Cosmological Magnetic Fields from Primordial Helicity

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Abstract

Primordial magnetic fields may account for all or part of the fields observed in galaxies. We consider the evolution of the magnetic fields created by pseudoscalar effects in the early universe. Such processes can create force-free fields of maximal helicity; we show that for such a field magnetic energy inverse cascades to larger scales than it would have solely by flux freezing and cosmic expansion. For fields generated at the electroweak phase transition, we find that the predicted wavelength today can in principle be as large as \(\sim 10\) kpc, and the field strength can be as large as \(\sim 10^{-10}\) G.
The origin of galactic and intergalactic magnetic fields is an unsolved problem [1]. The standard $\alpha$-$\Omega$ dynamo theory of galactic magnetic fields [2] has been criticized for not adequately taking into account the back reaction of the growing magnetic field, and in any case, the theory requires a seed field of unknown origin. There is now an extensive literature examining the possibility that magnetic fields were created by exotic processes in the early universe, although most such processes, if they work at all, produce fields on scales too small to be of interest to astronomy.

In magnetohydrodynamics, energy can be transferred from small to large scales by a process known as the inverse cascade. As shown in pioneering work by Pouquet and collaborators [3], a critical ingredient of the inverse cascade mechanism is the presence of substantial magnetic helicity (although non-helical cascades have also been investigated [4]). The idea that magnetic helicity may drive an inverse cascade from microphysical magnetic fields to large-scale cosmological fields has been advocated by Cornwall [5], and investigated subsequently by Son [6], who proposed scaling properties we will verify below. The present work has two goals: 1.) To work towards an analytic understanding of the cascade process as an initial-value problem appropriate for cosmology, by studying similarity solutions of the MHD equations in the presence of helicity; and 2.) To apply this understanding to proposed mechanisms which create helical primordial fields via pseudoscalar processes in the early universe. A preliminary account has been given in [7].

The early-universe processes we consider can be thought of as arising from the evolution of an electrically neutral pseudoscalar field $\phi$, coupled to electromagnetism through a Lagrange density

$$\mathcal{L} = -\frac{1}{4} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} = \phi \mathbf{E} \cdot \mathbf{B},$$

(1)

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual electromagnetic field strength tensor. (For simplicity we elide the distinction between electromagnetic and hypercharge fields; they are related by factors of order unity.) The pseudoscalar in question may represent an axion [8], a more general pseudo-Goldstone boson [9], or may model the effect of a chemical potential for right-handed electron number (in which case the chemical potential $\mu$ is given by the time derivative of $\phi$). This last scenario has been proposed by Joyce and Shaposhnikov [10], who showed that it could lead to fields at the electroweak phase transition with magnitude $B_{EW} \sim 10^{22}$ G and coherent over length scales $\lambda_{EW} \sim 10^{-8} H_{EW}^{-1} \sim 2 \times 10^{-9}$ cm.

The field equations for electromagnetism in the presence of the interaction (1) are (in
units with $c = 1$)
\[
\begin{align*}
\partial_t \mathbf{E} &= \nabla \times \mathbf{B} - \mathbf{J} - \dot{\phi} \mathbf{B} - \nabla \phi \times \mathbf{E} \\
\nabla \cdot \mathbf{E} &= \rho_E + \nabla \phi \cdot \mathbf{B} \\
\partial_t \mathbf{B} &= -\nabla \times \mathbf{E} \\
\nabla \cdot \mathbf{B} &= 0 ,
\end{align*}
\]
(2)
along with Ohm’s law, $\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where $\mathbf{v}$ is the velocity of the fluid. In the cosmological context of interest here, $\phi$ is effectively homogeneous ($\nabla \phi = 0$) and the charge density $\rho_E$ vanishes. The magnetic field then satisfies
\[
(\nabla^2 - \partial_t^2) \mathbf{B} = \sigma [\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B})] - \dot{\phi} \nabla \times \mathbf{B} .
\]
(3)

