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The Kondo Model with a Bulk Mass Term

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We introduce two massive versions of the anisotropic spin 1/2 Kondo model and discuss their integrability. The two models have the same bulk sine-Gordon interactions, but differ in their boundary interactions. At the Toulouse free fermion point each of the models can be understood as two decoupled Ising models in boundary magnetic fields. Reflection S-matrices away from the free fermion point are conjectured.

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Introduction

The s-d exchange model, often referred to as the Kondo model, has been extensively discussed since it was first used by Kondo [1] to calculate the effects of impurities on the resistivity of a metal. Various approaches have been applied to study this model, including renormalization group methods [2], the Bethe ansatz [3]-[9] and conformal field theory [10]-[12]. Traditionally the Kondo model contains a single coupling J. Allowing the exchange couplings to differ gives the anisotropic Kondo model. Regardless of whether there is an anisotropy or not, the Kondo model is a massless integrable system for arbitrary spin [13],[14].

In this paper we present two massive generalizations of the anisotropic spin 1/2 Kondo model and discuss their integrability. These models are bosonic boundary field theories related to the original fermionic system via bosonization. Such massive theories have possible applications to 1D impurity problems with an excitation gap. An example is a magnetic impurity in a superconductor, where the mass represents the BCS energy gap [15].

The motivation for considering our particular models came as follows. The massive boundary sine-Gordon model (MBSG) is a boundary field theory of a single boson ϕ with a bulk interaction in the action $G \int dxdt \cos(\beta\phi)$ and a boundary interaction $\lambda \int dt \cos(\beta(\phi - \phi_0)/2)$. The model is known to be integrable for arbitrary G, λ and ϕ_0. The massless limit $G \to 0$ is well defined, which defines the massless boundary sine-Gordon model [16]. On the other hand, the usual anisotropic Kondo model after bosonization has a boundary sine-Gordon like interaction but with the inclusion of spin operators $S_\pm$ at the boundary: $\lambda \int dt\ S_+ e^{i\beta\phi/2} + S_- e^{-i\beta\phi/2}$, and the bulk is just a free massless scalar field. If one compares the massless boundary sine-Gordon with the Kondo model, one finds that the same sine-Gordon spectrum of particles diagonalizes the boundary interaction; the difference is in the detailed form of the scattering matrices for reflection off the boundary [17]. So the question naturally arises whether it is possible to add a bulk sine-Gordon term to the Kondo theory and preserve the integrability, in analogy with the massive boundary sine-Gordon theory.

If the theories we define in this paper are indeed integrable this implies that the usual bulk sine-Gordon integrals of motion are not spoiled by the boundary interactions. In this work we do not establish the integrability of our models by studying these integrals of motion. Rather we take the following approach to studying the problem. In [18] the most general solution to the boundary Yang-Baxter equation corresponding to a massive sine-Gordon spectrum and no boundary degrees of freedom was described and related to the massive boundary sine-Gordon model. The known scattering description of the Kondo model also does not require boundary degrees of freedom for spin 1/2 due to screening. In the massive theory one might expect a competition between the gap effects and the screening, since the screened state is a property of a new infra-red fixed point, but the existence of the gap introduces an infra-red cutoff, and this may imply that this infra-red fixed point is never reached. However non-trivial renormalization group beta functions only occur at the isotropic point. Our analysis of the free fermion point indeed indicates that a variant of screening does occur in the massive case. We thus assume this continues to hold for the massive model, namely that the scattering description does not require any boundary degrees of freedom, at least in some sector. Then if the massive version of the Kondo model is integrable, since it involves a bulk sine-Gordon theory, its reflection S-matrix off the boundary must be contained in the general solution found by Ghoshal and Zamolodchikov (GZ) [18] up to overall scalar (CDD) factors. If one simply adds a bulk sine-Gordon term to the usual Kondo model, and specializes to the free fermion point, then the reflection amplitudes can be computed explicitly and it is found that they do not coincide with those of GZ, even up to CDD factors. However we found that a slight modification of the boundary interaction does match onto the GZ solution and it is this model for which we conjecture an extension away from the free fermion point. This latter model requires that the boundary and bulk couplings to not be independent. The first model on the other hand does not have the constraints of the boundary Yang-Baxter equation at the so-called reflectionless points, so in this situation we also conjecture an exact reflection S-matrix.

