Finite-length soliton solutions of the local homogeneous nonlinear Schrödinger equation

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Abstract

We found a new kind of soliton solutions for the 5-parameter family of the potential-free Stenflo-Sabatier-Doebner-Goldin nonlinear modifications of the Schrödinger equation. In contradistinction to the “usual” solitons like \( \cosh [\beta (x - kt)]^{-\alpha} \exp [i(kx - \omega t)] \), the new Finite-Length Solitons (FLS) are nonanalytical functions with continuous first derivatives, which are different from zero only inside some finite regions of space. The simplest one-dimensional example is the function which is equal to \( \cos [\gamma (x - kt)]^{1+\delta} \exp [i(kx - \omega t)] \) (with \( \delta > 0 \)) for \(|x - kt| < \pi/2 \gamma \), being identically equal to zero for \(|x - kt| \geq \pi/2 \gamma \).

The FLS exist even in the case of a weak nonlinearity, whereas the “usual” solitons exist provided the nonlinearity parameters surpass some critical values.

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1 Introduction

Recently, different authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] discovered an interesting multi-parametric nonlinear homogeneous modification of the Schrödinger equation in the coordinate representation. In the most general form this Stenflo–Sabatier–Doebner-Goldin (SSDG) equation can be written as (we confine ourselves to the case of a free motion and assume $\hbar = m = 1$)

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2 \psi + \Omega\{\psi\} \psi$$

(1)

where the local nonlinear functional $\Omega\{\psi\}$ is as a linear combination of terms $\Delta\psi/\psi$, $(\nabla\psi/\psi)^2$, $|\nabla\psi/\psi|^2$ and their complex conjugated counterparts, so that $\Omega\{\psi\}$ satisfies the homogeneity condition $\Omega\{\gamma \psi\} = \Omega\{\psi\}$ for an arbitrary complex constant $\gamma$. The specific choices of the complex coefficients in the linear combination correspond to the equations describing waves in plasmas with sharp boundaries and in nonlinear media [1, 2, 3, 4, 5]. However, trying to interpret (1) as a quantum mechanical equation one must worry about the conservation of probability. For this reason, the functional $\Omega\{\psi\}$ was chosen in [6, 7, 8] in an explicit real form:

$$\Omega\{\psi\} = \hat{D} \ln |\psi|,$$

where $\hat{D}$ is the second order differential operator $\hat{D}f = a\Delta f + b \cdot \nabla f + c \nabla f \cdot \nabla f$, with real parameters $a, b, c$. However, it was shown in [9, 10] that the normalization could be saved even in the presence of imaginary (antihermitian) nonlinear corrections of a special kind.

The most general parametrization was proposed in [11], where $\Omega\{\psi\}$ was written in terms of real and imaginary parts, $\Omega\{\psi\} = R\{\psi\} + iI\{\psi\}$, as follows,

$$I\{\psi\} = \frac{1}{2} \hat{D}\frac{\nabla^2(\psi^*\psi)}{\psi^*\psi},$$

(2)

$$R\{\psi\} = \hat{D} \sum_{j=1}^{5} \lambda_j \Lambda_j[\psi] = \hat{D} \sum_{j=1}^{5} c_j R_j[\psi],$$

(3)

where all the coefficients $\lambda_j$ and $c_j$ are real, and the functionals $\Lambda_j[\psi]$ or $R_j[\psi]$ are expressed in terms of the derivatives of the wave function or in terms of the probability density $\rho = \psi^*\psi$ and the probability current $j = (\psi^*\nabla \psi - \psi \nabla \psi^*)/2i$.
The coefficients $\lambda_j$ and $c_j$ are related as follows,

$$
\begin{align*}
\lambda_1 &= 2c_2, & c_1 &= \lambda_2 \\
\lambda_2 &= c_1, & c_2 &= \frac{1}{2} \lambda_1 \\
\lambda_3 &= 2c_5 - \frac{1}{2} c_3, & c_4 &= \lambda_5 - \lambda_1 - \lambda_3 \\
\lambda_4 &= c_4, & c_4 &= \lambda_4 \\
\lambda_5 &= 2c_2 + 2c_5 + \frac{1}{2} c_3, & c_5 &= \frac{1}{4} (\lambda_5 + \lambda_3 - \lambda_1)
\end{align*}
$$

More general homogeneous nonlinear functionals, which include as special cases the nonlocal terms proposed by Gisin [12] and by Weinberg [13], were given in [14, 15, 16, 17].

