Quark and Nuclear Matter  
in the  
Linear Chiral Meson Model*

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Abstract

The equation of state for quark and nuclear matter is encoded in the effective potential of a linear sigma model. We exploit an exact differential equation for its dependence upon the chemical potential $\mu$ associated to conserved baryon number. This equation describes the transitions from a nucleon gas to nuclear matter and from nuclear matter to quark matter within the same model. For vanishing temperature and increasing baryon density both phase transitions appear to be of first order. For high temperature the first order lines may end in second order endpoints with long range correlations.

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*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement # DE-FC02-94ER40818 and by the Deutsche Forschungsgemeinschaft.
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1 Introduction

Nucleons as bound states of quarks have a characteristic size $\sim (200 \text{ MeV})^{-1}$. For very high baryon density there is not enough space per baryon to form nucleons, and one expects a new state of strongly interacting matter — quark matter (see, e.g., [1] and references therein). For thermodynamic equilibrium at low temperature the situation is further complicated by a state which is intermediate between a gas of nucleons and quark matter. This state is nuclear matter and can be associated with a liquid of nucleons. Nuclear matter is expected to occur in neutron stars [2]. Also nuclei can be interpreted as droplets of nuclear matter.

Any analytical description of the equation of state for baryons at nonzero baryon density has to cope with the problem that the effective degrees of freedom change from nucleons at low density to quarks at high density. We attempt in this note a unified description of both the nuclear gas–liquid transition and the transition to quark matter. For this purpose we work within an effective linear meson model coupled to quarks and nucleons. It should describe the low momentum degrees of freedom of QCD for the range of temperatures and densities which are relevant for these phase transitions. Our main computational tool will be a new exact functional differential equation for the dependence of the effective action on the baryon chemical potential and an approximate solution to it. We will see many similarities but also important differences as compared to a mean field theory treatment.

Our main interest is the equation of state for quark and nuclear matter at nonzero baryon density and the order of the involved phase transitions. In addition to the nuclear and quark matter phases a number of interesting possibilities like the formation of meson condensates or strange quark matter [3–5] have been proposed. Recently, an extensive discussion has focused around the symmetry of the high density ground state, where the possible spontaneous breaking of color is associated with the phenomenon of color superconductivity [6–10]. In this note we adopt the assumption that possible additional condensates have only little influence on the transitions associated with the order parameter of chiral symmetry breaking\(^1\). We also concentrate mainly on the case of two quark flavors and neglect isospin violation\(^2\). We find that for low temperature both phase transitions are of first order, in accordance with indications from earlier investigations (cf. [11,7,9,12] and references therein). The phase transition from nuclear to quark matter tends to be much stronger (larger surface tension) than the gas–liquid nuclear transition. The first order character of these phase transitions has important implications. In particular, one may combine this with information about the high temperature phase transition for vanishing baryon density: One expects [9,12] an endpoint of the first order

\(^1\)It has recently been demonstrated [9] that condensates of quark Cooper pairs do not influence the behavior of the chiral condensate to a good approximation. First results indicate that the phenomenon of color superconductivity does not change the order of the high density transition and has only minor influence on the equation of state for quark matter [7–10].

\(^2\)Isospin violation and electromagnetism are important for nuclear matter in neutron stars. Our result for the equation of state is therefore not quantitatively realistic in all respects. Isospin violation can be incorporated in our formalism without conceptual difficulties. We have already included electromagnetism phenomenologically for the quantitative description of nuclei. For the general structure of the phase diagram isospin violation is a secondary effect.
critical line between quark and nuclear matter if the zero density, high temperature transition is a crossover (as for two flavor QCD with non–vanishing quark masses). Such an endpoint corresponds to a second order transition with an infinite correlation length that may lead to distinctive signatures in relativistic heavy ion collisions [13]. If the zero density transition turns out to be of first order in three flavor QCD, such an endpoint does not necessarily occur. (Endpoints are not excluded in this case, however, since the two first order regions could be disconnected.) The first order line for the gas–liquid nuclear transition exhibits a critical endpoint for a temperature of about 10 MeV. Signatures and critical properties of this point have been studied through measurements of the yields of nuclear fragments in low energy heavy ion collisions [14,15].

In quantum field theory the effects of a non–vanishing baryon density in thermal equilibrium or the vacuum are described by adding to the classical action a term proportional to the chemical potential \( \mu \),

\[
\Delta \mu S = 3i\mu \sum_j b_j \int_0^{1/T} dx_0 \int d^3 \vec{x} \bar{\psi}_j \gamma_0 \psi_j \equiv -3\frac{\mu}{T} B.
\] (1)

The index \( j \) labels all fermionic degrees of freedom which carry a non–vanishing baryon number \( b_j \) and a summation over spinor indices is assumed implicitly. For a description of the fermionic degrees of freedom in terms of quarks the sum is over \( N_c \) colors and \( N_F \) flavors, with \( b_j = 1/3 \). For nucleon degrees of freedom we may include protons, neutrons and strange baryons with \( b_j = 1 \). For our conventions, \( \mu \) corresponds to the chemical potential of quark number density.

The baryon number density \( n \) can be obtained from the \( \mu \)–dependence of the Euclidean effective action \( \Gamma \), evaluated at its minimum for fixed temperature \( T \) and volume\(^3 V \)

\[
 n = \frac{\langle B \rangle}{V} = -\frac{1}{3} \frac{\partial}{\partial \mu} \left. \frac{\Gamma_{\text{min}} T}{V} \right|_{T,V}.
\] (2)

We note that the Helmholtz free energy is \( F = \Gamma_{\text{min}} T + 3\mu n V \). Our aim is a computation of the difference of \( \Gamma_{\text{min}} \) between non–vanishing and vanishing \( \mu \). For \( T = 0 \) this is dominated by fermionic fluctuations with (spatial) momenta \( q^2 \leq \mu^2 \). For not too large \( \mu \) (say \( \mu \lesssim 600 \text{ MeV} \)) we can therefore work with an effective model for the low momentum degrees of freedom of QCD. This argument generalizes to moderate temperatures, say \( T \lesssim 200 \text{ MeV} \).

The minimum of the effective action corresponds to the minimum of the effective meson potential \( U = \Gamma T/V \) for constant fields. In addition to the fermionic degrees of freedom we include in our description the lightest (pseudo)scalar and (axial) vector mesons. In consequence, \( U \) is a function of a complex \( N_F \times N_F \) scalar field matrix \( \Phi \), which describes the nonets of scalar and pseudoscalar mesons and a similar matrix for the vector mesons. For a discussion of the chiral phase transition it will be sufficient to know the dependence of \( U \) on space and time independent fields which can acquire a vacuum expectation value. These are the real diagonal elements of \( \Phi \) which we denote by \( \sigma_j \), and the diagonal elements \( \omega_j \) of the zero component of \( ^3B \) counts the number of baryons minus antibaryons. For \( T \to 0 \) the factor \( T/V \) is simply the inverse volume of four–dimensional Euclidean space.
the vector meson matrix. In the limit of vanishing current quark masses the minimum of $U$ in a
high temperature or high density situation occurs at $\sigma = 0$. For low $T$ and $\mu$ spontaneous chiral
symmetry breaking is triggered by a non–vanishing expectation value $\sigma_j(\mu, T)$, corresponding
to the location of the minimum of $U(\sigma, \omega; \mu, T)$. (We adopt the convention through this work
that bars indicate locations of potential minima.) The explicit breaking of chiral symmetry
through non–vanishing current quark masses is described by a linear source term contained
in $U$ which induces nonzero $\sigma_j$ even in the “symmetric phase” [16–18]. The baryon density $n$,
energy density $\epsilon$ and pressure $p$ follow from $U(\mu, T) \equiv U(\sigma_j(\mu, T), \omega_j(\mu, T); \mu, T) = \epsilon - Ts - 3\mu n$
as
\[
\begin{align*}
  n & = -\frac{1}{3} \frac{\partial}{\partial \mu} U(\mu, T) ; \quad p = -U(\mu, T) \\
  \epsilon & \equiv \frac{E}{V} = U(\mu, T) + 3\mu n - T \frac{\partial U}{\partial T}(\mu, T) .
\end{align*}
\] (3)
Here we have normalized $U(0, 0) = 0$ corresponding to vanishing pressure in the vacuum.

For fluctuations in the momentum range $\vec{q}_H^2 < \vec{q}^2 < (600 \text{ MeV})^2$ we work within the
linear quark meson model [17–22] in an approximation which does not describe the effects of
confinement. For low momenta, i.e. $\vec{q}^2 < q_H^2$, this description therefore becomes inappropriate.
Three quarks are bound into color singlet nucleons. In this momentum range we describe the
fermionic degrees of freedom by baryons, while keeping the description of the bosons in terms
of the scalar field $\Phi$ and corresponding vector meson fields. The use of the same bosonic fields
for the whole momentum range will turn out to be an important advantage since it facilitates
the computation of the free energy in different ranges of $\mu$, corresponding in turn to different
baryon densities and a different picture for the relevant fermionic degrees of freedom. For
nuclear matter a typical value of the “transition momentum” is $q_H \gtrsim 260 \text{ MeV}$. We find that
the quark–hadron phase transition is substantially influenced by the change from quark to
baryon fields at $q_H$. The transition from a nucleon gas to nuclear matter can be described
realistically in terms of nucleon degrees of freedom.