In the massless cases the main distinction between the Kondo and boundary sine-Gordon theories is that sine-Gordon has flow between free boundary conditions in the ultra-violet and fixed boundary conditions in the infra-red, whereas the Kondo model maintains a free boundary condition throughout. Our conjectured reflection S-matrix for the modified massive Kondo model is in accordance with this, in that it corresponds to the reflection S-matrix for the boundary sine-Gordon theory at the free boundary condition times a CDD

factor.

We present our results as follows. In section 1 we define the two massive versions of the Kondo model that we consider. The first, simply referred to as massive Kondo (MK), is just a sine-Gordon model with the usual Kondo interaction at the boundary. The second, the modified massive Kondo model (mMK), has a slightly different boundary interaction. More importantly, in the mMK model the boundary coupling λ is not independent of the bulk coupling G, but satisfies $G \propto \lambda^2$. This is reminiscent of what happens in the boundary Toda theories [19]. In section 2 we study both models at the free fermion point, and relate them both to two decoupled Ising models in appropriate boundary magnetic fields. In section 3 we compare with the GZ solution and propose reflection S-matrices away from the free fermion point. Section 4 contains our results on boundary bound states for the various models. An appendix reviews the mapping of the Kondo model to a bosonic boundary field theory.

1. The Field Theory Models

The anisotropic Kondo Hamiltonian is

$$H^K = \sum_{\vec{k}\sigma} \epsilon(\vec{k}) c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} + \frac{J_z}{2} s_z \sum_{\vec{k}\vec{k}'\sigma\sigma'} c^\dagger_{\vec{k}\sigma} (\sigma^z)_{\sigma\sigma'} c_{\vec{k}'\sigma'} + \frac{J_\perp}{2} \sum_{\vec{k}\vec{k}'\sigma\sigma'} \left(s_x c^\dagger_{\vec{k}\sigma} (\sigma^x)_{\sigma\sigma'} c_{\vec{k}'\sigma'} + s_y c^\dagger_{\vec{k}\sigma} (\sigma^y)_{\sigma\sigma'} c_{\vec{k}'\sigma'} \right), \tag{1.1}$$

where $c^\dagger_{\vec{k}\sigma}$ are conduction electron creation operators with wave vector $\vec{k}$ and spin σ (up $\uparrow$ or down $\downarrow$), $\sigma^{x,y,z}$ are the Pauli matrices, $\vec{s}$ is the impurity spin operator (spin 1/2) at $\vec{r} = 0$ and J_z and $J_\perp$ are the exchange couplings. In the isotropic Kondo model $J_z = J_\perp$. For s-wave scattering near the fermi surface, the spin sector of (1.1) can be mapped onto the following bosonic boundary field theory

$$H^K = \frac{1}{2} \int_{-\infty}^0 dx \, \left((\partial_t \phi)^2 + (\partial_x \phi)^2 \right) + \lambda \left(S_+ e^{i\beta\phi(0)/2} + S_- e^{-i\beta\phi(0)/2} \right), \tag{1.2}$$

where λ is the coupling and β determines the anisotropy. The isotropic point corresponds to $\beta = \sqrt{8\pi}$. The series of steps leading to (1.2) are outlined in the appendix.

The boundary field theory (1.2) is our starting point, which we will simply refer to as the Kondo model. The parameters λ and β are considered to be arbitrary parameters, independent of their relation to the couplings J_z and $J_\perp$ (see appendix). This model is known to be integrable for any spin. However, integrability requires that the matrices S_i form a spin j representation of the q-deformed quantum algebra $su(2)_q$ [13], where $q = \exp(i\beta^2/8)$. The $su(2)_q$ relations are

$$[S_z, S_\pm] = \pm 2 S_\pm, \quad [S_+, S_-] = \frac{q^{S_z} - q^{-S_z}}{q - q^{-1}}. \tag{1.3}$$

For the isotropic case, $q = -1$ and (1.3) reduces to the usual $su(2)$ algebra. For spin 1/2, the difference between $su(2)_q$ and $su(2)$ is not important since the Pauli matrices also form a representation of $su(2)_q$ for arbitrary q.