It is not clear, until now, whether nonlinear corrections to the Schrödinger equation of the SSDG type have a physical meaning from the point of view of quantum mechanics (possible experiments which could verify the existence of such corrections were proposed in [18, 19], and the relations between the SSDG-equation and the master equation for mixed quantum states were studied in [15, 19, 20]). Nonetheless, the mathematical structure of the new family of nonlinear equations appears rather rich. In particular, studying this family resulted recently in discovering the nonlinear gauge transformations [21, 22, 23].

The aim of our article is to show another remarkable property of the SSDG equation, namely, the existence of a new type of soliton solutions, which are different from zero in a finite space domain even for arbitrarily small nonlinear coefficients. As far as we know, such kind of solitons was not discussed earlier.
2 Soliton solutions with linear phase

Looking for a *shape invariant* solution to the SSDG equation (1)–(3) with a *linear phase*,

\[ \psi(x, t) = g(x - vt)e^{i(kx - \omega t)}, \]

we obtain the following two equations for the real function \( g(x) \):

\[ (k - v) \frac{\nabla g}{g} = D \left[ \frac{\nabla^2 g}{g} + \left( \frac{\nabla g}{g} \right)^2 \right] \]

\[ (1 - \sigma) \frac{\nabla^2 g}{g} - \xi \left( \frac{\nabla g}{g} \right)^2 - 2\mu k \cdot \frac{\nabla g}{g} = k^2(1 + \eta) - 2\omega, \]

where the new coefficients are defined as

\[ \sigma = 2\tilde{D}\lambda_1 \equiv 4\tilde{D}c_2, \quad \xi = 2\tilde{D} (\lambda_3 + \lambda_5) \equiv 4\tilde{D} (c_2 + 2c_5), \]

\[ \eta = 2\tilde{D} (\lambda_5 - \lambda_3 - \lambda_1) \equiv 2\tilde{D}c_3, \quad \mu = 2\tilde{D} (\lambda_2 + \lambda_4) \equiv 2\tilde{D} (c_1 + c_4). \]

A general solution to eq. (5) in the one-dimensional case is

\[ g_D(x) = \{C_1 + C_2 \exp[(k - v)x/D]\}^{1/2}, \]

with arbitrary constants \( C_1 \) and \( C_2 \). However, the function \( g_D(x) \) cannot be normalized, thus in order to guarantee normalization we impose

\[ k = v, \quad D = 0, \]

i.e. soliton solutions can only exist in the absence of dissipative terms in the Hamiltonian.

The substitution

\[ g(x) = [f(x)]^\alpha, \quad \alpha = \frac{1 - \sigma}{1 - \sigma - \xi}, \]

eliminates the nonlinear term \( (\nabla g/g)^2 \) in eq. (6), such that

\[ \nabla^2 f - 2\kappa \cdot \nabla f + \gamma^2 f = 0, \]

where

\[ \gamma^2 = \left[ 2\omega - k^2(1 + \eta) \right] \frac{1 - \sigma - \xi}{(1 - \sigma)^2}, \quad \kappa = \frac{2\mu k}{1 - \sigma}. \]
Note that $\gamma^2$ is a free parameter, which may assume both positive and negative values, depending on the packet average energy

$$\langle E \rangle \equiv i \int_{-\infty}^{\infty} \psi^*(x, t) \frac{\partial \psi(x, t)}{\partial t} \, dx = \omega.$$ 

A general solution to eq. (8) in the one-dimensional case reads

$$f(x) = e^{sx} \left(C_1 e^{sx} + C_2 e^{-sx}\right), \quad s = \sqrt{\kappa^2 - \gamma^2}. \quad (10)$$