2 Chemical potential flow equation

We employ a new method for the computation of the $\mu$–dependent part of the effective action
that relies on an exact functional differential equation for $\Gamma$. This equation expresses the $\mu$–
derivative of $\Gamma$ in terms of the exact field dependent fermion propagator. We start from the
generating functional of the connected Green functions
\[
W[j] = \int D\chi \exp \left\{ -S[\chi] - \Delta_{\mu} S[\chi] + \int j \chi \right\}
\] (4)
where $\chi$ stands collectively for bosonic and fermionic fields with associated sources $j$ and $S$
is the action for $\mu = 0$. For our purpose it is convenient to subtract from the effective action
(defined by a Legendre transform) the $\mu$–dependent fermion bilinear (1):
\[
\Gamma[\varphi] = -W[j] + \int j \varphi - \Delta_{\mu} S[\varphi], \quad \varphi = \frac{\delta W}{\delta j} .
\] (5)
The $\mu$–dependence of $\Gamma$ arises only through $\Delta_\mu S$ and can be expressed by a trace over the connected two–point function. Using the manipulations of generating functions outlined in [23] in a context with fermions [24,25] one obtains the exact nonperturbative functional differential equation
\[ \frac{\partial}{\partial \mu} \Gamma = - \text{Tr} \left\{ \frac{\partial R_\mu}{\partial \mu} \left( \Gamma^{(2)} + R_\mu \right)^{-1} \right\} \] (6)
where
\[ R_{\mu,jj'}(q,q') = 3i\mu b_j \gamma^0 (2\pi)^4 \delta(q-q') \delta_{jj'} . \] (7)

We remind that $\Gamma$ is a functional of the meson and fermion fields, and the $\mu$–derivative on the left hand side of (6) is taken for fixed fields. The exact inverse propagator $\Gamma^{(2)}$ is the second functional derivative with respect to the fields. It is a matrix in the space of internal indices and momenta and involves fermions and bosons. Since $\Delta_\mu S$ only affects fermions, the trace is over fermionic indices only and contains a momentum integration. For a configuration with constant bosonic fields and vanishing fermion fields $\Gamma^{(2)}$ does not mix bosons and fermions and is diagonal in momentum space. We therefore only need the inverse fermion propagator
\[ \Gamma^{(2)}_{jj'}(q,q') = H_{jj'}(q)(2\pi)^4 \delta(q-q') \] (8)
in order to obtain an exact equation for the $\mu$–dependence of the effective potential
\[ \frac{\partial U}{\partial \mu} = - \sum_j 3b_j \int \frac{d^4q}{(2\pi)^4} \text{tr} i\gamma^0 \left[ H(q) + 3ib\mu \gamma^0 \right]^{-1} . \] (9)

Here tr denotes a Lorentz trace and $H_{jj'}$, $b_{jj'} = b_j \delta_{jj'}$ are matrices in the space of fermion species.

For arbitrary $\sigma_j$ and $\omega_j$ we approximate
\[ H_{jj'}(q) = \left[ q_\nu \gamma^\nu + m_j(\sigma;\mu,T)\gamma^5 + ib_j \Omega^{(j)0} \gamma^0 \right] \delta_{jj'} \] (10)
with
\[ \Omega^{(j)} = \sum_k g^{(j)}_{jk}(\sigma,T)\omega_k . \] (11)

For $N_F = 3$ the $\omega_k$ denote the analytic continuation of the zero component of the Euclidean $\omega$, $\phi$ and $\rho^0$ vector meson fields with coupling $g^{(j)}_{jk}$ to the fermion species $j$. We use our Ansatz for the fermion propagator only to compute the $\mu$–dependent contributions to the effective potential, i.e. we consider here the difference $U(\sigma,\omega;\mu,T) - U(\sigma,\omega;0,T)$. In fact, the computation of the contributions due to a non–vanishing chemical potential in many situations allows for quite crude approximations. This is based on the observation that strongly interacting fermions are often successfully described as freely propagating quasi–particles. In this case they acquire an effective “constituent” mass $m_j \gamma^5$ through a strong Yukawa coupling to mesonic vacuum expectation values. (The matrix $\gamma^5$ appears in the mass term as a consequence of our Euclidean conventions [23].) The meson self–interactions, on the other hand, may turn out to be

\[ ^4\text{We mention that in the presence of a local gauge symmetry this equation is manifestly gauge invariant.} \]
quite complicated. They are needed, however, only for the $\mu$–independent part of the effective potential and for the $\mu$–dependence of Yukawa couplings and wave function renormalizations which will be omitted in a first computation.

Our approximation consists in neglecting the dependence of the fermion wave function renormalization on momentum, $\sigma$, $\mu$ and $T$. We also neglect a possible difference in normalization of the quark kinetic term and the baryon number current. Similarly, we have omitted the momentum dependence of the mass term as well as the momentum dependence of the contribution $\sim \gamma^0$. We observe that the term $\sim \Omega^{(j)}$ can be combined with $R_{\mu}$ such that $\mu$ is replaced in the propagator $(\Gamma^{(2)} + R_{\mu})^{-1}$ by an effective chemical potential

$$\mu^{(j)}_{\text{eff}} = \mu + \frac{1}{3} \Omega^{(j)}(\omega, \sigma; T).$$  

With the approximation (10) the evolution equation for the $\mu$–dependence of the effective meson potential reads

$$\frac{\partial U}{\partial \mu} = - \sum_j 3b_j \int \frac{d^4q}{(2\pi)^4} \text{tr}\left\{ i\gamma^0 \left( \vec{q} + m_j \gamma^5 + 3ib_j \mu^{(j)}_{\text{eff}} \gamma^0 \right)^{-1} \right\}. \tag{13}$$

The remaining trace over spinor indices is easily performed

$$\frac{\partial U}{\partial \mu} = -2 \sum_j \int \frac{d^3\vec{q}}{(2\pi)^3} K_j, \quad K_j = 6ib_j \int \frac{dq^0}{2\pi} \frac{(q^0 + 3ib_j \mu^{(j)}_{\text{eff}})}{(q^0 + 3ib_j \mu^{(j)}_{\text{eff}})^2 + \vec{q}^2 + m_j^2}. \tag{15}$$

For non–vanishing temperature the $q^0$–integration is replaced by a sum over Matsubara frequencies

$$\int \frac{dq^0}{2\pi} \rightarrow T \sum_{n \in \mathbb{Z}}$$

with $q^0 = 2\pi(n + 1/2)T$ and, correspondingly, $\delta(q - q') \rightarrow \delta(q - q')\delta_{nn'}/(2\pi T)$. This results in

$$K_j = 3b_j \left[ e^{\frac{\sqrt{\vec{q}^2 + m_j^2} - 3b_j \mu^{(j)}_{\text{eff}}}{T}} + 1 \right]^{-1} - \left[ e^{\sqrt{\vec{q}^2 + m_j^2} + 3b_j \mu^{(j)}_{\text{eff}}}/\sqrt{\vec{q}^2 + m_j^2} \right]^{-1} \tag{17}$$

where the two terms are proportional to the fermion and anti–fermion contributions to $n$.

We will concentrate here mainly on $T = 0$ where the $q^0$–integration yields a step function:

$$K_j = 3b_j \Theta(9b_j^2(\mu^{(j)}_{\text{eff}})^2 - (q^2 + m_j^2)). \tag{18}$$

The remaining $\vec{q}$–integration is therefore cut off in the ultraviolet, $\vec{q}^2 < 9b_j^2(\mu^{(j)}_{\text{eff}})^2 - m_j^2$, and only involves momenta smaller than the Fermi energy $3b_j \mu^{(j)}_{\text{eff}}$. As it should be, it is dominated by modes with energy $\sqrt{\vec{q}^2 + m_j^2}$ near the Fermi surface.
3 Quark and nucleon degrees of freedom

We will assume that $\partial U/\partial \mu$ can be expressed as a simple sum of the contribution from quarks with momenta $q^2 > q_H^2$ and that of baryons with momenta $q^2 < q_H^2$. We first consider the range of momenta with $q^2 \geq q_H^2$ for which we use an effective linear quark meson model [17–22]. Here the quark mass term $\sim \gamma^5$ arises through a Yukawa coupling $h_a$ to the expectation value of the $\sigma$–field, $m_a = h_a(\sigma; \mu, T)\sigma_a$, where $a$ labels the $N_F$ different quark flavors. Since the quark description breaks down for small momenta, we restrict the integration over $q^2$ in (14) to the range $q^2 > q_H^2$. We therefore infer for the quark contribution to the $\mu$–dependence of the effective potential (for $\mu > 0$ and $T = 0$)

$$
\frac{\partial U^Q}{\partial \mu} = - \frac{N_c}{3\pi^2} \sum_a \left[ \left( \mu_{\text{eff}}^{(a)} - h_a^2 \sigma_a^2 \right)^{3/2} - q_H^3 \right] \Theta \left( \mu_{\text{eff}}^{(a)} - h_a^2 \sigma_a^2 - q_H^2 \right).
$$

(19)

For the low momentum range $q^2 < q_H^2$ where the fermionic degrees of freedom are the lightest baryons rather than quarks we repeat the steps leading from (6) to (19). The trace now involves a sum over proton, neutron and the strange baryons of the lowest mass octet but no color factor. This yields a contribution (again for $T = 0$)

$$
\frac{\partial U^B}{\partial \mu} = - \frac{1}{\pi^2} \sum_j \left\{ \left( 9\mu_{\text{eff}}^{(j)} - m_j^2 \right)^{3/2} \Theta \left( 9\mu_{\text{eff}}^{(j)} - m_j^2 \right) \Theta \left( m_j^2 + q_H^2 - 9\mu_{\text{eff}}^{(j)} \right) \right\}
$$

(20)

where $j$ now labels the individual baryons of the lowest lying octet. They are thought of as composites of three constituent quarks. The baryon density can be directly inferred from eqs. (19), (20) as

$$
n = - \frac{1}{3} \left. \left( \frac{\partial U^Q}{\partial \mu} + \frac{\partial U^B}{\partial \mu} \right) \right|_{\sigma = \sigma, \omega = \sigma}
$$

(21)

since the partial derivatives of the effective potential with respect to $\sigma$ and $\omega$ vanish at the $\mu$–dependent potential minimum $(\sigma, \omega)$.

