A Note on Entanglement Entropy and Conformal Field Theory

Dmitri V. Fursaev

Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, 141 980 Dubna, Russia
e-mail: fursaev@thsun1.jinr.ru

Abstract

It is pointed out that the entanglement entropy of quantum fields near the horizon of a two-dimensional black hole can be derived by means of the conformal field theory. This can be done in a way analogous to the computation of the entropy of BTZ black holes. The important feature of the considered case is that the degrees of freedom of the conformal theory are states localized in the physical space-time.
In the last months there was a large interest to the conjectured correspondence between field theories on asymptotically anti-de Sitter (AdS) space-times and conformal field theories (CFT) on the boundary of AdS [1]–[3]. In particular, this correspondence is a key moment for understanding why the entropy of BTZ black holes [4] can be computed in the framework of the boundary CFT (see [5], [6] and also references therein).

It should be noted however that in the Strominger’s computations of the entropy [7] CFT is a dual theory. Consequently, it does not explain what are the degrees of freedom of the black hole itself and where are they localized (for a discussion of this issue see, e.g., [8]). To put it in another way, the equivalence between bulk and boundary theories is well established on the level of thermodynamics\(^1\), but still there is no a definite opinion about the microscopic structure of the bulk gravity corresponding to states of CFT.

The aim of this note is to draw an attention to a situation when the degrees of freedom of the conformal field theory can be microscopic states in the physical space-time. This example is black holes in two dimensions. The entanglement entropy of quantum fields which appears due to the presence of the black hole horizon can be correctly reproduced in the leading approximation by means of 2D CFT. This happens for a very simple reason: near the horizon all quantum fields are effectively massless and thus conformally invariant.

The fact that CFT can be used for computation of the entanglement (or geometric) entropy is not new. This was demonstrated in [10] by Holzhey, Larsen, and Wilczek who suggested several elegant methods to compute the entropy by using CFT and the replica method. We want to show how come to the same result in a different way which is parallel to the computations by Strominger [7] of the BTZ black hole entropy. As a practical application, our results may be useful for derivation of the entropy of two-dimensional black holes (a recent research in this direction can be found in [11]).

To begin with let us recall why the masses have no effect on the entanglement entropy in the leading approximation. The exterior region of a static two-dimensional black hole is described by the metric

\[
ds^2 = -g(x)dt^2 + g^{-1}(x)dx^2, \quad x_h < x \leq x_b, \tag{1}
\]

where coordinates \(x_h\) and \(x_b\) correspond to the positions of the horizon and the boundary, respectively. On the horizon \(g(x_h) = 0\). For non-extremal black holes \(g'(x_h)\) is not zero, and one can define the “surface gravity” constant \(k = g'(x_h)/2\). Let us consider as an example a scalar field \(\phi\) on space-time (1). The equation for \(\phi\) is

\[
(-\nabla^2 + \xi R + m^2)\phi(x) = 0 \quad . \tag{2}
\]

It is reduced to a relativistic analog of the Schroedinger equation for wave functions \(\phi_\omega\) of single-particle excitations with frequencies \(\omega\). By making in (2) the substitution\(^1\)

\(^1\)This means that the Euclidean Liouville action [9] induced by CFT on the boundary correctly reproduces the classical free energy of the BTZ black hole, see, e.g., [5].
\( \phi(t, x) = \exp(-i\omega t)\phi_\omega(x) \) one finds

\[
\bar{H}^2 \phi_\omega = \omega^2 \phi_\omega ,
\]

\[
\bar{H}^2 = -\partial_y^2 + g(\xi R + m^2) ,
\]

where coordinates \( y \) and \( x \) are related as \( dy = dx/g \). As follows from (4), near the horizon all mass terms can be neglected because of the factor \( g \) and the single-particle Hamiltonian \( \bar{H} \) is just \( \sqrt{-\partial_y^2} \). The same property is true for other fields\(^2\). In fact, \( \bar{H} \) is the Hamiltonian of single-particle excitations on the ultrastatic space-time

\[
d\bar{s}^2 = -dt^2 + dy^2 ,
\]

which is conformally related to (1). In (5) the position of the horizon is at infinity. Thus, one is dealing with massless fields on an infinite space. The entropy of such fields is infrared divergent quantity. By making the size of the system finite and equal to \( l \) one easily finds the free-energy \( F \), energy \( E \) and entropy \( S \) at the temperature \( \beta^{-1} \)

\[
F = -\frac{\pi}{6\beta^2} l , \quad E = \frac{\pi}{6\beta^2} l , \quad S = \frac{\pi}{3\beta} l .
\]

The finite size \( l \) is equivalent to introducing a cutoff near the horizon at some proper distance \( \epsilon \). Usually, to find the relation between \( l \) and \( \epsilon \), metric (1) is represented in another form

\[
ds^2 = e^{-\varphi}(-\kappa^2 \rho^2 dt^2 + d\rho^2) .
\]

Then, at small \( \epsilon \),

\[
l = \frac{1}{k} \left( \frac{1}{2} \varphi_h + \ln \frac{\rho_h}{\epsilon} \right) ,
\]

where \( \varphi_h \) is the value of the conformal factor at the horizon and \( \rho_h \) is the boundary value of \( \rho \). The entanglement entropy corresponding to a Hartle-Hawking vacuum is evaluated at the Hawking temperature \( \beta H^{-1} = k/(2\pi) \). For massless fields, calculated in this way \( S \), see (6), represents the exact result for the entropy.

