Strange Quark Mass Dependence of the Tau Hadronic Width

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Abstract

The perturbative quark mass corrections to the tau hadronic width are studied to $O(\alpha_s^3 m_q^2)$. Including up to dimension four corrections, we get $m_s(4 \text{ GeV}^2) = (143 \pm 42) \text{ MeV}$ [$m_s(1 \text{ GeV}^2) = (193 \pm 59) \text{ MeV}$]. Possible improvements to reduce the theoretical uncertainty are pointed out.

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1 Introduction

The measurements of the tau hadronic width,

$$R_{\tau} \equiv \frac{\Gamma [\tau^- \to \text{hadrons} \nu_\tau(\gamma)]}{\Gamma [\tau^- \to e^- \nu_e \nu_\tau(\gamma)]},$$

(1)

and related observables have reached a high precision status, which will further improve with the foreseen data from heavy–flavour factories. This will allow for much more detailed studies of the Standard Model low–energy dynamics and in particular of QCD.

One of the important issues that can be addressed with these measurements is the determination of the strange quark mass. The latest Review of Particle Physics (PDG) [1] quotes the (MS) running strange quark mass at 2 GeV to be in between 60 MeV and 170 MeV. This large range reflects mainly the uncertainty in the hadronic input needed to determine $m_s$ within QCD sum rules and the spread of values obtained within lattice QCD.

The strange quark mass induces a sizeable correction [2] to the semi-inclusive $\tau$ decay width into Cabibbo–suppressed modes. This can be used to perform a determination of $m_s$. Preliminary values, extracted from the ALEPH $\tau$ data, have been already reported in Refs. [3, 4, 5]. The obvious and very interesting advantage of this determination is that the hadronic input does not depend on any extra hypothesis; it is a purely experimental issue. Therefore, the major part of the uncertainty will eventually come from the theoretical input, which can be treated within QCD using the Operator Product Expansion (OPE) at the tau mass scale.

It is then very important to have a detailed study of what do we know at present, within QCD, on the quark mass corrections to $R_{\tau}$ and related observables. This subject has been analyzed recently in Refs. [6, 7, 8, 9, 10].

2 Theoretical Framework

The theoretical framework needed to study the hadronic $\tau$ decay involves two-point correlation functions of vector $V_{ij}^\mu \equiv \bar{q}_j \gamma^\mu q_i$ and axial–vector quark currents $A_{ij}^\mu \equiv \bar{q}_j \gamma^\mu \gamma^5 q_i$, $(i, j = u, d, s)$:

$$
\begin{align*}
Π_{V,ij}^{\mu\nu}(q) &\equiv i \int d^4x \, e^{iqx} \left\langle 0 | T \left\{ V_{ij}^{\mu}(x) V_{ij}^{\nu}(0) \right\} | 0 \right\rangle ; \\
Π_{A,ij}^{\mu\nu}(q) &\equiv i \int d^4x \, e^{iqx} \left\langle 0 | T \left\{ A_{ij}^{\mu}(x) A_{ij}^{\nu}(0) \right\} | 0 \right\rangle ; \\
Π_{V(A),ij}^{\mu\nu}(q^2) &\equiv \left( q^\mu q^\nu - q^2 g^{\mu\nu} \right) Π_{V(A),ij}^T(q^2) \\
&\quad + q^\mu q^\nu Π_{V(A),ij}^T(q^2).
\end{align*}
$$

(2)

The tau hadronic width can be expressed as an integral over the invariant mass $s$ of the final–state hadrons, of the spectral functions $\text{Im} Π^T(s)$ and $\text{Im} Π^L(s)$, with adequate phase space factors:

$$
R_{\tau} = 12\pi \int_0^{M^2_\tau} ds \, \frac{d^2}{M^2_\tau} \left( 1 - \frac{s}{M^2_\tau} \right)^2 \\
\times \left[ \left( 1 + 2\frac{s}{M^2_\tau} \right) \text{Im} Π^T(s) + \text{Im} Π^L(s) \right].
$$

(3)
Moreover, according to the quantum numbers content of the two–point function correlators
\[ \Pi_J(s) \equiv |V_{ud}|^2 \left[ \Pi_{V,ud}(s) + \Pi_{A,ud}(s) \right] + |V_{us}|^2 \left[ \Pi_{V,us}(s) + \Pi_{A,us}(s) \right], \] (4)
we can decompose \( R_\tau \) into
\[ R_\tau \equiv R_{\tau,V} + R_{\tau,A} + R_{\tau,S}, \] (5)
where \( R_{\tau,V} \) and \( R_{\tau,A} \) correspond to the first two terms in Eq. (4), while \( R_{\tau,S} \) contains the remaining Cabibbo–suppressed contributions.