We first consider the non–strange contributions, i.e. those from the up and down quarks as well as from the two nucleons. The effects of the strange quark and the strange baryons will be considered below. We neglect isospin violation, $\sigma_u = \sigma_d \equiv \sigma$, $h_u = h_d \equiv h$ and parameterize the nucleon mass as $m_N(\sigma) \equiv 3h_N(\sigma)\sigma$. Furthermore, we assume a universal coupling $g^{(\omega)}$ of the $\omega$–meson to the up and down quarks and the nucleons. Isospin conservation implies a vanishing expectation value for the $\rho^0$–meson and we neglect the coupling of the $\phi$ to the non–strange fermions, implying

$$
\mu_{\text{eff}} \equiv \mu_{\text{eff}}^{(u)} = \mu_{\text{eff}}^{(d)} = \mu_{\text{eff}}^{(p)} = \mu_{\text{eff}} = \mu + \frac{g^{(\omega)}\omega}{3}.
$$

(22)

It is obvious that this picture is only a crude approximation to the binding of quarks into nucleons. A nucleon description should work well for $h_N^2\sigma^2$ near $\mu_{\text{eff}}^2$, since only low momentum
degrees of freedom contribute in this range. On the other hand, the quark description becomes
important for $h^2\sigma^2 \ll \mu_{\text{eff}}^2 - q_H^2$. In a more realistic description the $\Theta$-functions in (19), (20)
would become smooth. Furthermore, the characteristic quark–baryon transition momentum $q_H$ may depend on $\sigma$. Indeed, a baryon description for the low momentum degrees of freedom is
necessary for $\sigma$ not too far from its vacuum expectation value $\sigma_0$. On the other hand, baryons
do not seem to be meaningful degrees of freedom in a situation of chiral symmetry restoration
at $\sigma = 0$.

For $h_N$ of the same order as $h$, eq. (20) results in an important enhancement of $\partial U/\partial \mu$ in the
range $\sqrt{\mu_{\text{eff}}^2 - \frac{2}{9}q_H^2} < h_N |\sigma| < \mu_{\text{eff}}$ as compared to the contribution from the quarks. This is due
to the fact that more energy levels fall below the Fermi energy $3\mu_{\text{eff}}$ for the baryons. This large
“nucleon enhancement” by a factor of about 27 is the basic mechanism which leads to separate
gas–liquid and hadron–quark phase transitions. Despite this enhancement one observes that
“nucleon enhancement” by a factor of about 27 is the basic mechanism which leads to separate
gas–liquid and hadron–quark phase transitions. Despite this enhancement one observes that
$\partial U/\partial \mu$ is continuous in $\sigma$ and $\mu$.

With $\frac{\partial}{\partial \mu_{\text{eff}}} = \frac{\partial}{\partial \mu}$ we can easily rewrite eqs. (19), (20) as flow equations for $\mu_{\text{eff}}$. In the
approximation of $\mu$–independent Yukawa couplings $h = h(\sigma)$, $h_N = h_N(\sigma)$ and $q_H = q_H(\sigma)$
these differential equations can be integrated analytically. We define

$$U(\sigma, \omega; \mu, T) \equiv U_0(\sigma; T) + U_\omega(\sigma, \omega; T) + 2U_\mu(\sigma, \omega; \mu, T) + U^{(s)}(\sigma, \omega; \mu, T)$$ (23)

where $2U_\mu(\sigma, \omega; \mu, T)$ entails the $\mu$–dependent contribution from the two lightest quarks ($2U^{(q)}_\mu$)
as well as proton and neutron ($2U^{(n)}_\mu$), whereas $U^{(s)}(\sigma, \omega; \mu, T)$ denotes that of the strange
quark and the lightest strange baryons which will be added later. The $\mu$–independent part of
the potential is thus given by $U_0 + U_\omega$ with $U_\omega$ the $\omega$–dependent contribution. For $T = 0$ we obtain

$$U_\mu = U^{(q)}_\mu + U^{(n)}_\mu,$$ (24)

$$U^{(q)}_\mu(\sigma, \omega; \mu, 0) = -\frac{1}{4\pi^2} \left[ \mu_{\text{eff}} \left( \mu_{\text{eff}}^2 - \frac{5}{2}h^2\sigma^2 \right) \sqrt{\mu_{\text{eff}}^2 - h^2\sigma^2} \right. + \frac{3}{2}h^4\sigma^4 \ln \frac{\mu_{\text{eff}} + \sqrt{\mu_{\text{eff}}^2 - h^2\sigma^2}}{q_H + \sqrt{h^2\sigma^2 + q_H^2}}$$ (25)

$$\left. - q_H \left( 4q_H^2\mu_{\text{eff}} - (3q_H^2 + \frac{3}{2}h^2\sigma^2)\sqrt{h^2\sigma^2 + q_H^2} \right) \Theta(\mu_{\text{eff}}^2 - h^2\sigma^2 - q_H^2) \right] \Theta(\mu_{\text{eff}}^2 - h^2\sigma^2 - q_H^2),$$

$$U^{(n)}_\mu(\sigma, \omega; \mu, 0) = -\frac{27}{4\pi^2} \left[ \mu_{\text{eff}} \left( \mu_{\text{eff}}^2 - \frac{5}{2}h_N^2\sigma^2 \right) \sqrt{\mu_{\text{eff}}^2 - h_N^2\sigma^2} \right. + \frac{3}{2}h_N^4\sigma^4 \ln \frac{\mu_{\text{eff}} + \sqrt{\mu_{\text{eff}}^2 - h_N^2\sigma^2}}{h_N|\sigma|} \right] \Theta(\mu_{\text{eff}}^2 - h_N^2\sigma^2) \Theta(h_N^2\sigma^2 - \frac{1}{9}q_H^2 - \mu_{\text{eff}}^2)$$

$$+ \left. q_H \left( 4q_H^2\mu_{\text{eff}} - (\frac{1}{9}q_H^2 + \frac{1}{2}h_N^2\sigma^2)\sqrt{h_N^2\sigma^2 + \frac{1}{9}q_H^2} \right) \right] \Theta(\mu_{\text{eff}}^2 - h_N^2\sigma^2) \Theta(h_N^2\sigma^2 - \frac{1}{9}q_H^2 - \mu_{\text{eff}}^2)$$
Therefore baryons can only exist for sufficiently small average quark kinetic energies or momenta. Very roughly, the relevant critical kinetic energy is expected to be proportional to 

\[ m \sim \sqrt{2} \sigma \] 

where \( \sigma \) is the pion decay constant. Since \( m \) from the effective potential as \( \sigma \), we find that the quantitative aspects of the quark–hadron transition depend on \( \sigma \). On the other hand, we will find that the constante \( q_H(\sigma) \) always tends to zero for small enough \( \sigma \) there should be a critical value \( \sigma_c \) for which \( q_H(\sigma_c) = 0 \). We will not use baryons for \( \sigma < \sigma_c \) and take \( q_H(\sigma < \sigma_c) = 0 \). For our purpose we will be satisfied with a crude approximation\(^5\) where we neglect the \( \sigma \)-dependence of \( q_H \) in the range of \( \sigma \) relevant for nuclear physics, \( \sigma > \sigma_H \)

\[
q_H(\sigma) = \frac{q_H}{\sigma_H^2 - \sigma_c^2} \sqrt{(\sigma^2 - \sigma_c^2)(2\sigma_H^2 - \sigma_c^2 - \sigma^2)} \Theta(\sigma - \sigma_c) \Theta(\sigma_H - \sigma) + q_H \Theta(\sigma - \sigma_H) .
\]  

(27)

For a wide range of \( \sigma_c \) and \( \sigma_H \) our results for the gas–liquid transition will turn out to be independent of the precise values of these two quantities. For definiteness we take \( \sigma_c = 15 \text{ MeV} \), \( \sigma_H = 25 \text{ MeV} \). We expect that the constant \( q_H \) should have the size of a typical QCD scale, i.e., around 200 MeV. On the other hand, we will find that the quantitative aspects of the quark–hadron transition depend on \( \sigma_c \), \( \sigma_H \) and \( q_H \) which parameterize in our crude approximation the effects of confinement.

It is interesting to note that for only two light quark flavors (\( N_c N_F = 6 \)) the \( \mu \)-dependent contribution to the potential at the origin and therefore to the energy density reads for arbitrary finite \( h(\sigma, \mu) \)

\[
\epsilon_\mu^{(0)} = -6U_\mu(\sigma = 0, \omega; \mu, 0) = \frac{3}{2\pi^2} \mu_\text{eff}^4 = \left(\frac{3}{2}\right)^{7/3} \pi^{2/3} (\mu^{(0)})^{4/3} .
\]  

(28)

This has the simple interpretation of the total energy of six massless quarks with all energy levels filled up to the Fermi energy \( \mu_\text{eff} \). Furthermore, for \( \sigma \) and \( \mu \) in the range relevant for nuclear physics and for sufficiently large \( q_H \), i.e. \( q_H^2 > 9(\mu_\text{eff}^2 - h_N^2 \sigma^2) \), the contribution \( U_\mu^{(0)} \) is simply the mean field result for a nucleon meson model, whereas \( U_\mu^{(q)} \) vanishes. Our approach gives a new motivation for the approximate validity of mean field theory from the truncation of an exact flow equation. Furthermore, it offers the possibility of a systematic improvement, e.g., by taking the \( \mu \)-dependence of \( h_N \) into account. Despite this similarity, our method goes beyond mean field theory in an important aspect: For the free energy only the difference between vanishing and non–vanishing chemical potential is described by mean field theory. Since \( \Gamma(\mu = 0) \) is the generating functional for the propagators and vertices in vacuum it can, in principle, be directly related to measured properties like meson masses and decays. This is very important, since mean field theory does not give a very reliable description of the vacuum properties.