For the case that $\sigma = 0$ it has been shown that solutions of (3) can be unstable to exponential growth in the magnetic field [11]; Garretson, Field and Carroll [9] considered field production by such a mechanism during inflation, concluding that the resulting amplitudes were too small to be of astrophysical interest. If one ignores the displacement current $\partial_t \mathbf{E}$ and sets $\dot{\phi} = 0$, (3) reduces to the induction equation of dissipative MHD. In the situation of interest here the conductivity is non-negligible; the electric field changes slowly on the timescales of interest, so we neglect $\partial_t \mathbf{E}$; and the bulk velocity of the fluid is small, so we neglect $\mathbf{v}$ as well. Hence the appropriate form of (3) is
\[
(\partial_t - \eta \nabla^2) \mathbf{B} = -\eta \dot{\phi} \nabla \times \mathbf{B} ,
\]
(4)
where $\eta = 1/\sigma$ is the resistivity.

It is useful at this point to go to Fourier space and decompose $\mathbf{B}(\mathbf{k})$ into modes of definite helicity (or equivalently, circular polarization), $\mathbf{B}(\mathbf{k}) = B_+ \hat{u}_+ + B_- \hat{u}_-$; here $\hat{u}_\pm = \hat{u}_1 \pm i \hat{u}_2$, where $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3 = \mathbf{k}/k$ form a right-handed orthonormal basis. Then solutions to (4) are of the form
\[
B_\pm (\mathbf{k}, t) = B_\pm (\mathbf{k}, 0) \exp \left[ -\eta \int k (k \pm \dot{\phi}) \, dt \right] ,
\]
(5)
For $\dot{\phi}$ negative, the $B_+$ modes will grow exponentially if $k$ is less than $-\dot{\phi}$, with maximum growth rate for $k = -\dot{\phi}/2$, while the $B_-$ modes decay away. When $\dot{\phi} = 0$ the field undergoes Ohmic decay, although Joyce and Shaposhnikov show that this effect is unimportant for the relevant wavelengths in their scenario [10].
The expectation value of the magnetic energy density of an isotropic plasma in a volume $V$ can be expressed in terms of an energy spectrum $e^M(k)$ as

$$E^M = \frac{1}{2}V^{-1} \int_V B^2 \, d^3x = \int_0^\infty e^M(k) \, dk , \quad (6)$$

where

$$e^M(k) = 2\pi k^2 \langle B(k) \cdot B^*(k) \rangle . \quad (7)$$

Similarly, the expectation value of the magnetic helicity (or Chern-Simons) density can be written

$$H^M = V^{-1} \int_V A \cdot B \, d^3x = \int_0^\infty h^M(k) \, dk , \quad (8)$$

where

$$h^M(k) = 4\pi k^2 \langle A(k) \cdot B^*(k) \rangle . \quad (9)$$

Note that while $e^M(k)$ is non-negative, $h^M(k)$ can be of either sign. The helicity and energy spectra satisfy an inequality:

$$|h^M(k)| \leq 2k^{-1}e^M(k) . \quad (10)$$

We say the field is “maximally helical” if, for every $k$, $h^M(k)$ is of the same sign and saturates this inequality. (See [12] for further discussion.)

In Coulomb gauge ($\mathbf{k} \cdot \mathbf{A} = 0$), the modes of the vector potential and the magnetic field are related by $B_\pm = \pm kA_\pm$. The energy and helicity spectra are then

$$e^M(k) = 4\pi k^2 \left( |B_+|^2 + |B_-|^2 \right) ,$$

$$h^M(k) = 8\pi k \left( |B_+|^2 - |B_-|^2 \right) . \quad (11)$$

Since the $B_+$ modes are amplified and the $B_-$ modes suppressed by the evolution of $\phi$, only $B_+$ will contribute; such a field satisfies $h^M(k) = 2k^{-1}e^M(k)$, and is therefore maximally helical.

If the spectrum of a maximally helical magnetic field is strongly peaked around some wavenumber $k_p$, the configuration will be force-free: $\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} = 0$. This can be seen by considering

$$\nabla \times \mathbf{B}(\mathbf{x}) = \int kB_+(k)\hat{u}_+ e^{i\mathbf{k} \cdot \mathbf{x}} \, d^3k \sim k_p \mathbf{B}(\mathbf{x}) . \quad (12)$$
Force-freedom has been verified in numerical simulations [13]. The force-free condition plays an important role in the evolution of the fields, working for example to protect them from the Silk damping discussed in [14].