In particular,

(a) For $\gamma^2 < 0$, function (10) goes to infinity when $x \to \pm \infty$ (if both constants $C_1$ and $C_2$ are positive), so a normalizable solution $g(x)$ (eq. (7)) exists only under the condition $\alpha < 0$, i.e., for parameters $\sigma$ and $\xi$ satisfying the inequalities $\sigma < 1$, $\xi > 1 - \sigma$ or $\sigma > 1$, $\xi < 1 - \sigma$ (in other words, these parameters must be located between the straight lines $\sigma = 1$ and $\sigma + \xi = 1$ in the $\sigma \xi$-plane). This means that only strong nonlinearity can give “usual” soliton solutions with exponentially decreasing tails, whose simplest representative ($\mu = 0$) reads

$$g_\ast(x) = [\cosh(\beta x)]^{-|\alpha|}. \quad (11)$$

This conclusion agrees with the results of studies [6, 7, 8], where exponentially confined solitons were found for the nonlinear functionals like $\Omega\{\psi\} = a \Delta(\ln|\psi|)$. Similar solutions to the special cases of the SSDG equation with complex coefficients were studied in [3, 4, 5]. A large family of exact solutions corresponding to the most general Doebner–Goldin parametrization (2)-(3) was found in [24, 25, 26]. However, that family does not contain the solitons with a linear phase.

For example, the solution given in [25] has the same amplitude factor as in (11), but its phase is proportional to $\ln[g(x - kt)]$, so it does not converge to the plane-wave solution of the linear Schrödinger equation when the nonlinear coefficients $D$ and $\tilde{D}$ go to zero.

(b) For $0 < \gamma^2 < \kappa^2$, function (10) goes to infinity only for $x \to +\infty$, while for $x \to -\infty$ it goes to zero (or vise versa). In this case, we cannot obtain a normalizable solution in the form (7) for any value of $\alpha$.

(c) Quite different situation arises when $\gamma^2 > \kappa^2$, then expression (10) shall contain trigonometric functions, and (making a shift of the origin, if necessary) we arrive at a solution to eq.

5
(6) in the form
\[ g_\delta(x) = [e^{\kappa x} \cos(\tilde{\gamma} x)]^{1+\delta} \] (12)
with \( \tilde{\gamma} = \sqrt{\gamma^2 - \kappa^2} \geq 0 \) and
\[ \delta = \frac{\xi}{1 - \sigma - \xi} . \] (13)
At first glance, we have a problem when \( f < 0 \), since function \( f^\alpha \) is ill-defined in this case (unless the exponent \( \alpha \) is an integer). But we notice that if \( \alpha > 1 \) (i.e. \( \delta > 0 \)), then the function \( g(x) = [f(x)]^\alpha \) turns into zero together with its derivative \( g'(x) \) at \( \tilde{\gamma} x = \pi/2 \). This means that there exists an integrable solution with a continuous first derivative, which is localized completely inside a finite domain:
\[
\psi_\delta(x,t) = \begin{cases} 
\cos [\tilde{\gamma}(x - kt)]^{1+\delta} \exp [(1+\delta)\kappa(x - kt) + i(kx - \omega t)] & \text{if } |\tilde{\gamma}(x - kt)| < \pi/2 \\
0 & \text{if } |\tilde{\gamma}(x - kt)| \geq \pi/2
\end{cases}
\]
It is remarkable that such a “finite-length soliton” (FLS) exists for an arbitrarily weak nonlinearity, since the requirement \( \delta > 0 \) implies the inequalities
\[ 0 < \xi < 1 - \sigma \] (14)
which can be satisfied for small values of \( \xi \) and \( \sigma \) (another possibility is \( 0 > \xi > 1 - \sigma \), but it demands \( \sigma > 1 \), meaning a stronger nonlinearity). In terms of the coefficients \( \lambda_j \) and \( c_j \), condition (14) reads
\[
\tilde{D} (\lambda_3 + \lambda_5) > 0 , \quad 2\tilde{D} (\lambda_1 + \lambda_3 + \lambda_5) < 1 \\
\tilde{D} (c_2 + 2c_5) > 0 , \quad 8\tilde{D} (c_2 + c_5) < 1 .
\]
It was shown in [11] that the SSDG equation is Galilean invariant provided that (i) \( c_1 + c_4 = 0 \) and (ii) \( c_3 = 0 \). In our notation this means \( \mu = \eta = 0 \). Thus we arrive at the 3-parameter family of homogeneous local nonlinear functionals admitting Galilean-invariant and spatially confined soliton solutions to eq. (1):
\[
\Omega\{\psi\} = \frac{1}{2} \left\{ \sigma \text{Re} \frac{\nabla^2 \psi}{\psi} + \nu \text{Im} \left[ \nabla \cdot \left( \nabla \frac{\psi}{\psi} \right) \right] + \xi \left[ \text{Re} \frac{\nabla \psi}{\psi} \right]^2 + \sigma \left[ \text{Im} \frac{\nabla \psi}{\psi} \right]^2 \right\} \] (15)
\[
= \frac{1}{8} \left\{ 2\sigma \frac{\nabla^2 \rho}{\rho} + 4\nu \nabla \cdot \left( \frac{\mathbf{j}}{\rho} \right) + (\xi - \sigma) \left( \frac{\nabla \rho}{\rho} \right)^2 \right\} . \] (16)
Note that parameter $\nu = 2\tilde{D}\lambda_2 = 2\tilde{D}\lambda_1$ does not make any influence on the discussed solutions, thus only derivatives of the density $\rho$ and not of the current density $j$ are important for soliton solutions. The only crucial parameter is $\xi$, so, the simplest 1-parameter nonlinear functional admitting FLS solution reads ($\nu = \sigma = 0$),