\(^5\)For our choice \( \partial q_H(\sigma)/\partial \sigma \) is continuous at \( \sigma = \sigma_H \).
4 Meson interactions

In order to discuss possible phase transitions as \( \mu \) is increased beyond a critical value we need information about the effective potential for \( \mu = 0 \). For a vacuum without spontaneous symmetry breaking relatively accurate information about \( U_0(\sigma; T) = U_0(\sigma, \omega = 0; \mu = 0, T) \) for all relevant \( \sigma \) could be extracted from the knowledge of meson masses and interactions. Also the approximation (10) for the fermionic propagator would presumably be reasonable for arbitrary \( \sigma \). In case of spontaneous chiral symmetry breaking the situation is more complex: The effective potential \( U_0 \) becomes convex because of fluctuations which interpolate between the minima of the “perturbative” or “coarse grained” potential [26,27]. Masses and interactions give only information about the “outer region” of the potential which is not affected by this type of fluctuations. In parallel, the simple form of the fermionic propagator (10) becomes invalid in the “inner region” for small \( \sigma \) because of a complex momentum dependence [26,27] and the breakdown of the approximation of a constant Yukawa coupling. In consequence, \( U_0(\sigma; T) \) should rather be associated with a coarse grained effective potential. For a suitable coarse graining scale\(^6\) \( k \) the effect of the omitted fluctuations with momenta smaller than \( k \) is expected to be small near the \( \mu \)-dependent minimum of \( U \). Around the minimum we can therefore continue to associate \( U_0(\sigma; T) \) with the effective potential and relate its properties to the measured masses and decay constants. On the other hand, we do not have much information about the shape of \( U_0(\sigma; T) \) for \( \sigma \approx 0 \). This uncertainty in the appropriate choice of \( U_0(\sigma; T) \) is one of the main shortcomings of our method. In practice, we interpolate the partly known polynomial form of \( U_0(\sigma; T) \) form the outer region (which includes the minimum characterizing the vacuum) to the inner region for small \( \sigma \). By continuity, this should be quite reasonable for nuclear matter since the relevant values of \( \sigma \) are not much smaller than the vacuum expectation value \( \sigma_0 \). For quark matter, the uncertainties are more important.

We investigate first the two flavor case with a potential of the form

\[
U_0(\sigma; T) \equiv 2m_\pi^2(T)\left[\sigma^2 - \sigma_0^2(T)\right] + 2\lambda(T)\left[\sigma^2 - \sigma_0^2(T)\right]^2 +\]

\[
+ \frac{4}{3}\frac{\gamma_3(T)}{\sigma_0^3(T)}\left[\sigma^2 - \sigma_0^2(T)\right]^3 + \frac{\gamma_4(T)}{\sigma_0^4(T)}\left[\sigma^2 - \sigma_0^2(T)\right]^4\]

\[
+ \frac{4}{5}\frac{\gamma_5(T)}{\sigma_0^5(T)}\left[\sigma^2 - \sigma_0^2(T)\right]^5 - 2j\sigma + c(j, T) \tag{29}
\]

where

\[
j = 2m_\pi^2(0)\sigma_0(0) , \quad c(j, 0) = 2j\sigma_0(0) . \tag{30}
\]

In the remainder of this work we mainly consider \( T = 0 \) and use \( \lambda \equiv \lambda(0) \), \( U(\sigma; \mu) \equiv U(\sigma; \mu, 0) \) etc. The meson field is normalized such that \( \sigma_0 = \sigma_0(0) \) is related to the pion decay constant by \( \sigma_0 = f_\pi/2 = 46.5 \text{ MeV} \). This means that the pions have a standard kinetic term (as derived from \( L_{\text{kin}} = \text{Tr} \partial_\mu \Phi \bar{\Phi} \partial^\mu \Phi \)). Because of higher order kinetic invariants [20] the kinetic term

---

\(^6\)The coarse graining scale \( k \) is chosen such that \( U_k \) is approximately \( k \)-independent for \( |\sigma| \) around \( |\Phi| \) or larger, whereas the approach to convexity for \( |\sigma| < |\Phi| \) and \( k \to 0 \) has not yet set in.
for the sigma meson, $\mathcal{L}_{\text{kin},\sigma} = 2Z_\sigma \partial_\mu \sigma \partial^\mu \sigma$ can involve a wave function renormalization $Z_\sigma$ different from the one for the pions. The potential (29) arises from a fifth order polynomial in the invariant $\rho = \text{Tr} \Phi \dagger \Phi = 2\sigma^2$ with an additional source term $-\frac{1}{4} \text{Tr} j(\Phi + \Phi^\dagger)$, where $j$ is proportional to the renormalized current quark mass (say at 1 GeV). The only violation of the chiral $SU_L(2) \times SU_R(2)$ symmetry arises from this source and in the chiral limit of vanishing current quark masses the last two terms in eq. (29) should be dropped. The coupling $\lambda$ is related to the $\sigma$–mass $m_\sigma$ by $\tilde{m}_\sigma^2 = Z_\sigma m_\sigma^2 = m_\pi^2 + 4\lambda \sigma^2_0$.

Without the complications of confinement (i.e., for $q_H = 0$) the quark–hadron phase transition in the chiral limit ($j = 0$) would occur for $\mu_{\text{eff}} = \mu_0$.

Finally, we determine the expectation value of $\omega$ by observing the identity

$$\frac{\partial U_0}{\partial \omega} = \frac{2}{3} g(\omega) \frac{\partial U_\mu}{\partial \mu}.$$  

It follows from first differentiating eqs. (19), (20) with respect to $\omega$ and then performing the $\mu$–integration. For the $\mu = 0$ contribution we only take into account a $\sigma$– and $T$–dependent mass term

$$U_\omega = -\frac{1}{2} M_\omega^2(\sigma, T) \omega^2.$$  

The solution of the $\omega$–field equations for arbitrary $\sigma$, $T$, $\mu$ obeys

$$\omega(\sigma, \mu, T) = \frac{2g(\omega)}{3M_\omega} \frac{\partial U_\mu}{\partial \mu}(\sigma, \omega; \mu, T).$$  

We note that at the potential minimum $\omega$ is proportional to the (non–strange) baryon density with a negative coefficient. This implies that the coupling to $\omega$ reduces the effective chemical potential [11]. In the following we will always assume that $\omega(\sigma, \mu, T)$ is inserted such that $\mu_{\text{eff}}$ becomes a function of $\sigma$, $\mu$ and $T$. The field equation which determines $\sigma$ can be expressed in terms of partial $\sigma$–derivatives at fixed $\mu_{\text{eff}}$

$$\frac{\partial U_0}{\partial \sigma}(\sigma) - M_\omega(\sigma) \frac{\partial M_\omega}{\partial \sigma}(\sigma) \omega^2 + 2 \frac{\partial U_\mu}{\partial \sigma}(\mu_{\text{eff}}) \sigma = 0.$$  

5 Meson–baryon interactions

A crucial ingredient for any quantitative analysis is the sigma–nucleon coupling $h_N$. We first investigate if chiral symmetry and the observed value of the pion nucleon coupling place any
restrictions on this coupling. For this purpose we employ a derivative expansion of the most general effective Lagrangian which is bilinear in the nucleon doublet field $\Psi_N$ and involves scalar and pseudoscalar fields contained in the $2 \times 2$ matrix $\Phi$

$$\mathcal{L} = \frac{1}{2} \left[ \mathcal{W}_{\text{NR}} F(\Phi^\dagger, \rho) \Phi \Psi_{\text{NL}} - \mathcal{W}_{\text{NL}} F(\Phi^\dagger, \rho) \Phi^\dagger \Psi_{\text{NR}} + \mathcal{W}_{\text{NL}} G_1(\Phi^\dagger, \rho) i\gamma^\mu \partial_\mu \Psi_{\text{NL}} + \mathcal{W}_{\text{NR}} G_1(\Phi^\dagger, \rho) i\gamma^\mu \partial_\mu \Psi_{\text{NR}} \right. \\
+ \left. \mathcal{W}_{\text{NR}} \Phi^\dagger G_2(\Phi^\dagger, \rho) i\gamma^\mu (\partial_\mu \Phi) \Psi_{\text{NL}} + \mathcal{W}_{\text{NR}} \Phi G_2(\Phi^\dagger, \rho) i\gamma^\mu (\partial_\mu \Phi^\dagger) \Psi_{\text{NR}} + \text{h.c.} \right] \tag{36}.$$  

Here we have imposed $\mathcal{P}$ and $\mathcal{C}$ symmetry and used $\Psi_{\text{NL}} = (1 + \gamma_5)\Psi_N/2$. With the standard decomposition

$$\Phi = \sigma \xi^2 = \sigma U \quad , \quad \xi = \exp \left( \frac{i}{4\sigma} \vec{\tau} \vec{\pi} \right) \tag{37}$$

one finds

$$\mathcal{L} = 3h_N(\sigma) \sigma \gamma_5 N + Z_N(\sigma) \gamma_5 \left( i\gamma^\mu \partial_\mu - \gamma^\mu v_\mu + G_A(\sigma) \gamma^\mu \gamma^5 a_\mu \right) N$$

$$- \frac{i}{2\sigma^2} \left[ G_A(\sigma) - 1 \right] \sigma(\partial_\mu \sigma) \gamma^\mu N \tag{38}$$

where

$$h_N(\sigma) = F(\sigma^2, 2\sigma^2)/3 \quad , \quad Z_N(\sigma) = G_1(\sigma^2, 2\sigma^2)$$

$$G_A(\sigma) = 1 - \frac{2G_2(\sigma^2, 2\sigma^2)\sigma^2}{G_1(\sigma^2, 2\sigma^2)} \tag{39}$$

and

$$v_\mu = -\frac{i}{2} \left( \xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger \right)$$

$$a_\mu = -\frac{i}{2} \left( \xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger \right) = \frac{1}{4\sigma} \vec{\tau} \partial_\mu \vec{\pi} + \ldots. \tag{40}$$

Normalization of the baryon number current requires $Z_N(\sigma_0) = 1$ and we neglect the $\sigma$-dependence of $Z_N$ in the following. The strength of the linear pion–nucleon coupling is fixed by $g_A = G_A(\sigma_0)$ and bares no relation to the function $h_N(\sigma)$. We may expand $h_N(\sigma)$ around $\sigma_0$

$$h_N(\sigma) = h_N(\sigma_0) + \frac{g_N}{\sigma_0^2} \left( \sigma^2 - \sigma_0^2 \right) + \ldots. \tag{41}$$

With $m_N = 3h_N(\sigma_0)\sigma_0 = 939 \text{ MeV}$ we find $h_N(\sigma_0) = 6.73$. Linearizing $m_N(\sigma) = 3h_N(\sigma_0)\sigma$ around $\sigma_0$ then yields $m_N(\sigma) = 3h_N + \epsilon_G$ with $\tilde{h} = h_N(\sigma_0) + 2g_N$, $\epsilon_G = -6g_N\sigma_0$. The linear sigma–nucleon coupling $\tilde{h}$ is a free parameter which is expected to be in the vicinity of $h_N(\sigma_0)$. We will determine it below from the properties of nuclear matter. Since $\tilde{h}$ also appears in the scattering of nucleons a comparison with experiment may serve as a test for our model.
6 Nuclear matter and the nuclear phase transition