We turn now to the principles behind the inverse cascade. Pouquet et al. [3] studied MHD turbulence in the eddy-damped quasi-normal Markovian (EDQNM) approximation, in which eddy damping is used to close the nonlinear equations in Fourier space. These equations preserve the ideal invariants of total energy and magnetic helicity. Pouquet et al. found numerically that if the helicity $H^M$ is injected at a constant rate $\dot{H}^M$, $H^M$ grows linearly with time and the helicity spectrum peaks sharply at a wave number $k_p(t) \propto 1/t$. Here we develop a semi-analytic understanding of this result which can be generalized to our problem, in which helicity is injected at some initial time, after which $\dot{H}^M = 0$.

To this end we look for similarity solutions of the form

$$h^M(k, t) = g(t)s(\xi), \quad \text{(13)}$$

where the shape function $s(\xi)$ depends on the wavenumber scaled to the peak:

$$\xi = \frac{k}{k_p(t)}. \quad \text{(14)}$$

We normalize the shape function by $\int s(\xi) d\xi = 1$, which allows us to express the time dependence of (13) as

$$g(t) = \frac{H^M(t)}{k_p(t)}. \quad \text{(15)}$$

Given this ansatz and $H^M(t)$, a solution will be fully specified by the shape $s(\xi)$ and the peak wavenumber $k_p(t)$.

Pouquet et al. [3] show that, in the context of the EDQNM approximation, nonlocal MHD effects lead to an evolution equation for the helicity of the form

$$\partial_t h^M(k) = \alpha_k \varepsilon^M(k) \quad \text{(16)}$$

(cf. their eq. [3.4]), where

$$\alpha_k = \frac{4}{3} \int_{k/a}^{\infty} \theta_{kqq} q^2 h^M(q) dq \quad \text{(17)}$$

(cf. their eq. [3.6], and note that their conventions differ slightly from ours). Here, $a$ is a small dimensionless parameter (taken to be 0.26 in [3]) and the inverse decay constant $\theta_{kqq}$ is given by $\theta_{kqq} = (\mu_k + 2\mu_q)^{-1}$, where the eddy-damping rate $\mu_k$ is given by

$$\mu_k = \frac{4}{3} \int_{k/a}^{\infty} \theta_{kqq} q^2 h^M(q) dq \quad \text{(17)}$$
\[ \mu_k = (\nu + \eta)k^2 + 0.36 \left( \int_0^k p^2 e_p dp \right)^{1/2} + \sqrt{\frac{2}{3}} k \left( \int_0^k e_p dp \right)^{1/2}. \]  

(18)

The first term represents damping by viscosity \( \nu \) and resistivity \( \eta \). In astrophysical applications \( \nu \) and \( \eta \) are typically of order \( 10^{-6} \) times the succeeding terms, so this effect is important only at very large wavenumbers; we will ignore it henceforth. The second term parameterizes damping due to self-distortion, and the third that due to the nonlinear interaction of Alfvén waves.

The condition of maximal helicity, \( e^M(k) = \frac{1}{2} k h^M(k) \), is preserved under the evolution of the fields [3,13]. We can therefore use (13) to express (18) as

\[ \mu_k = (H^M k_p^3)^{1/2} F(\xi), \]  

(19)

where

\[ F(\xi) = 0.36 \left( \int_0^\xi \zeta^3 s(\zeta) d\zeta \right)^{1/2} + 0.82 \xi \left( \int_0^\xi \zeta s(\zeta) d\zeta \right)^{1/2}. \]  

(20)

With these definitions, the evolution equation (16) can be written

\[ \frac{\dot{H}^M}{H^M} - \frac{\dot{k}_p}{k_p} \left( 1 + \frac{s'}{s} \right) = \left( \frac{1}{2} H^M k_p^3 \right)^{1/2} G(\xi), \]  

(21)

where \( s' \equiv ds/d\xi \) and

\[ G(\xi) = \frac{4}{3} \xi \int_{\xi/a}^\infty \frac{\zeta^2 s(\zeta)}{F(\xi) + 2F(\zeta)} d\zeta. \]  

(22)

