This makes it possible to measure the vertical beam size in the same way as the horizontal beam size. This exchanges the characteristics of measurement in the horizontal direction and the vertical direction.

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