Let us turn to the zero temperature properties of nuclear matter in our picture. For $\mu = 0$ the effective potential or free energy $U$ has its minimum at $\sigma_0 = f_\pi/2 = 46.5$ MeV. The potential in the region near $\sigma_0$ is not altered as long as $\mu_{\text{eff}}$ remains small enough (cf. eq. (20)). For a range of $3\mu$ somewhat below the nucleon mass a new minimum of $U$ occurs at $\sigma_{\text{nuc}}(\mu) < \sigma_0$, with a potential barrier between both minima. For a certain range of $\mu$ the local minimum at $\sigma_{\text{nuc}}(\mu)$ and the global minimum at $\sigma_0$ coexist. As $\mu$ increases, the value of $U(\sigma_0)$ is lowered whereas $U(\sigma_0) = 0$ remains fixed as long as the effective chemical potential is smaller than a third of the nucleon mass, $\mu_{\text{eff}} < h_N(\sigma_0)\sigma_0$. There is a critical value $\mu_{\text{nuc}}$ for which the two minima at

$$\sigma_{\text{nuc}} \equiv \sigma_{\text{nuc}}(\mu_{\text{nuc}})$$

and $\sigma_0$ are degenerate, $U(\sigma_{\text{nuc}}, \mu_{\text{nuc}}) = U(\sigma_0, \mu_{\text{nuc}}) = 0$. The corresponding critical potential is plotted in figure 1. Both phases have equal, vanishing pressure $p = -U$ and can coexist. We observe that the phase transition between the vacuum ($\sigma = \sigma_0$) and nuclear matter ($\sigma = \sigma_{\text{nuc}}$) is clearly of first order. For small temperature this corresponds to the transition between a gas of nucleons and nuclear matter which may be associated with a nuclear liquid.

![Figure 1: The critical effective potential for the nuclear liquid–gas transition corresponding to the parameter set A of table 1.](image)

For a quantitative description we concentrate mainly on large values of $q_H$ where this transition happens in a region with $\mu_{\text{eff}}^2 < h_N^2(\sigma_{\text{nuc}})\sigma_{\text{nuc}}^2 + q_H^2/9$. In this case the transition from nucleon to quark degrees of freedom does not affect nuclear properties and the phase transition from a nucleon gas to nuclear matter. We recover the $\sigma$–$\omega$–model of nuclear physics [28,29,2], in a context where chiral symmetry breaking and constraints from meson masses and decays...
are properly incorporated. For large enough $q_H$ the critical baryon density of the nuclear liquid is given by eq. (20)

$$n_{\text{nuc}} = \frac{18}{\pi^2} \left[ \mu_{\text{eff}}(\mu_{\text{nuc}}, \bar{\sigma}_{\text{nuc}}) - h^2_{\text{nuc}}(\bar{\sigma}_{\text{nuc}}) \bar{\sigma}_{\text{nuc}}^2 \right]^{3/2} .$$  \hspace{1cm} (43)

We will see below that one can identify $n_{\text{nuc}}$ with the baryon density in nuclei $n_{\text{nuc}} = 1.175 \times 10^6 \cdot \text{MeV}^3$ up to small corrections. Furthermore, the baryon number independent contribution to the binding energy per nucleon in a large sample of nuclear matter is known from the mass formula for nuclei: $\beta_{\text{nuc}} = -16.3 \text{MeV}$. In our context one finds $\beta_{\text{nuc}} = 3\mu_{\text{nuc}} - m_N$ and for realistic models the gas–liquid transition should therefore occur for $\mu_{\text{nuc}} = 307.57 \text{MeV}$. Eq. (43) then yields a quantitative relation between the effective chemical potential in nuclear matter $\mu_{\text{eff}, \text{nuc}} = \mu_{\text{nuc}} - \frac{1}{3} g^{(\omega)} \frac{m_N}{\omega^2} n_{\text{nuc}}$.

$$\mu_{\text{eff}, \text{nuc}} = \mu_{\text{nuc}} - \frac{1}{3} \frac{g^{(\omega)} m_N}{\omega^2} n_{\text{nuc}} .$$  \hspace{1cm} (44)

For a $\sigma$–independent $\omega$–mass $M_\omega = 783 \text{MeV}$ typical values for the above ratios for $m_N(\bar{\sigma}_{\text{nuc}})/m_N$ are $g^{(\omega)} = (12.61, 11.68, 10.66, 9.52, 8.21)$. From the value of nuclear density we can compute the Fermi momentum $q_{\text{nuc}} = 259 \text{MeV}$. This yields for this scenario a lower bound $q_H > q_{\text{nuc}} = 259 \text{MeV}$. (For quantitative computations we take $q_H = 1.2 q_{\text{nuc}}$.)

<table>
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<th>$h$</th>
<th>$g^{(\omega)}$</th>
<th>$\gamma_3$</th>
<th>$\gamma_4$</th>
<th>$\gamma_5$</th>
<th>$\mu_0$ MeV</th>
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<tr>
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<td>8.74</td>
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<td>55</td>
<td>0</td>
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</table>

Table 1: Coupling constants for three different parameter sets. The linear sigma–nucleon coupling $\hat{h}$, the coupling $g^{(\omega)}$ of the $\omega$–meson to the $u, d$–quarks and the nucleons, the meson self–interactions $\gamma_3$, $\gamma_4$, $\gamma_5$ and the scale $\mu_0$ are defined in sections 4 and 5.

An important quantity for the equation of state is the compression modulus

$$K = 9 n^2 \frac{d^2}{dn^2} \left( \frac{\epsilon}{n} \right) = 9 \left( \frac{dp}{dn} - 2 \frac{p}{n} \right) .$$  \hspace{1cm} (45)

The value at the phase transition

$$K_0 = 9 \left. \frac{dp}{dn} \right|_{n_{\text{nuc}}} = 2\gamma n_{\text{nuc}} \left. \frac{d\mu}{dn} \right|_{n_{\text{nuc}}} .$$
\[\frac{d\sigma}{dn} = \frac{\tilde{h} m_N(\sigma)}{\mu_{\text{eff}, \text{nuc}}} \left[ \frac{\partial^2 U_0}{\partial \sigma^2}(\sigma) + 2 \frac{\partial^2 U_\mu}{\partial \sigma^2}(\sigma) + \frac{6}{\pi^2} \hbar^2 m_N^2(\sigma) \sqrt{9 \mu_{\text{eff}, \text{nuc}}^2 - m_N^2(\sigma)} \right]^{-1}. \] 

Combining eqs. (47) and (46) the compression modulus yields information about \(\frac{\partial^2 U_0}{\partial \sigma^2}(\sigma)\) in addition to \(U_0(\sigma)\) and \(\frac{\partial U_\mu}{\partial \sigma}(\sigma)\) which are determined (for given \(m_N(\sigma)\) and \(\tilde{h}\)) by the condition \(U(\sigma) = 0\) and the field equation (35).

For any given value of \(\gamma_5\) the system of equations provides a mapping between the parameters \((\tilde{h}, g^{(\omega)}, \gamma_3, \gamma_4)\) and the quantities \((n_{\text{nuc}}, \beta, K_0, m_N(\sigma))\). For a demonstration of the range of values for various quantities of interest we report our results for two parameter sets with different \(\gamma_5\) (A and B) in tables 1–3. (For both sets \(\beta = -16.3\) MeV and \(n_{\text{nuc}} = \sigma^{(n)}\).)

In figure 2 we have plotted the binding energy per nucleon, \(\beta = \epsilon/n - m_N\), as a function of density corresponding to the parameter set A. For values of \(n\) larger than approximately 1.7 the details of the transition from nuclear to quark degrees of freedom become important and we don’t expect our results to remain quantitatively reliable. Similarly, our results for the baryon density as a function of pressure are displayed in figure 3. Figures 1 and 2 can be combined to yield the equation of state \(\epsilon(p)\) for \(n < 1.5\sigma^{(n)}\).
It is instructive to consider also an extreme scenario where the characteristic quark–baryon transition momentum $q_H$ takes on its lower bound

$$q_H(\sigma_{\text{nuc}}) = q_H = q_{\text{nuc}} = 259 \text{ MeV} \quad (48)$$

as exemplified in set $C$ of tables 1–3. Because of the $\Theta$–function in eq. (20) the nucleon contribution to the density does not increase as $\mu_{\text{eff}}$ exceeds the critical value given by eq. (44). On the other hand, there is a range of $\mu_{\text{eff}}$ for which the quark fluctuations (19) do not yet contribute to the baryon density. For this range the density $n$ will not depend on any other parameter of the model and $n_{\text{nuc}} = n(n)$ is guaranteed by eq. (48). Details of the potential in the vicinity of $\sigma_{\text{nuc}}$ are now affected by the transition from nucleon to quark degrees of freedom. For the simple choice (27), however, they do not depend on $\sigma_c$ or $\sigma_H$ provided both are smaller than $\sigma_{\text{nuc}}$. Because of the gap in $\mu_{\text{eff}}$ between the nucleon Fermi surface and the onset of quark fluctuations many properties become very simple. The minimum occurs within the range $\sigma_q < \sigma_{\text{nuc}} < \sigma_{nf}$. Here $\sigma_q$ corresponds to the onset of quark fluctuations

$$\sigma_q(\mu) \equiv \frac{1}{\hbar} \sqrt{\mu_{\text{eff}}^2 - q_H^2(\sigma_q)} \quad (49)$$

whereas $\sigma_{nf}$ denotes the maximal value of $\sigma$ for which all nucleon levels with $\vec{q}^2 \leq q_H^2$ are filled

$$\sigma_{nf}(\mu) \equiv \frac{1}{\hbar N(\sigma_{nf})} \sqrt{\mu_{\text{eff}}^2 - \frac{1}{9} q_H^2(\sigma_{nf})} \quad (50)$$

For values of $\sigma^{(\text{nuc})}(\mu)$ between $\max(\sigma_q, \sigma_H)$ and $\sigma_{nf}$ the baryon density is independent of $\mu$

$$n(\mu) = \frac{2}{3\pi} \frac{q_H^3}{\pi^{(n)}} \quad (51)$$
Figure 3: Baryon density as a function of pressure in the vicinity of the nuclear gas–liquid transition. Parameters correspond to set A in table 1.