Equation (21) is valid for \( k \sim k_p \), in particular for \( k = k_p(\xi = 1) \), where \( s' = 0 \) by definition. Thus, \( k_p(t) \) solves the differential equation

\[ \frac{d}{dt} \ln \left( \frac{H^M}{k_p} \right) = \left( \frac{1}{2} H^M k_p^3 \right)^{1/2} G(1), \]  

(23)

where \( G(1) \) is a dimensionless constant of order unity. Pouquet et al. [3] considered the case \( \dot{H}^M = \text{const} \), so \( H^M = \dot{H}^M t \), in which case the solution to (23) is

\[ k_p(t) = k_p(t_i) t_i/t, \]  

(24)

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provided that

\[ G(1) = \left[ \frac{1}{8} \dot{H}^M k_p^3(t_i) t_i^3 \right]^{-1/2}. \]  

(25)


Reassured that we can find similarity solutions consistent with the numerical results, we turn to the \( \dot{H}^M = 0 \) case, relevant to cosmology once the pseudoscalar \( \phi \) has stopped evolving. Then (21) implies that similarity solutions will once again exist, with

\[ k_p(t) = k_p(t_i) \left( \frac{t_i}{t} \right)^{2/3}, \]  

(26)

provided that

\[ G(1) = \left[ \frac{9}{8} H^M K^3(t_i) t_i^2 \right]^{-1/2}. \]  

(27)

Note that (26) verifies a result of Son [6], derived from different arguments.

In addition to the peak wavelength, we also want to know the rms magnetic field, \( B_{\text{rms}} = \langle B^2 \rangle^{1/2} \). For our maximally helical fields we have

\[ \langle B^2 \rangle = 2 \int_0^\infty e^M(k) \, dk = \int_0^\infty kh^M(k) \, dk = H^M k_p(t) \int_0^\infty \xi s(\xi) \, d\xi. \]  

(28)

The integral in the last expression is independent of time. Thus, when \( H^M = \text{constant} \), (26) implies

\[ B_{\text{rms}}(t) = B_{\text{rms}}(t_i) \left( \frac{t_i}{t} \right)^{1/3}. \]  

(29)

The preceding discussion has assumed a flat spacetime background; it is straightforward to adapt these results to an expanding Robertson-Walker spacetime with metric \( ds^2 = -dt^2 + R^2(t)dx^2 = R^2(t^*)(-dt^*2 + dx^2) \), where \( R \) is the scale factor and \( t^* \) is the conformal time. In the radiation-dominated era, the complete set of MHD equations is conformally invariant (see for example [15]), and our results will apply directly to the conformally transformed quantities

\[ B_{\text{rms}}^* = R^2 B_{\text{rms}}, \quad k_p^* = R k_p. \]  

(30)

Thus, the inverse cascade will be characterized by \( k_p^*(t^*) \propto (t^*)^{-2/3} \) and \( B_{\text{rms}}^*(t^*) \propto (t^*)^{-1/3} \), where the conformal time in the radiation-dominated era is given by \( t^* = 2(t_{\text{EQ}}/R_{\text{EQ}}) t^{1/2} \).
in which the subscript EQ refers to the epoch of matter-radiation equality. The physical quantities at this epoch are therefore related to their initial values by

\[
B_{\text{rms}}(t_{\text{EQ}}) = R_{\text{EQ}}^{-2} B_{\text{rms}}^*(t_*)^{-1/3} \\
= R_{\text{EQ}}^{-2} \left( \frac{t_*}{t_{\text{EQ}}} \right) B_{\text{rms}}^*(t_*) \\
= R^2(t_i) \left( \frac{t_i}{t_{\text{EQ}}} \right)^{1/6} B_{\text{rms}}(t_i)
\]

(31)

and

\[
k_p(t_{\text{EQ}}) = R_{\text{EQ}}^{-1} k_p^*(t_*)^{2/3} \\
= R_{\text{EQ}}^{-1} \left( \frac{t_*}{t_{\text{EQ}}} \right) k_p^*(t_*) \\
= \frac{R(t_i)}{R_{\text{EQ}}} \left( \frac{t_i}{t_{\text{EQ}}} \right)^{1/3} k_p(t_i)
\]