1.1. The Massive Kondo Models

We are interested in a massive generalization of (1.2), which in the massless limit reproduces the Kondo behaviour. In particular, the massless limit should give the spin 1/2 Kondo scattering matrix. This has led us to consider two models.

The first model, the MK model, has the Hamiltonian

$$H^{MK} = \frac{1}{2} \int_{-\infty}^0 dx \, \left((\partial_t \phi)^2 + (\partial_x \phi)^2 - G \cos(\beta\phi) \right) + \lambda \left(S_+ e^{i\beta\phi(0)/2} + S_- e^{-i\beta\phi(0)/2} \right). \tag{1.4}$$

The bulk part is simply the sine-Gordon model. This model was studied in [20] at the free fermion point ($\beta = \sqrt{4\pi}$) and the reflection S-matrix was calculated. We will re-derive the free fermion reflection amplitudes and discuss their extension to arbitrary β. It is known that the bulk sine-Gordon spectrum also diagonalizes the boundary interaction [18], thus (1.4) is perhaps the most obvious massive generalization. The bulk spectrum consists of solitons for $\sqrt{4\pi} \le \beta < \sqrt{8\pi}$, and for $\beta < \sqrt{4\pi}$ there are also breathers. (For $\beta > \sqrt{8\pi}$ the bulk energy is unbounded.) The soliton and breather masses are functions of G and β. We will argue that unlike the Kondo model, (1.4) is not integrable for arbitrary β, but only at certain so-called reflectionless points.

This brings us to the second model, the mMK model, with the Hamiltonian

$$H^{mMK} = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 + (\partial_x\phi)^2 - G\cos(\beta\phi)\right) \\ -i\lambda\left(\cos\frac{\beta}{2}(\phi(0)-\hat{\phi})S_+ - S_-\cos\frac{\beta}{2}(\phi(0)-\hat{\phi}^*)\right), \tag{1.5a}$$

where

$$\hat{\phi} = -\frac{2}{\beta}\left(\frac{\pi}{2} - i\hat{\phi}_0\right), \tag{1.5b}$$

and the coupling λ is related to the bulk coupling G

$$\lambda^2 \propto G, \quad \lambda = \sqrt{2M} \text{ at } \beta = \sqrt{4\pi}. \tag{1.5c}$$

Here $\hat{\phi}_0$ is an arbitrary real parameter and M is the fermion (soliton) mass. Let us compare this with the MBSG model

$$H^{MBSG} = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 + (\partial_x\phi)^2 - G\cos(\beta\phi)\right) + \lambda\cos\frac{\beta}{2}(\phi(0)-\phi_0). \tag{1.6}$$

The MBSG model is integrable for all values of λ, ϕ_0 and G, and all can be considered as independent [18]. In the mMK model there is only one free coupling $\hat{\phi}_0$, with λ being a function of the bulk coupling G, i.e. not a free parameter. We believe that the mMK model is integrable for all values of β and conjecture a soliton reflection S-matrix in section 3, which in the massless limit recovers the Kondo model. Note that for both massive models, the scaling dimension of the bulk coupling G is twice that of the boundary coupling λ. Because of the cosines and exponentials, these couplings have anomalous dimensions, resulting in a scaling dimension of $2 - \beta^2/4\pi$ for G and $1 - \beta^2/8\pi$ for λ.

2. Boundary Reflection S-Matrices at the Free Fermion Point

In this section we derive the reflection S-matrices at the free fermion point starting from the action. A similar procedure was used in [21] for the MBSG model. For the MK model this was first done in [20]. Here we repeat that calculation in a more general context, and by making a change of basis give a clearer understanding into the structure of the reflection S-matrix.