Also, for \( \max(\sigma_q, \sigma_{H_H}) < \sigma < \sigma_{n_f} \) one finds that \( \partial U_\mu/\partial \mu \) is independent of \( \sigma \) and the constant shift (44) between \( \mu_{\text{eff}}(\sigma, \mu) \) and \( \mu \) holds for all \( \sigma \). The relation between \( n_{\text{nuc}} \) and \( q_H(\sigma_{\text{nuc}}) = q_H \) is such that up to a Fermi momentum \( q_H(\sigma_{\text{nuc}}) = q_H \) all levels are filled with nucleons (or bound quarks). In this crude picture the higher momentum levels (corresponding to a larger baryon number in a fixed volume) would have to be filled by free constituent quarks. This leads to a particularly simple explanation why nuclear density is almost independent of all other parameters characterizing the state of nuclear matter at \( T = 0 \), like pressure, baryon number or the \( Z/B \) ratio of a nucleus. Typical parameter values and corresponding characteristics of nuclear matter for this scenario can be found in the tables as set C.

For a given value of \( \sigma \) one finds in this scenario a range \( \sigma_{n_f} < \sigma_{\text{eff}} < \sigma_q \) with constant \( \partial U_\mu/\partial \mu \) where

\[
\mu_{n_f}(\sigma) = \sqrt{h^2_{\text{f}}(\sigma)\sigma^2 + \frac{1}{9}q_{\text{HH}}^2(\sigma)} \\
\mu_q(\sigma) = \sqrt{h^2\sigma^2 + q_{\text{HH}}^2(\sigma)}. \tag{52}
\]

In this range \( \mathcal{U} \) is independent of \( \mu \) and \( U_\mu \) has the simple form

\[
U_\mu(\sigma; \mu) = U_\mu(\sigma; \bar{\mu}) - \frac{1}{\pi^2} (\mu - \bar{\mu}) q_{\text{HH}}^3(\sigma). \tag{53}
\]

Here \( \bar{\mu} \) is a fixed reference value within the interval \( [\mu_{n_f}, \mu_q] \), and we remind the reader that \( q_H(\sigma > \sigma_{H}) = q_H \). We note that the location of the potential minimum at \( \mathcal{U}^{(\text{nuc})}(\mu) \) is independent of \( \mu \). Using \( \bar{\mu} = \mu_{\text{nuc}} \), the pressure and energy density of nuclear matter are

\[
p = 2 \frac{q_{\text{HH}}^3}{\pi^2} (\mu - \mu_{\text{nuc}}). \tag{54}
\]
\[ \epsilon = \frac{2}{\pi^2} q_n^3 \mu - p = (m_N + \beta) n . \]  

The \((T = 0)\) equation of state for nuclear matter for this extreme saturation scenario

\[ \frac{\partial \epsilon}{\partial p} = 0 , \quad \frac{\partial n}{\partial p} = 0 , \]  

implies a diverging compression modulus \(K_0\) and is therefore not realistic. Nevertheless, it is well conceivable that the true behavior of nuclear matter is somewhere between the simple version of the \(\sigma-\omega\) model and the extreme saturation scenario. In the language of the \(\sigma-\omega\) model this would be expressed through the momentum–dependence of couplings and wave function renormalizations (form factors).

Despite the substantial difference in the compression modulus the three scenarios (A)–(C) all show a similar value of \(\sigma_{\text{nuc}} \simeq 30\,\text{MeV}\) and therefore a nucleon mass in nuclear matter around 700\,MeV, as supported by some experimental evidence \[31,2\]. Also the critical value \(\mu_{\text{eff,nuc}} \simeq 250\,\text{MeV}\) is very similar for these three models.

7 Droplet model for nuclei

We describe nuclei as droplets of nuclear matter in a surrounding vacuum. For quantitative estimates of their properties we have to take into account that because of the surface tension the pressure inside the droplet is different from zero. The nucleus is at equilibrium if the pressure equals the derivative of the sum of surface and Coulomb energy\(^{7}\) with respect to the volume

\[ p = \frac{\partial E_{\Sigma}}{\partial V} + \frac{\partial E_c}{\partial V} . \]

The surface tension \(\Sigma\) can be expressed (thin wall approximation) in terms of the potential as

\[ \Sigma = 2 \left( \frac{Z}{Z_n} \right)^{1/2} \int_{\sigma_0}^{\sigma_{\text{nuc}}} d\sigma \sqrt{2U(\sigma ; \mu)} . \]  

We use the phenomenological relation \(Z/B \simeq (2 + 0.0153B^{2/3})^{-1}, \alpha = 1/137,\) and \(\kappa\) should be very close to one. This leads to a volume– and therefore baryon number dependent pressure

\[ p = \frac{4}{3} \left( \frac{3}{\pi} \right)^{1/3} q_{\text{nuc}}^3 \Sigma B^{-1/3} - \frac{4}{45\pi^2} \left( \frac{3}{\pi} \right)^{1/3} \alpha \kappa q_{\text{nuc}}^4 Z^2 B^{-4/3} \]  

and a total binding energy per nucleon

\[ \frac{E_B}{B} = 3\mu(p) - m_N - \frac{p}{n} + \frac{E_{\Sigma}}{B} + \frac{E_c}{B} = \beta + \frac{E_{\Sigma}}{B} + \frac{E_c}{B} - \frac{9}{2K_0} \frac{p^2}{n^2_{\text{nuc}}} + \ldots . \]

\(^{7}\)We neglect here the asymmetry effect from the proton–nucleon mass difference.
Neglecting the pressure term the mass formula for nuclei yields the “experimental” values

\[
\sum^{(n)} = 4.22 \cdot 10^4 \text{MeV}^3 \\
\kappa^{(n)} = 0.96 .
\] (61)

For \( B = 208(12) \) one finds \( p = 0.9(5.9) \cdot 10^6 \text{MeV}^4 \). The pressure therefore contributes to \( E/B \) only very little, \( \Delta(E/B) = -0.012(-0.53) \text{MeV} \) and can indeed be neglected for large \( B \). For small \( B \) eqs. (59), (60) result in an interesting correction to the mass formula. Comparison with fig. 3 shows that the baryon density is indeed almost independent of \( B \) for large nuclei. (Note that \( \pi^{(n)} \) formally corresponds to \( B \to \infty \).) For small nuclei the baryon density is enhanced.

The value of the surface tension as computed from eq. (58) for different parameter sets can be found in table 2. This result is quite reasonable in view of the uncertainties, first from the proper choice of a coarse grained potential in (58) (cf. [32–34]), and second from the choice of parameters in \( U_0 \). In fact, the successful explanation of the small ratio \( (\Sigma^{(n)})^{1/3}/q_{\text{nuc}} \) is encouraging. In summary, our simple approach gives a very reasonable picture for nuclei.

8 Two flavor quark hadron phase transition at \( T = 0 \)

At the critical chemical potential \( \mu_{\text{nuc}} \) the free energy \( U(\sigma;\mu) \) shows degenerate minima at \( \sigma_0 \), with vanishing density for \( T = 0 \), and at \( \sigma_{\text{nuc}} \) where nuclear matter density \( n_{\text{nuc}} \) is reached. In this section we consider densities higher than \( n_{\text{nuc}} \). For sufficiently high density one expects, and we observe, a further transition from nuclear matter to quark matter. This transition is related to a third distinct minimum of the effective potential \( U \).

For \( \mu \) increasing beyond \( \mu_{\text{nuc}} \) the density of nuclear matter increases beyond \( n_{\text{nuc}} \) and \( \bar{\sigma}^{(\text{nuc})}(\mu) \) decreases (scenarios \( A, B \)). In our crude picture this continues until \( \mu_{\text{eff}} \) reaches the value \( \mu_{\text{st}}(\bar{\sigma}^{(\text{nuc})}) \) (cf. eq. (52)). At the corresponding density the equation of state becomes very stiff, similar to scenario \( C \) discussed in section 6 (eq. (56)). The density can further increase because of quark contributions only once \( \mu_{\text{eff}} \) becomes larger than \( \mu_0(\bar{\sigma}^{(\text{nuc})}) \). For the parameter set \( C \) corresponding to the extreme saturation scenario \( \mu_{\text{eff}} \) must first exceed \( \mu_0(\bar{\sigma}^{(\text{nuc})}) = 334 \text{MeV} \) (\( \mu > 383 \text{MeV} \)) before the density can increase beyond nuclear density. On the other hand, the quarks always contribute to \( \partial U/\partial \mu \) at \( \sigma = 0 \). For \( \mu_{\text{eff}} > q_H \) and \( \mu_{\text{eff}} > \mu_{\text{st}}(\bar{\sigma}_{\text{nuc}}) \) the potential at \( \sigma = 0 \) decreases faster with \( \mu \) than for the nuclear matter phase at \( \bar{\sigma}^{(\text{nuc})} \) (cf. eqs.(17), (18) with \( q_H(\sigma = 0) = 0 \) and \( q_H(\bar{\sigma}_{\text{nuc}}) = q_H \)).