$$\Omega \{ \psi \} = \frac{\xi}{2} \left[ \text{Re} \left( \frac{\nabla \psi}{\psi} \right) \right]^2 = \frac{\xi}{8} \left( \frac{\nabla \rho}{\rho} \right)^2, \quad 0 < \xi < 1,$$

(17)

The explicit form of all FLS-solutions is as follows,

$$\psi_{k\gamma}(x, t) = \begin{cases} [f_{\gamma}(x - k t)]^{1+\delta} e^{i(kx - \omega_{k\gamma} t)} & \text{if } |x - kt| \in \mathcal{R}^{(+)}(f_{\gamma}) \\ 0 & \text{if } |x - kt| \notin \mathcal{R}^{(+)}(f_{\gamma}) \end{cases},$$

(18)

where $f_{\gamma}(x)$ is any positive solution to the Helmholtz equation $(\nabla^2 + \gamma^2) f = 0$ with an arbitrary real constant $\gamma$, and $\mathcal{R}^{(+)}(f_{\gamma})$ is the internal part of a space region bounded by a closed surface (in 3 dimensions) or a closed curve (in 2 dimensions) determined by the equation $f_{\gamma}(x) = 0$ (in principle, this region may be multi-connected). To avoid any ambiguity, we define the nonlinear functional $\Omega \{ \psi \}$ for $\psi = 0$, assuming $\Omega \{ \psi \} \psi = 0$ on such points. Although the solution (18) is non-analytical, it has continuous first derivatives in all points of the space.

Under the Galilean invariance symmetry ($\eta = 0$), the frequency $\omega_{k\gamma}$ (eq. (9)) equals

$$\omega_{k\gamma} = \frac{1}{2} k^2 + \frac{1}{2} \gamma^2 \frac{(1 - \sigma)^2}{1 - \sigma - \xi},$$

(19)

so, the usual dispersion relation of linear Quantum Mechanics ($\omega_k = \frac{1}{2} k^2$) is modified by an additional constant term (proportional to $\gamma^2$) that may be interpreted as an “internal energy” of the wave packet (18) due to its confinement, whereas $k^2/2$ is the energy of the “center-of-mass” motion. For $\gamma = 0$, then $\nabla^2 f_0 = 0$, and considering $f_0 = 1$ we obtain a plane wave solution to the Schrödinger equation with the wave number $k$.

