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The Caldirola-Kanai Theory from "Brownian" Path Integral Quantum Mechanics

Luiz C. Botelho and Edson P. da Silva

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

Departamento de Física
Universidade Federal Rural do Rio de Janeiro
23851-180 - Itaguaí, RJ - Brazil

ABSTRACT

We deduce the classical one-dimensional Caldirola-Kanai action from quantum mechanics, at the W.K.B. limit, formulated as an electronic flux interacting with an environment in the usual Langevin-Brownian phenomenological path integral framework.

Key-words: Brownian motion; Path integral.

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It is an interesting problem in quantum mechanics to introduce interactions leading to damping in the equations of motion, at the classical limit of the associated quantum systems ([1],[4]).

In this brief report, we propose a phenomenological framework for the above mentioned problem by considering instanton configurations associated to a path integral ([2]) describing the W.K.B. leading limit of a quantum particle wave function interacting with a white noise reservoir with a friction interaction of a Brownian type.

Let us start by writing the one dimensional particle Schroedinger wave equation in a convenient way to implement the hydrodynamical analogy with quantum mechanics:

$$\Psi(x, t) = \rho^2(x, t) e^{\frac{i}{\hbar} S(x, t)} \quad (1)$$

$$\frac{\partial S(x, t)}{\partial t} + \frac{1}{2m} \left| \frac{d}{dx} S \right|^2(x, t) + \frac{\hbar^2}{2m} \frac{\frac{d^2}{dx^2} \rho(x, t)}{\rho(x, t)} + V(x, t) = 0 \quad (2)$$

$$\frac{\partial(\rho^2(x, t))}{\partial t} + \frac{d}{dx}(\rho^2(x, t) \frac{d}{dx} S(x, t)) = 0 \quad (3)$$

In order to write the above Schroedinger equation as a hydrodynamical equation suitable to introduce dissipation as usually implemented in classical fluid fluxes, we consider a W.K.B. pure phase approximation for Eqs.(1)-(3) and relevant for probing the classical limit of the system

$$\Psi(x, t) \sim \exp\left(\frac{i}{\hbar} S(x, t)\right) \quad (4)$$

By substituting Eq.(4) into Eq.(2), we get as our basic phenomenological dynamical equation, the inviscid Burger equation for the electronic flow defined by the wave function phase gradient $v(x, t) = \frac{d}{dx} S(x, t)$, namely:

$$\frac{\partial}{\partial t} v(x, t) + \frac{1}{2m} \left(v \frac{d}{dx} \right) v = - \frac{d}{dx} V(x, t) \quad (5)$$

At this point it is worth to remark that the quantum mechanical probability conservation law, Eq.(3), will be neglected at the zeroth W.K.B. limit, Eq.(4), together with the quantum

potential $\frac{\hbar^2}{2m} \frac{\frac{d^2}{dx^2} \rho(x,t)}{\rho(x,t)}$ which has the meaning of describing a complicated pressure term in the quantum hydrodynamical interpretation ([4]).

In order to modelling, phenomenologically, the friction with a thermal reservoir, we follow the usual Langevin-Brownian procedure by introducing the viscosity stress tensor on the right-hand side of Eq.(5), as a damping term of the form $-\nu v(x,t)$. The reservoir fluctuation interaction will be modelled by random impurities producing a pure Gaussian stochastic force stirring the electronic flux $v(x,t)$ ([5]).

As a consequence of the above made model assumptions, we are lead to consider the following Beltrami-Burger equation, in the range $0 \leq t \leq \infty$, describing our damped quantum electronic system in the phase approximation, Eq.(4),

$$\frac{\partial v(x,t)}{\partial t} + \frac{1}{2m} \left(v \frac{d}{dx} \right) v(x,t) = -\nu v(x,t) + F(x,t) - \left(\frac{d}{dx} V \right)(x,t) \quad (6)$$

Here the phenomenological damping term, ν , is temperature dependent and the stochastic force possesses the non-trivial two-point correlation form below, similar to phenomenological studies of Brownian motion ([5]),

$$\langle F(x,t) F(x',t') \rangle = \delta(x - x') \delta(t - t') \quad (7)$$

Proceeding as in Refs ([2], [3]), the space time characteristic functional of the random "forced" (Brownian) electronic flux, Eq.(6), is given by the following Wyld-Rosen path integral

$$\begin{aligned} Z[J(x,t)] = & \frac{1}{Z[0]} \int \mathcal{D}^F[\Pi(x,t)] \mathcal{D}^F[v(x,t)] \times \\ & e^{i \int_{-\infty}^{+\infty} dx \int_0^{\infty} dt \Pi(x,t) \left(\frac{\partial v}{\partial t} + \frac{1}{2m} \left(v \frac{d}{dx} \right) v + \nu v + \frac{d}{dx} V \right)(x,t)} \times \\ & e^{[-\frac{1}{2} \int_{-\infty}^{+\infty} dx \int_0^{\infty} dt \Pi^2(x,t)]} e^{[i \int J(x,t) v(x,t) dx dt]} \end{aligned} \quad (8)$$

Following Ref.[2], we consider the classical (instanton) equations of motion associated with the path integral, Eq.(8), with $J(x,t) \equiv 0$,

$$\frac{\partial v_{ins}(x,t)}{\partial t} + \frac{1}{2m} \left(v_{ins}(x,t) \frac{d}{dx} \right) v_{ins}(x,t) + \frac{d}{dx} V(x) + \nu v_{ins}(x,t) = \Pi_{ins}(x,t) \quad (9)$$

$$-\frac{\partial \Pi_{ins}(x, t)}{\partial t} - \frac{1}{2m}(v_{ins}(x, t)\frac{d}{dx})\Pi_{ins}(x, t) + \nu \Pi_{ins}(x, t) = 0 \quad (10)$$