At sufficiently high \( \mu \) the effective potential (23) develops a new minimum near the origin at

\[
\bar{\sigma}^{(\text{qm})}(\mu) \simeq \frac{m_0^2(\mu)}{m_0^2(\mu)} \sigma_0 .
\] (62)

Here the mass parameter

\[
m_0^2(\mu) = \frac{1}{4} \left. \frac{\partial^2 U}{\partial \sigma^2} \right|_{\sigma=0} = \frac{3}{4\pi^2} \hbar^2 \mu_{\text{eff}}^2 - \overline{m}^2 ,
\] (63)
corresponds to the curvature of the potential at the origin. At this minimum the effective quark mass is small, $h\sigma^{(qm)} = h\sigma_0 m_0^2 / m_0^2(\mu)$ and vanishes in the chiral limit $m_\pi \to 0$. We identify the corresponding phase with quark matter. For a vanishing current quark mass ($\lambda = 0$, $m_\pi = 0$) chiral symmetry is restored in this phase. As $\mu$ increases, the height of the potential for the quark matter phase $U^{(qm)}(\mu) = U(\sigma^{(qm)}(\mu); \mu)$ decreases much faster than the one for the nuclear matter phase $U^{(nuc)}(\mu) = U(\sigma^{(nuc)}(\mu); \mu)$, where we remind that decreasing $U$ corresponds to increasing the pressure $p = -U$. This can be seen directly from eqs. (19), (20), since $\sigma^{(qm)} \ll \sigma^{(nuc)}$ and $\partial U / \partial \sigma(\sigma, \mu) = 0$. One concludes that for large enough $\mu$ the absolute minimum of $U$ is always given by eqs. (62)--(64).

Away from the chiral limit the quark–hadron phase transition is not characterized by a change of symmetry in our model\(^8\). It could therefore be of first order or a crossover. (A second order transition would require an additional tuning of parameters.) A first order transition is guaranteed if a range of $\mu$ exists for which $m_0^2(\mu)$ is positive and substantially larger than $m_\pi^2$ whereas $\sigma^{(nuc)}(\mu)$ remains of the same order of magnitude as $\sigma_0$. In this case one has $\sigma^{(qm)}(\mu) \ll \sigma^{(nuc)}(\mu)$ and the mass term at $\sigma^{(qm)}(\mu)$ is well approximated by $m_0^2(\mu)$ and therefore positive. By definition the mass term at $\sigma^{(nuc)}(\mu)$ is also positive. Two local minima of $U$ coexist for this range of $\mu$. As $\mu$ is increased further the mass term at $\sigma^{(qm)}(\mu)$ monotonically grows (cf. eq. (63)) thus excluding a crossover. Typical values for $\overline{m}$ from an extrapolation of the polynomial potential (29) for the parameter sets $(A, B, C)$ are (833.4, 709.7, 584.9) MeV. For these values a first order transition would be guaranteed for $m_0(\mu_{eff}) \approx 400$ MeV or $h \mu_{eff} / 260$ MeV $> (13.10, 11.82, 9.92)$ if $\sigma^{(nuc)}(\mu_{eff})$ remains of order $\sigma_0$. We use here a vacuum constituent quark mass of 330 MeV or a Yukawa coupling $h = 7.1$. For low enough values of $\overline{m}$ (as, for instance, in scenario C) the bound can be met for rather low values of $\mu_{eff}$. A first order transition occurs then independently of other details of the potential.

Actually, the most natural scenario is a first order transition at a critical value $\mu_{eff, qm}$ which is lower than $\mu_q(\sigma^{(nuc)})$. In this case the quarks do not contribute in the nuclear matter phase and nucleons are absent in the quark matter phase. Typical values of $\mu_{eff, qm}$ for this situation are somewhat above $q_H$, say, $\mu_{eff, qm} \approx (300 - 400)$ MeV. The baryon density in the quark phase at this transition would be around three times nuclear density. This occurs naturally for values of $\mu_0$ somewhat below $\mu_{eff, qm}$. A recent investigation of the coarse grained effective potential in the framework of the average action for a nonvanishing baryon chemical potential [35] finds values of $\mu_0$ only slightly above the constituent quark mass. This can be interpreted as first direct information about the potential $U_0$ near the origin. It strongly supports the above scenario.

In order to estimate the critical value $\mu_{eff, qm}$ for the quark hadron phase transition in this scenario we equate the pressure in the quark matter phase (for the approximation $m_\pi = 0$)

$$ p^{(qm)} = \frac{1}{2\pi^2} \left( \mu_{eff}^4 - \mu_0^4 \right) $$

\(^8\)As mentioned in the introduction, we do not take into account in the present approach the possible spontaneous breaking of color at high density [6–10].
with the one in the nuclear matter phase

\[ p^{(\text{nuc})} = \frac{2}{\pi^2} q_{\text{H}}^3 \mu_{\text{eff}} - \frac{2}{3\pi^2} q_{\text{H}}^3 \sqrt{m_{\text{N}}^2 + p^2} + p . \]  

(66)

Here \( m_{\text{N}} \) and \( p \) are the nucleon mass and pressure corresponding to \( \mu_{\text{eff}} = \mu_{\text{nf}} \), respectively. For \( q_{\text{H}} \) not much larger than \( q_{\text{nuc}} \) one may neglect \( p \). Inserting two typical sets of values, \( \mu_0 = 320 \text{ MeV}, q_{\text{H}} = 1.05(1.2)q_{\text{nuc}}, m_{\text{N}} = 0.7(0.6)m_{\text{N}}, \) one obtains \( \mu_{\text{eff,qm}} = 390(440) \text{ MeV}. \) This corresponds to a critical baryon density in the quark matter phase

\[ n_{\text{qm}} = 3.4(4.9)n_{\text{nuc}} . \]  

(67)

The results of a quantitative analysis for the polynomial potential (29) with the parameter sets A–C are reported in table 3. One always finds a first order transition between nuclear and quark matter. The sets A and B with high values of \( \mu_0 \) and \( m \) lead, however, to relatively large values of the critical chemical potential \( \mu_{\text{qm}} \) as well as \( \mu_{\text{eff,qm}} \) and the associated critical baryon density \( n_{\text{qm}} \) in the quark matter phase. One should remember, though, that the observed meson masses and decays contain only very limited information about the behavior of \( U_0 \) near the origin. We do not expect a polynomial expansion of \( U_0 \) around \( \sigma_0 = 0 \) to lead to a very good approximation of the potential in the vicinity of the origin. It is certainly possible to extrapolate a form of \( U_0 \) which is compatible with nuclear physics constraints in the region \( 0.6\sigma_0 < \sigma < 1.5\sigma_0 \) to a wide range of parameters \( \mu_0 \) and \( m \) characterizing the behavior of \( U_0 \) near \( \sigma = 0 \). Furthermore, a possible \( \sigma \)-dependence of \( g(\omega) \) or \( M_\omega \) would substantially affect the ratio \( \mu_{\text{eff}}/\mu \). In particular, a smaller value of \( g(\omega)/M_\omega \) for the quarks (near \( \sigma = 0 \)) would enhance the effective chemical potential for given \( \mu \) in the quark phase, thereby shifting the transition to lower values of \( \mu \). We conclude that the values in table 3 (especially those corresponding to sets A and B) should be considered as an illustration of the uncertainties still inherent in the polynomial extrapolation rather than as actual predictions (which we expect closer to eq. (67)). This uncertainty is reduced significantly once independent information about the behavior of \( U_0 \) near \( \sigma = 0 \) becomes available as, for example, from ref. [35].

<table>
<thead>
<tr>
<th>( \mu_{\text{qm}} ) MeV</th>
<th>( \mu_{\text{eff,qm}} ) MeV</th>
<th>( n_{\text{qm}} )</th>
<th>( n_{\text{nuc}} )</th>
<th>( \sigma_{\text{qm}} ) MeV</th>
<th>( M_{\text{q,qm}} ) MeV</th>
<th>( \Sigma_{\text{qm}} ) 10^6 MeV</th>
</tr>
</thead>
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<td>A 975.5</td>
<td>530</td>
<td>8.6</td>
<td>2.2</td>
<td>15.8</td>
<td>5.9</td>
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<tr>
<td>B 859.7</td>
<td>484</td>
<td>6.5</td>
<td>2.2</td>
<td>15.4</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td>C 511</td>
<td>370</td>
<td>2.9</td>
<td>5</td>
<td>34</td>
<td>1.7</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Critical quantities for the quark–hadron phase transition. The values for the chemical potentials \( \mu_{\text{qm}} \) and \( \mu_{\text{eff,qm}} \), the baryon density in the quark matter phase \( n_{\text{qm}} \), the order parameter \( \sigma_{\text{qm}} \), the effective quark mass \( M_{\text{q,qm}} \) and the surface tension \( \Sigma_{\text{qm}} \) for the sets A and B should be interpreted as an illustration of the uncertainties of a polynomial extrapolation of the potential \( U_0 \) to the origin \( \sigma = 0 \).
We have also computed the surface tension $\Sigma_{qm}$ for the quark–hadron transition at the critical $\mu_{qm}$. It turns out to be much larger than the one between the nucleon gas and nuclear matter. The quantitative value is given in table 3. The surface tension depends, however, strongly on the details of the transition from quark to nucleon degrees of freedom (e.g., $\sigma_c$ and $\sigma_H$). Stability of nuclear matter requires the critical chemical potential $\mu_{qm}$ for a possible quark–hadron transition to be above $\mu_{nucl}$ as realized for our parameters. At the quark hadron phase transition the quark mass $M_q$, $\mu_{qm}$ in quark matter is much smaller than in nuclear matter. Nevertheless, it is substantially larger than the current quark mass. We quote the value of the order parameter $\sigma \equiv \sigma(qm)(\mu_{qm})$ for the quark phase in table 3 together with the corresponding quark mass. For $\mu = \mu_{qm}$ the values in the nuclear matter phase are $\sigma(nuc)(\mu_{qm}) \simeq 24$ MeV, $M(nuc \mu_{qm}) \simeq (165 - 170)$ MeV. Since $\mu_{qm}$ may exceed the effective strange quark mass, the strange quarks could play a role for this transition in real QCD. We investigate this in the following.