Exploiting the analytic properties of \( \Pi_J(s) \), we can rewrite (3) as a contour integral in the complex \( s \)-plane around the circle \( |s| = M_\tau^2 \) running counter–clockwise:
\[ R_\tau = -i\pi \oint_{|s|=M_\tau^2} ds \left( 1 - \frac{s}{M_\tau^2} \right)^3 \times \left\{ 3 \left( 1 + \frac{s}{M_\tau^2} \right) D^{L+T}(s) + 4D^L(s) \right\}. \] (6)

We used integration by parts to rewrite \( R_\tau \) in terms of the logarithmic derivative of the relevant correlators
\[ D^{L+T}(s) \equiv -s \frac{d}{ds} \left[ \Pi^{L+T}(s) \right], \]
\[ D^L(s) \equiv \frac{s}{M_\tau^2} \frac{d}{ds} \left[ s \Pi^L(s) \right], \] (7)
which satisfy homogeneous renormalization group (RG) equations.

For large enough Euclidean \( s \), \( D^{L+T}(s) \) and \( D^L(s) \) are calculable within QCD and we can organise the contributions in a series of higher dimensional contributions, using the OPE. One can then express \( R_\tau \) as an expansion in inverse powers of \( M_\tau^2 \) [2]:
\[ R_\tau \equiv 3 \left[ |V_{ud}|^2 + |V_{us}|^2 \right] S_{EW} \left[ 1 + \delta_{EW}^{(2)} + \delta^{(0)} \right] + \sum_{D=2,4,\ldots} \left( \cos^2(\theta_C) \delta_{ud}^{(D)} + \sin^2(\theta_C) \delta_{us}^{(D)} \right), \]
where \( \sin^2(\theta_C) \equiv |V_{us}|^2 / [|V_{ud}|^2 + |V_{us}|^2] \) and we have pulled out the electroweak corrections \( S_{EW} = 1.0194 \pm 0.0040 \) [11] and \( \delta_{EW}^{(0)} \approx 0.0010 \) [12]. The dimension–zero contribution \( \delta^{(0)} \) is purely perturbative [2, 13, 14, 15] and equal for the vector and axial–vector parts. The symbols \( \delta_{ij}^{(D)} \equiv [\delta_{V,ij}^{(D)} + \delta_{A,ij}^{(D)}] / 2 \) are the average of the vector and axial–vector contributions of the dimension \( D \geq 2 \) operators appearing in the corresponding OPE.

3 Quark–Mass Corrections

The largest contribution of the strange quark mass appears in \( \delta_{us}^{(2)} \). This contribution was studied extensively in [6] and we shall report here the main results found. Taking for simplicity
\[ m_u = m_d = 0, \] the dimension two corrections can be written as \( (a \equiv \alpha_s/\pi) \)

\[ \delta^{(2)}_{us} \equiv -8 \frac{m_u^2(M^2_f)}{M^2_f} \Delta[a(M^2_f)], \tag{8} \]

\[ \Delta[a] \equiv \frac{1}{4} \left\{ 3 \Delta^{L+T}[a(M^2_f)] + \Delta^L[a(M^2_f)] \right\}, \tag{9} \]

where

\[ \Delta^J[a(M^2_f)] = \sum_{n=0} \tilde{d}_n^J(\xi) B_n^J(a_\xi). \tag{10} \]

Here \( \xi \) is an arbitrary scale factor (of order unity), \( a_\xi \equiv a(\xi^2 M^2_f) \) and the coefficients \( \tilde{d}_n^J(\xi) \) are constrained by the homogeneous RG equations satisfied by the corresponding \( D^J(s) \). The question is how well can we predict \( \Delta[a] \) within QCD.

The coefficients \( \tilde{d}_n^J(\xi) \) are known to \( O(a^3) \) for \( J = L \) and \( O(a^2) \) for \( J = L+T \). The functions \( B_n^J(a_\xi) \) contain the contour integrations

\[ B_n^{L+T}(a_\xi) \equiv \frac{-1}{4\pi i} \oint_{|x|=1} \frac{dx}{x^2} (1 + x)(1 - x)^3 \times \left( \frac{m(-\xi^2 M^2_F x)}{m(M^2_f)} \right)^2 a^n(-\xi^2 M^2_F x); \tag{11} \]

and

\[ B_n^{L}(a_\xi) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - x)^3 \times \left( \frac{m(-\xi^2 M^2_F x)}{m(M^2_f)} \right)^2 a^n(-\xi^2 M^2_F x). \tag{12} \]

They have been calculated exactly [6], using the RG to four loops; i.e. with the first four expansion coefficients of the QCD beta and gamma functions.