A straightforward calculation leads to the following solutions for $t > 0$:

$$\Pi_{ins}(x, t) = 0 \quad (11)$$

$$v_{ins}(x, t) = e^{-\nu t} \hat{v}_{ins}(x, t) \quad (12)$$

with $\hat{v}_{ins}(x, t)$ satisfying the equation

$$\frac{\partial \hat{v}_{ins}(x, t)}{\partial t} + \frac{1}{2me^{\nu t}}(\hat{v}_{ins}\frac{d}{dx})\hat{v}_{ins}(x, t) + e^{\nu t}(\frac{d}{dx}V)(x) = 0. \quad (13)$$

Now, proceeding backward of the steps taken at the beginning of this brief report, we can write the effective Schroedinger equation for the wave function associated with the phase-flux instanton, Eq.(12),

$$i\hbar \frac{\partial \Psi_{ins}(x, t)}{\partial t} = (-\frac{\hbar^2}{2me^{\nu t}} \frac{d^2}{dx^2} + e^{\nu t} V(x)) \Psi_{ins}(x, t) \quad (14)$$

It has a time dependent mass term and a linear forcing potencial as a consequence of our proposed Brownian-Ohmic friction enviroment interaction. It is worth to remark that effective Schroedinger equation, Eq.(14), may be formally associated with the well known Caldirola-Kanai Lagrangean([6]):

$$\mathcal{L}_{classical}(x, t) = \frac{1}{\hbar} \int_0^t d\sigma e^{\nu\sigma} [\frac{1}{2}m(\frac{dx}{d\sigma})^2 - V(x(\sigma))] \quad (15)$$

As a consequence of the above exposed results, we get our main result, i.e, the damped classical equation of motion for the quantum particle interacting with our Brownian reservoir model, at the W.K.B. ($\hbar \rightarrow 0$) limit.

$$\frac{d^2 x_{cl}(t)}{dt^2} + \nu \frac{dx_{cl}(t)}{dt} = -\frac{d}{dx} V(x_{cl}(t)). \quad (16)$$

Finally, and for completeness of our previous path integral study, let us briefly discuss the Fokker-Planck approach ([7]) for our Beltrami-Burger stochastic equation, Eq.(6), where for simplicity we have taken $V(x) \equiv 0$.

The Fokker-Planck probability distribution function in a Fourier domain is given by the equal time occupation random variable, i.e,

$$P((x_1, \dots, x_n); (p_1, \dots, p_n); t) = \langle \exp(i \sum_{k=1}^N p_k v(x_k; t)) \rangle \quad (17)$$

where the average $\langle \dots \rangle$ is defined by the stochastic process of the electronic fluid velocity induced by the Gaussian stochastic force with general spatial correlation

$$\langle F(x, t) F(x', t') \rangle = K(x - x') \delta(t - t') \quad (18)$$

The Fokker-Planck equation is easily deduced by following closely Ref.[6], i.e,

$$\begin{aligned} -i \frac{\partial}{\partial t} P((x_1, \dots, x_n); (p_1, \dots, p_n); t) = \\ \sum_{\ell=1}^N [p_{\ell} \frac{\partial}{\partial p_{\ell}} (\frac{1}{p_{\ell}} \frac{\partial}{\partial x_{\ell}}) - \nu p_{\ell} \frac{\partial}{\partial p_{\ell}}] + \sum_{\ell=1, \ell'=1}^N (K(x_{\ell} - x_{\ell'}) p_{\ell} p_{\ell'}) P((x_1, \dots, x_n); (p_1, \dots, p_n); t) \end{aligned} \quad (19)$$

The main remark related to this Fokker-Planck approach is that Eq.(9) is a closed partial differential equation. Let us map Eq.(9) into a deterministic many-particle Schroedinger equation. In order to implement such a mapping, we introduce mixed coordinates, i.e,

$$p_j + x_j = u_j \quad ; \quad p_j - x_j = v_j \quad (20)$$

The Fokker-Planck Eq.(19) takes, thus, the following closed form(after analytic continuation $v \rightarrow iv$).

$$\begin{aligned} -i \frac{\partial}{\partial t} P((u_1, \dots, u_n); (iv_1, \dots, iv_n); t) = \\ [\sum_{\ell=1}^n \frac{1}{4} [\frac{\partial^2}{\partial u_{\ell}^2} + \frac{\partial^2}{\partial v_{\ell}^2} - (\frac{2}{u_{\ell} + iv_{\ell}}) (\frac{\partial}{\partial u_{\ell}} + i \frac{\partial}{\partial v_{\ell}}) - \frac{\nu}{2} (u_{\ell} + iv_{\ell}) (\frac{\partial}{\partial u_{\ell}} - i \frac{\partial}{\partial v_{\ell}})] \\ + \frac{1}{4} \sum_{\ell=1, \ell'=1}^n (u_{\ell} + iv_{\ell}) (u_{\ell'} + iv_{\ell'}) K(u_{\ell} - u_{\ell'} + i(v_{\ell'} - v_{\ell}))] P((u_1, \dots, u_n); (iv_1, \dots, iv_n); t) \end{aligned} \quad (21)$$

At this point we remark that it is straightforward to implement a numerical, "low viscosity" ($\nu \ll 1$), perturbative calculations by considering a slowly varying (even function) correlation function of the form (see Ref.[2]):

$$\begin{aligned}
K(x) &\sim K(0) - \frac{\ell_0}{2} x^2; \quad |x| \ll \left(\frac{K(0)}{\ell_0}\right)^{1/2} \equiv L \\
K(x) &\sim 0; \quad |x| \gg L
\end{aligned}
\tag{22}$$

Numerical studies of Eq.(21) are in progress and will appear elsewhere.

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