9 Strangeness

We first have to generalize the potential $U_0$ for the three flavor situation. A phenomenological analysis of the potential $U_0(\sigma,\sigma_s)$ in terms of $SU(3)_L \times SU(3)_R$ invariants can be found in ref. [20]. We simply report here that for a range of allowed parameters the value $\sigma_{bs}(\sigma)$ which minimizes the potential for arbitrary $\sigma$ depends only very little on $\sigma$. We concentrate on this particularly simple situation where $\sigma_{bs}(\sigma)$ can be identified with its vacuum value [20]

$$\sigma_{bs} = \left[2 \left(\frac{Z_\pi}{Z_K}\right)^{1/2} \frac{f_K}{f_\pi} - 1\right] \sigma_0 \simeq (1.82 - 1.90) \sigma_0 = (84.6 - 88.3) \text{ MeV}$$

(68)

as long as the density of strange baryons remains zero. Here $Z_\pi$ and $Z_K$ denote the pion and kaon wave function renormalizations, respectively. The remaining $\sigma$–dependence can then be parameterized by the potential (29), i.e. $U_0(\sigma,\sigma_{bs}) \equiv U_0(\sigma)$.

For the contribution of the strange quarks to the baryon number dependence of the potential $U_0^{(Q_s)}$ we use eq. (19) with $\sigma_a = \sigma_s$ and $h_a = h_s$. For the low momentum modes we include the $\Lambda$, $\Sigma$ and $\Xi$ baryons with masses

$$m_\Lambda = m_N + \left(1 - \frac{\xi}{3}\right) \Delta, \quad m_\Sigma = m_N + (1 + \xi) \Delta$$

$$m_\Xi = m_N + 2 \Delta, \quad \xi = 0.31$$

(69)

From $\Delta(\sigma_0,\sigma_{bs}) = 189$ MeV we find $\tilde{h}_s \simeq \tilde{h} \simeq 5$. The density of strange baryons as well as $U^{(s)}_\mu$ vanishes for $3\mu_{eff}(\Lambda) < m_\Lambda(\sigma,\sigma_s)$. We note that the coupling of the $\omega$ to the $\Lambda$–baryon differs from the one to the nucleons implying for a vanishing strange baryon density $3\mu_{eff}(\Lambda) = (1 - \frac{\xi}{3}) \mu + (2 + \frac{\xi}{3}) \mu_{eff}$. For nuclear matter this yields $3\mu_{eff,nuc}(\Lambda) \simeq 810$ MeV. Since the strangeness content of nuclei is essentially zero we conclude that $3\mu_{eff,nuc}$ should be smaller than or around the $\Lambda$ mass.

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in nuclear matter, i.e. \( m_{\Lambda,\text{nucl}} = m_\Lambda(\sigma_{\text{nucl}}, \mu_{\text{nucl}}) \sim m_\Lambda(\sigma_{\text{nucl}}, \sigma_{0\text{b}}) = m_{\Lambda,0} - 2.1 \tilde{h}(\sigma_0 - \sigma_{\text{nucl}}) = (957, 944, 947) \text{ MeV} \). (The three values are for the parameter sets A, B, C.) This is the case and one concludes that strangeness does not play a role for nuclear matter.

For the quark–hadron phase transition we have to compare

\[
\mu_{\text{eff},s} = \mu - \frac{1}{3} \frac{g^{(s)2}}{M_\phi^2} n_s
\]  

(70)

with the strange quark mass. Here

\[
n_s = -\frac{1}{3} \left( \frac{\partial U^{(Q,s)}}{\partial \mu} + \frac{\partial U^{(B,s)}}{\partial \mu} \right)
\]

\[
n - n_s = -\frac{2}{3} \frac{\partial U_\mu}{\partial \mu}
\]

(71)

are the densities of strange and non–strange baryons, respectively. We may estimate the relevant strange quark mass either from \( h_s \sigma_{0\text{b}} = h \sigma_0 + \Delta \simeq 520 \text{ MeV} \) (\( h_s = 6 \)) or assume \( h_s \simeq h \simeq 7 \) which yields \( h_s \sigma_{0\text{b}} = 610 \text{ MeV} \).

From table 3 we infer that typical values of \( \mu_{\text{qm}} \) for the two flavor calculation could exceed this range of values (cf. sets A, B). In this case one would conclude that the strange quark content in the quark phase plays a role, \( n_s > 0 \). A reliable calculation of the ratio \( n_s/(n - n_s) \) would require, however, more detailed information about \( U_0 \) and \( \mu_{\text{qm}} \). For \( n_s > 0 \) the strange quark fluctuations will affect the quark–hadron phase transition. Nevertheless the main qualitative aspects of this transition are expected to remain similar to the two flavor case. From set C we learn that it is also conceivable that \( n_s \) remains zero at the quark–hadron phase transition.

### 10 High temperature

At non–vanishing temperature one infers for the quark contribution to the \( \mu \)–dependence of the effective meson potential (neglecting strangeness)

\[
U^{(n)}(\sigma; \mu, T) = -\frac{3}{\pi^2} \int_{q_H}^{\infty} dq \ q^2 T \times \left\{ \ln \frac{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2} - \mu}{T} \right\}}{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2}}{T} \right\}} + \ln \frac{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2} + \mu}{T} \right\}}{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2}}{T} \right\}} \right\}
\]

(72)

whereas the nucleon contribution reads

\[
U^{(n)}(\sigma; \mu, T) = \frac{1}{\pi^2} \int_0^{q_H} dq \ q^2 T \times \left\{ \ln \frac{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2} - 3\mu}{T} \right\}}{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2}}{T} \right\}} + \ln \frac{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2} + 3\mu}{T} \right\}}{1 + \exp \left\{ -\frac{\sqrt{q^2 + 9h^2\sigma^2}}{T} \right\}} \right\}
\]

(73)
with \( U_\mu \equiv U^{(q)}_\mu + U^{(n)}_\mu \) generalizing eqs. (24)–(26) for \( T > 0 \). Instead of a quantitative numerical investigation we only report here a few qualitative properties. First, the smoothening of the \( \Theta \)–function for \( T \neq 0 \) (compare eqs. (17) and (18)) implies a nonzero baryon density even for small \( \mu \). The equilibrium state for low \( T \) and low \( \mu \) can be associated with a gas of nucleons. For low enough temperature the phase transition to nuclear matter remains of first order, with a discontinuity in the baryon density. For increasing temperature the density contrast between the nucleon gas and liquid becomes smaller and finally disappears at a second order endpoint of the critical line. The fate of the quark–hadron phase transition for increasing temperature is not obvious. In fig. 4 we depict a qualitative phase diagram which is suggested by our zero temperature results.

![Phase diagram](image)

**Figure 4:** Schematic diagram for the phases of strongly interacting matter inferred from this work. Solid lines correspond to first order transitions, points represent second order transitions and dashed lines indicate crossover behavior. We have not addressed transitions within the quark matter phase as related to color superconductivity [7–10].

If the zero density high temperature quark–hadron transition is of first order, one expects an odd number of endpoints in the phase diagram. If, instead, it is an analytical crossover, the low temperature quark–hadron transition line is expected to end in a second order transition \([9,12]^{9}\). We should mention, however, that a crossover from hadron to quark matter at high density and \( T = 0 \) would modify the topology of the phase diagram.

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\(^{9}\)All endpoints are expected in the Ising universality class. Our simple truncation is not supposed to yield the correct critical exponents.
11 Conclusions

We have presented here a new method for the computation of the dependence of the free energy on the chemical potential. It is based on an approximate solution to an exact functional differential equation. This method allows us to put mean field theory into a more systematic context. Chiral symmetry is explicitly implemented and phenomenological information about pion masses and decay constants is taken into account. Expressions which are close to mean field theory describe the difference in the free energy between vanishing and non–vanishing chemical potential. They can be considered as the leading order in a series of systematic truncations of the exact differential equation (6). On the other hand, the free energy for vanishing chemical potential is not reliably described by mean field theory. Many relevant characteristics of this quantity can, however, be inferred from observation.

We use quark and nucleon degrees of freedom in their appropriate momentum ranges. For high momenta, $\vec{q}^2 > q_H^2$, the quark meson model gives a useful approximation. For small momenta, $\vec{q}^2 < q_H^2$, the effects of confinement have to be taken into account and we describe the carriers of baryon number as nucleons. Our simple model leads to a unified description of the nucleon gas, nuclear matter and quark matter phases. The appearance of three phases of strongly interacting matter is related to three distinct minima of the effective potential for the $\sigma$–field. (Typically only two coexist simultaneously.) This rich structure is a consequence of the fact that more energy levels fall below the Fermi energy for nucleons than for quarks. This results in a substantial enhancement of the $\mu$–dependence of the free energy due to low momentum nucleon fluctuations. The inclusion of nucleon degrees of freedom is crucial for a description of the liquid–gas nuclear transition. It also shifts the transition from nuclear to quark matter to a larger chemical potential and baryon number density, as compared to a description of the fermionic fluctuations in terms of quarks alone. This is due to a non–zero pressure at the coexistence between quark and nuclear matter and may give a qualitative explanation why standard Nambu–Jona-Lasinio type models typically predict a transition to quark matter at comparably low densities.

The gas–liquid nuclear transition can be described within the present model in a quantitative way, at least for low temperature. Our model gives a successful description of the nuclear droplet model, consistent with the observed nuclear density, the binding energy per nucleon, the compression modulus and the nucleon mass in nuclear matter. It explains why nuclear density is approximately independent of the baryon number of a nucleus. We also obtain a realistic value for the nuclear surface tension and we have computed small corrections to the baryon density and the mass formula for nuclei due to nonvanishing pressure. Isospin violation and electromagnetism can be incorporated easily in our model. This should give a reliable equation of state for neutron stars in the region of moderate densities. Our model predicts coupling constants which directly enter the effective nucleon–nucleon potential. Comparison with nucleon scattering experiments will provide an interesting test in the future.

We find a first order phase transition between nuclear and quark matter at high density and vanishing temperature. We emphasize, however, that some important information is still missing for a quantitative understanding of the quark–hadron transition: The first problem
concerns the appropriate formulation of a coarse grained effective potential and a determination of its shape. Within the linear quark meson model we have addressed this issue in the context of the average action [35]. The second loose end is a more detailed understanding of the change from quark to baryon effective degrees of freedom. This concerns primarily the behavior of nuclear matter at densities much larger than nuclear density. It is therefore also relevant for a quantitative description of the quark–hadron transition. This second problem is common to all analytical descriptions of baryonic matter at very high density. Whereas for the high temperature chiral phase transition the effects of confinement are presumably only minor [18] they play an important role for the high density quark hadron transition.

References