The perturbative behaviour of \( \Delta^{L+T}[a] \) and \( \Delta^L[a] \) has been studied in Ref. [6]. For \( \xi = 1 \) and \( a = 0.1 \) (the actual value is around 0.11), one finds the following loop series:

\[ \Delta^{L+T}[0.1] = 0.7824 + 0.2239 + 0.0823 \]

\[ -0.000060 \tilde{d}_3^{L+T}(1) + \cdots \tag{13} \]

\[ \Delta^L[0.1] = 1.5891 + 1.1733 + 1.1214 \]

\[ +1.2489 + \cdots. \tag{14} \]

While the \( L + T \) series converges very well \([\tilde{d}_3^{L+T}(1) \text{ is expected to be of } O(300)]\), the \( L \) series behaves very badly. The combined final expansion for \( \Delta \),

\[ \Delta[0.1] = 0.9840 + 0.4613 + 0.3421 \]

\[ + [0.3122 - 0.000045 \tilde{d}_3^{L+T}(1)] + \cdots, \tag{15} \]

looks still acceptable because \( \Delta^{L+T} \) is weighted by a larger factor in Eq. (9).

The dependence on the renormalization–scale factor \( \xi \) is shown [6] in Figures 1 \( (\Delta^{L+T}) \), 2 \( (\Delta^L) \) and 3 \( (\Delta) \). The factor \( \xi \) should be around one in order not to get large logarithms. We have
also to keep $a_\xi$ within the radius of convergence of the perturbative expansion [15], i.e. $\xi > 0.5$. Again, the $\Delta^{L+T}$ series behaves very well. In fact, one can use some minimal sensitivity criterion to predict the unknown $\tilde{d}_3^{L+T}(1)$. On the contrary, the $\Delta^L$ series is monotonically decreasing and changes a 65% between $\xi = 1$ and $\xi = 2$. Finally, $\Delta$ reflects the bad $\Delta^L$ behaviour and is again monotonically decreasing. Our best estimate of $\Delta[0.1]$ is [6]:

$$\Delta[0.1] = 2.1 \pm 0.6,$$

where the central value is for $\xi = 1$ and the error is a combination of both the loop series expansion uncertainty and the truncation scale dependence.

4 Results

ALEPH has recently presented [5] a preliminary measurement of the Cabibbo–suppressed width of the $\tau$, $R_{\tau,S} = 0.1607 \pm 0.0066$. Moreover, they have extracted from their data the SU(3)–breaking quantity, $R_{\tau,V} + R_{\tau,A} = R_{\tau,S} |V_{ud}|^2 - |V_{us}|^2 = 0.413 \pm 0.126$,

which directly measures the effect of the strange quark mass. The error includes the experimental uncertainty from $R_{\tau,S}$ as well as the one from the relevant quark mixing factors.

Including up to dimension four contributions, Eq. (17) implies [7]:

$$m_s[M^2_{\tau}] = (149 \pm 44) \text{ MeV},$$
$$m_s[4 \text{GeV}^2] = (143 \pm 42) \text{ MeV},$$
$$m_s[1 \text{GeV}^2] = (193 \pm 59) \text{ MeV},$$

where the error, at the tau mass scale, splits into 22 MeV from the experimental uncertainty and 22 MeV from the theoretical one. At present, the resulting error on $m_s$ is slightly larger than the usually quoted uncertainties from QCD Sum Rules [17] and lattice [18] determinations.
Nevertheless, the $R_{τ,S}$ result has the potential to be more precise when better data will become available. Notice that the lower value in (18) is already larger than the PDG quoted lower bound and excludes some of the lattice results [18].

The main theoretical uncertainty originates in the bad perturbative behaviour of the longitudinal series $Δ^L$. Therefore, an experimental determination of the separate $J = L$ and $J = L + T$ pieces would allow a much more precise analysis.

For the time being, let us assume that we have just the full final hadron mass distribution of $R_{τ,S}$ (a first measurement of this distribution has already been presented in Ref. [5]). In that case we could still reduce the theoretical uncertainty, through a judicious choice of weight factors (i.e. moments) in Eqs. (3) and (6), which could improve the convergence of the perturbative series (the phase space factors are partly responsible for the bad perturbative behaviour of the $J = L$ contribution). One could reach in this way a theoretical precision for the strange quark mass of the order or below 10 MeV [7]. Obviously, a measurement of the energy distribution $R_{τ,S}(s)$ would also decrease the experimental uncertainties considerably. We conclude then that there are good prospects for performing a precise determination of the strange quark mass from $τ$ decays.

References

Figure 3: Variation of $\Delta[0.1]$ with the renormalization–scale factor $\xi$, to four loops.