The concrete shapes of the FLS-packets in 2 and 3 dimensions may be quite diverse. The most symmetric solutions are given by eq. (18) with $f_{\gamma}(x)$ in the form of $J_0(\gamma |x|)$ or $\sin(\gamma |x|)/|x|$ (in 2 and 3 dimensions, respectively). However, there exist also asymmetric packets with functions $f_{\gamma}(x)$ proportional to $J_m(\gamma |x|) \cos(m \varphi)$ or $j_l(\gamma |x|) Y_{lm}(\theta, \varphi)$, where $J_m(x)$ is the Bessel function, $j_l(x)$ is the spherical Bessel function (proportional to the Bessel function with the semi-
integral index), and \( Y_{lm}(\vartheta, \varphi) \) is a real-valued analog of the spherical harmonics, \( \vartheta, \varphi \) being the usual angular variables.

### 3 Discussion

Although the substitution (7) was used already in [8, 11], the existence of the FLS solutions was not noticed before, perhaps, because the authors of the cited papers were looking for solutions of the stationary Schrödinger equation or for usual exponentially confined solitons. It should be noted that the substitution (7) linearizes only the equation (6) for the real amplitude of the special form of solution (4), but not eq. (1) as a whole. As was shown in [21, 22], the nonlinear gauge transformation (NGT)

\[
\psi \mapsto \psi' = |\psi| \exp \left[ i (z^* \ln \psi + z \ln \psi^*) \right]
\]

(where \( z \) is a complex parameter) transforms any SSDG equation into an equation of the same kind, but with another set of coefficients, and all equations can be classified in accordance with the possible values of 5 invariants of the NGT family. In the case of eq. (1) with the functional (17) the invariants are as follows (we use the same notation as in [22]):

\[
\tau_1 = \tau_4 = 0, \quad \tau_2 = \frac{1}{8}, \quad \tau_3 = -1, \quad \tau_5 = -\frac{\xi}{16}.
\]

If we had \( \tau_5 = 0 \), then the whole SSDG equation could be linearized by means of a suitable NGT. It is the nonzero value of parameter \( \xi \) that prevents the linearization and makes possible the existence of the FLS solutions. It is interesting to note in this connection, that various special cases of the general SSDG equation were considered before the general structure of the equation was found in [8, 11], but the coefficients were chosen in such a way that the parameter \( \xi \) was almost always taken equal to zero [27, 28, 29]. The only exception is Ref. [30], where the only nonzero coefficient in the \( \lambda_j \)-parametrization is \( \lambda_5 \); however, in this case not only \( \xi \neq 0 \), but \( \eta \neq 0 \), too, so the Galilean invariance symmetry is absent.

Turning to a possible physical meaning of the FLS solutions, we may say that they realize the “dream of De Broglie”, in a sense that they permit to identify a quantum particle with a
nonspreading wave packet of finite length travelling with a constant velocity in the free space. Earlier, the only proposed nonlinear equation that resulted into a nonspreading wave packet solution for a free particle was the one proposed by Bialynicki-Birula and Mycielski [31] (BBM), with the nonlinear term \( \Omega\{\psi\} = -b \ln |\psi|^2 \) (it was shown recently that this term can arise if one applies to the SSDG equation a nonlinear gauge transformation with time-dependent coefficients [22]). The solitons of the BBM-equation are Gaussian wave packets (gaussons) whose constant width is inversly proportional to the nonlinear coefficient \( b \). In contrast to the gaussons, the FLS of the SSDG equation have the width \( \gamma^{-1} \) as a free parameter, independent of the nonlinear coefficients. The most attractive feature of FLS solutions is that they exist for an arbitrarily weak nonlinearity. Consequently, the superposition principle of quantum mechanics, which is verified, from the experimental point of view, with a limited accuracy, does not rule out the nonlinear terms like (17) immediately. On the contrary, new experiments on the verification of (non?)linearity of quantum mechanics could be proposed, which would take into account the FLS phenomenon.

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