(32)

It is possible that additional inverse cascade occurs in the matter-dominated era following \( t_{\text{EQ}} \). Pending further study, we will assume here that the effect of this is negligible, and so the following estimates of the characteristic scale of the field today are lower limits. Thus, for \( t > t_{\text{EQ}} \) the field is frozen in, the wavenumber scales as \( R^{-1} \), and the magnetic field as \( R^{-2} \). Their values today are thus

\[
B_{\text{rms}}(t_0) = \frac{R^2(t_i)}{R_0^2} \left( \frac{t_i}{t_{\text{EQ}}} \right)^{1/6} B_{\text{rms}}(t_i)
\]

(33)

and

\[
k_p(t_0) = \frac{R(t_i)}{R_0} \left( \frac{t_i}{t_{\text{EQ}}} \right)^{1/3} k_p(t_i)
\]

(34)

In summary, a maximally helical primordial magnetic field created at time \( t_i \) (during radiation domination), with initial amplitude \( B_{\text{rms}}(t_i) \) and initial coherence length \( \lambda(t_i) = 2\pi/k_p(t_i) \), will undergo an inverse cascade, increasing its length scale by a factor \((t_{\text{EQ}}/t_i)^{1/3}\) over and above stretching due to the expansion of the universe, while its amplitude is diluted by an additional factor \((t_i/t_{\text{EQ}})^{1/6}\).

For purposes of illustration, let us consider the fate of a magnetic field created at the electroweak phase transition (\( T_{\text{EW}} = 200 \text{ GeV} \)), so \( t_i = t_{\text{EW}} = 6 \times 10^{-12} \text{ sec} \). We express the initial coherence length of the field in terms of the Hubble radius, \( \lambda(t_{\text{EW}}) = f_{\lambda} H_{\text{EW}}^{-1} = 0.4 f_{\lambda} \text{ cm} \), and the initial amplitude in terms of the total energy density, \( B_{\text{rms}}(t_{\text{EW}}) = f_B \sqrt{8\pi \rho_{\text{EW}}} = \ldots \)
$2 \times 10^{25} f_B$ Gauss, where $f_{\lambda}$ and $f_B$ are dimensionless factors less than unity. (We have switched here from Lorentz-Heaviside units to CGS in order to express the magnetic field in Gauss.) The field today will be coherent over scales

$$\lambda(t_0) = 5 \times 10^{22} f_{\lambda} \text{ cm} = 20 f_{\lambda} \text{ kpc} ,$$

with amplitude

$$B_{\text{rms}}(t_0) = 4 \times 10^{-10} f_B \text{ Gauss} .$$

In the electroweak case, then, the characteristic length scale of the field has been amplified by a factor of $(t_{\text{EQ}}/t_{\text{EW}})^{1/3} = 6 \times 10^7$ more than would be expected for a frozen-in configuration.

If the initial field is coherent over the Hubble radius at the electroweak scale ($f_{\lambda} \sim 1$) and comparable in energy to the total energy ($f_B \sim 1$), this results in a length scale of $\sim 20$ kpc and an amplitude of $\sim 10^{-9}$ Gauss. According to Dolag et al. [16], the primordial field required to explain Faraday rotation measures of the Coma cluster is $\sim 10^{-9}$ Gauss. The observations are consistent with scales of $\sim 60$ kpc. Thus, if $f_{\lambda} \sim f_B \sim 1$, primordial helicity could explain the fields in the Coma cluster.

However, Joyce and Shaposhnikov [10] have estimated that their scenario for helical field generation at the electroweak scale results in fields with $f_{\lambda} \sim 10^{-8}$ and $f_B \sim 2 \times 10^{-3}$. While intriguing (for example as a candidate seed field for a galactic dynamo), these parameters fall short of providing a sufficient explanation for the fields seen in galaxies today. It is therefore worth considering variations on this mechanism, perhaps with different dynamics for the pseudoscalar $\phi$, or field creation at a later epoch such as the QCD scale. Mechanisms which create large-amplitude fields without appreciable helicity will undergo significantly less (if any) inverse cascade, and thus have a difficult time leading to fields of astrophysical significance in the present universe.

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REFERENCES


