Gravitational Clustering to All Perturbative Orders

Elcio Abdalla\textsuperscript{a} and Roya Mohayaee\textsuperscript{b}
\textsuperscript{a}Instituto de Física, USP, São Paulo
\textsuperscript{b}HEP, ICTP, Trieste

We derive the time evolution of the density contrast to all orders of perturbation theory, by solving the Einstein equation for scale-invariant fluctuations. These fluctuations are represented by an infinite series in inverse powers of the radial parameter. In addition to the standard growing modes, we find infinitely many more new growing modes for open and closed universes.

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The large scale structure of the universe is believed to have grown, due to gravitational instability, from small primordial density fluctuations. These fluctuations are fully characterized by the density contrast, \( \delta(t, \vec{x}) = \delta \rho(t, \vec{x})/\rho(t) \). At any given time, the connected correlation functions of the density contrast, determine the power spectrum and the space distribution of the matter in the universe [1–5]. The time dependence of the density contrast, on the other hand, determines if the inhomogeneities would indeed grow, or would just oscillate or decay, in the first place.

In this article, we are primarily concerned with the time evolution of the density contrast. In almost all standard methods, this is determined by the continuity, Euler and Poisson equations in the Newtonian regime [1,6–10] and by the Einstein equation in the relativistic era [1,6–10]. In the nonlinear regime, the perturbation is carried to second order and new growing modes containing nonlocal terms are obtained for dust universes [1,5,11]. The local nonlinear terms have also been studied to all orders for the flat universe [4,12]. A single complete treatment covering all possible cases and also extending to an arbitrary perturbative order does not exist.

Our starting point is not the usual equations of fluid hydrodynamics but the Einstein equation which naturally covers the present era too. We expand the Einstein equation around the background Friedmann universe by scale-invariant fluctuations. These fluctuations are expressed as an infinite series in inverse powers of the radial parameter. This method, which assumes spherical symmetry, allows us to obtain the full expression for the growth rate of density inhomogeneities in Newtonian, relativistic, zero and non-zero curvatures, linear and local nonlinear regimes in a single treatment. Thus in this way, we make major simplifications on current perturbative methods. In a flat universe, the leading growing modes are given by an infinite series in the matter and radiation-dominated eras. For the open and closed universes we find, in addition to the modes presented in the literature [8,11], infinitely many more new growing modes which are also given by infinite sums. In the nonlinear regime, we also find new growing modes for closed and open universes in the extreme limits of small and large times.

Contrary to the standard analysis, where the density contrast is often taken to be a random Gaussian field, we take it to be a scale-invariant quantity. That the standard growing modes are contained in our results, is a further solid evidence that the time evolution of the density contrast is unaffected by its statistical properties.

Since, the inverse radial parameter now plays the role of the perturbation parameter, the order by which the structures at different scales are formed is easily determined. Our perturbation scheme implies a bottom-up or a hierarchical clustering scenario for the formation of structures and thus a universe dominated by cold dark matter [8].

To account for the inhomogeneity of space-time, we use the spherically symmetric metric,

\[
\text{diag}(g_{\mu\nu}) = \begin{pmatrix} -1, & R_p^2(t, r)/1 - kr^2, & R_p^2(t, r)r^2, & R_p^2(t, r)r^2 \sin^2 \theta \end{pmatrix},
\]

where the scale factor \( R_p(t, r) \) is a function of both time and coordinate. This metric does not assume homogeneity and is contained in the Tolman metric for a pressureless universe [13], although not restricted to this era. We assume that the inhomogeneities are given by scale-invariant fluctuations and expand the scale factor as

\[
R_p(t, r) = R(t) + \sum_{n=0}^{\infty} \frac{\delta R_n(t)}{r^n},
\]

around the background Friedmann-Robertson-Walker scale factor, \( R(t) \). This is a genuine perturbation of the metric and not just a gauge mode. A solution \( \zeta_\mu \) of \( \delta g_{\mu\nu} = \zeta_{\mu\nu} + \zeta_{\nu\mu} \) where \( \delta g_{\mu\nu} \) is the metric perturbation, such that \( \delta g_{00} = 0 \) and \( \delta g_{rr} \sim t^a r^{-n} \), for a general \( a \) and \( n \), cannot be constructed [14].

The time-time and radial-radial components of the Einstein equation are expanded as in (2) and, on using the equation of state \( P = \omega \rho \), are reduced to the following second-order recursive differential equation [15]:

\[
\delta(t, \vec{x}) = \delta \rho(t, \vec{x})/\rho(t).
\]
where summations over the indices $n, m, l$ and $s$ are implied and $\delta R_0$ represents $R + \delta R_0$ when $n = 0$. We have solved the above equation by a Maple program to very large orders for a flat universe, i.e. $k = 0$ [16]. The solutions are substituted back in the time-time component of the Einstein equation and a series expression for large orders for a flat universe, solved the above equation by a Maple program to very large orders for a flat universe, i.e. $k = 0$ [16].

The solutions can be used in the Einstein equation (3) for open and closed universes can also be solved analytically at low orders and by a fully algebraic Maple program at higher orders [15] in the limits of small and large times. At small times, the first growing modes $\delta_+ \sim t^{2/3}$ and $\delta_+ \sim t$ in the matter and radiation-dominated eras occur at the lowest order in the perturbation series, i.e. at the $r$-independent order [15]. At higher orders the solution contains oscillatory and or decaying modes only. Therefore, open and closed universes behave as a flat universe at small times in the linear regime, as expected. However, this statement is not completely true. In a flat universe the first growing modes appear at the third, $r^{-3}$, rather than the first, order in the perturbation expansion, as expressed by equations (4) and (5). In other words, at small times the inhomogeneities will enter the nonlinear phase faster in the open and closed universes than in the flat universe. Therefore, unlike common expectations, structures can grow faster in an open universe than in a flat universe.

At large times, a new growing mode arises in an open universe. Taking the asymptotic limit of the hypergeometric solutions to the Einstein equation [15] we obtain

$$\delta_+ \sim \frac{t^{3\omega/2}}{r}. \tag{8}$$

This result indicates that, at very large times, inhomogeneities will only grow in an open universe if the radiation pressure is non-vanishing. However, since the baryonic matter cannot grow in the radiation-dominated era due to its strong coupling to the radiation, the above mode is only relevant for the growth of the perturbation in the non-baryonic dark matter component.

Nonlinear-nonlocal growing modes [1] which would arise if radial coordinates, in expression (3), were taken to be at different points, shall be discussed in future works.

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[16] The complete algebraic Maple programs can be found at the site http://www.fma.if.usp.br/~fractals/perturbation.