Let us now show how to carry out the computations of the entanglement entropy along the lines of Ref. [7]. We consider \( N \) fields, which may be scalars or spinors or both. According to (6), in the leading order

\[
E = N \frac{\pi}{6\beta^2} l , \quad S = N \frac{\pi}{3\beta} l .
\]

The corresponding CFT near the horizon is characterized by the central charge \( c = N \), see [9]. The relation between the Hamiltonian of the system and generators of the Virasoro algebra follow from the representation of the metric (5) in the form

\[
d\bar{s}^2 = \left(\frac{l}{\pi}\right)^2 (-d\eta^2 + dz^2) = \left(\frac{l}{\pi}\right)^2 dudv ,
\]

\(^2\)A discussion of the general situation in four-dimensional space-times can be found, for instance, in [12].
\[ u = \frac{z + \eta}{2}, \quad v = \frac{z - \eta}{2}. \]  

(11)

Consequently,

\[ \partial_t = \frac{\pi}{2l}(\partial_u - \partial_v). \]  

(12)

In (10) the coordinate \( z \) ranges from 0 to \( \pi \). This corresponds to a theory on an interval where the points \( z = 0 \) and \( z = \pi \) are independent. In order to carry out the computations it is convenient to pass to a theory where \( z \) is a periodic coordinate. This can be done if one considers two equivalent CFT’s on the intervals with the length \( \pi \) and makes from them a CFT on a circle by gluing together the ends of the intervals. In the obtained theory \( z \) has the periodicity \( 2\pi \).

One has two copies of the Virasoro algebra where the elements \( L_n \) and \( \bar{L}_n \) can be defined in a standard way, as the generators of the coordinate transformations, \( \delta u = e^{inu} \) and \( \delta v = e^{inv} \), respectively. As a result of relation (12), the Hamiltonian \( H \) of the system which generates transformations along the Killing time \( t \) is represented as

\[ H = \frac{\pi}{2l}(L_0 - \bar{L}_0). \]  

(13)

Similarly, translations of the system along \( y \) are generated by the momentum

\[ P = \frac{\pi}{2l}(L_0 + \bar{L}_0). \]  

(14)

Because the system is at rest the average momentum is zero. On the other hand, the average value of \( H \) coincides with the energy \( E \) in (9). This fixes the average values \( h \) and \( \bar{h} \) of \( L_0 \) and \( \bar{L}_0 \), respectively. In the given quantum state

\[ h = -\bar{h} = \frac{N l^2}{6 \beta^2}. \]  

(15)

In the limit when \( l \) is very large (\( \epsilon \) goes to zero), \( h \gg 1 \) and one can use Cardy’s formula to estimate the degeneracy of \( L_0 \) and \( \bar{L}_0 \). In this approximation the total degeneracy \( D \) is

\[ \ln D = 2\pi \sqrt{\frac{c}{6}} + 2\pi \sqrt{\frac{|\bar{h}|}{6}} \]  

(16)

and by taking into account that in our case the central charge \( c = N \) we find

\[ \ln D = 2N \frac{\pi l}{3 \beta}. \]  

(17)

Finally, we have to remember that \( D \) is the number of states of the system with the doubled Hilbert space which results from the trick with the periodization of the coordinate \( z \). The real number of states of the system we are interested in is \( \sqrt{D} \). Thus, the entropy is

\[ S = \frac{1}{2} \ln D \]  

(18)

and it coincides exactly with the required value in Eq. (9). To get the entanglement entropy of the fields in the Hartle-Hawking state one has to put \( \beta = \beta_H \) in (18).
We have shown how to calculate the entanglement entropy by means of CFT in the way which is close to computations of the entropy of three-dimensional BTZ black holes [7]. The difference between our case and three-dimensional one is that here the degrees of freedom of CFT are the physical microscopic states of the theory. It is worth pointing out that in the above computations the entropy was determined by the degeneracy of states at the given energy, i.e., as the entropy of a microcanonical ensemble. On the other hand, the results of [10] are based on using the replica method and they are equivalent to calculation of the entropy of a canonical ensemble.

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References