The bulk action for both models is

$$S_{\text{bulk}} = \frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 - (\partial_x\phi)^2 + G\cos(\beta\phi)\right). \tag{2.1}$$

In the massless limit, the scalar field can be separated into its right and left moving components

$$\phi(z^+, z^-) = \varphi(z^+) + \overline{\varphi}(z^-), \quad z^\pm = t \pm x, \tag{2.2}$$

in terms of which the Neumann boundary condition $\partial_x\phi = 0$ at $x = 0$ becomes

$$\partial_{z^+}\varphi(z^+) = \partial_{z^-}\overline{\varphi}(z^-), \quad (x = 0). \tag{2.3}$$

This implies

$$\varphi(z^+) = \overline{\varphi}(z^-) - \frac{\sigma}{\sqrt{4\pi}}, \quad (x=0), \tag{2.4}$$

where σ is a constant related to the zero modes of ϕ [21], [22]. The massless boson bulk theory is equivalent to a massless Dirac theory through bosonization/fermionization. Let $\psi_\pm(z^+)$ and $\overline{\psi}_\pm(z^-)$ respectively be the left and right chiral components of the Dirac fermion with $U(1)$ charge ± 1. Then the bosonization relations read

$$\psi_\pm(z^+) = e^{\pm i\sqrt{4\pi}\varphi(z^+)}, \quad \overline{\psi}_\pm(z^-) = e^{\mp i\sqrt{4\pi}\overline{\varphi}(z^-)}, \quad \psi_- = (\psi_+)^\dagger, \ \overline{\psi}_- = (\overline{\psi}_+)^\dagger. \tag{2.5}$$

In terms of the fermions the (free) boundary condition becomes

$$\psi_\pm = e^{\mp i\sigma}\overline{\psi}_\mp, \quad (x=0) \tag{2.6}$$

which breaks the charge symmetry.

For the massive case, the bulk theory (2.1) is equivalent to a free massive Dirac theory at $\beta = \sqrt{4\pi}$ [23],[24]. The fields $(\psi_\pm, \overline{\psi}_\pm)$ are now no longer chiral. These can still be related to ϕ as in (2.5), but now φ and $\overline{\varphi}$, with $\phi = \varphi + \overline{\varphi}$, are not simply the left and right moving components [25]. The fermions $\psi_\pm$ and $\overline{\psi}_\pm$ correspond to the solitons in the sine-Gordon model. Considering the massive system as a perturbation of the massless one, we use the same boundary conditions (2.4) and (2.6). The massive fermionic action which enforces (2.6) takes the form

$$S = S_{\rm bulk} + S_{\rm bc} \tag{2.7}$$

$$S_{\rm bulk} = S_{\rm kin} + S_{\rm mass} \tag{2.8a}$$

$$S_{\rm kin} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left[i\psi_-(\partial_t - \partial_x)\psi_+ + i\psi_+(\partial_t - \partial_x)\psi_- + i\overline{\psi}_-(\partial_t + \partial_x)\overline{\psi}_+ + i\overline{\psi}_+(\partial_t + \partial_x)\overline{\psi}_- \right] \tag{2.8b}$$

$$S_{\rm mass} = 2i \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\, M\big(\psi_-\overline{\psi}_+ - \overline{\psi}_-\psi_+\big) \tag{2.8c}$$

$$S_{\rm bc} = -i \int_{-\infty}^{\infty} dt\, \big(e^{i\sigma}\psi_+\overline{\psi}_+ + e^{-i\sigma}\psi_-\overline{\psi}_-\big), \tag{2.9}$$

where M is the soliton mass. The action $S_{\rm bulk}$ is simply a Dirac action. $S_{\rm bc}$ is required to produce the boundary conditions. In varying S with respect to all the fields, the boundary terms give (2.6). Note that the ψ's are different from the original fermions occurring in the Kondo model. We went from (1.1) to the bosonic boundary field theory for ϕ (1.2) by bosonizing (see appendix), and now we have (i) added a mass term and (ii) re-fermionized, going from ϕ to a new set of Dirac fermions. The mass term (2.8c) is not a mass term for the original Kondo fermions. (In the Kondo model a similar transformation gives the resonant model [26],[32].) The action (2.7) is common to both the MK and mMK model.

We now specify the interaction with the spin operators $S_\pm$. It will be convenient to consider a more general boundary interaction, with the MK and mMK interactions being special cases. Consider the following action

$$S_{\rm int} = -\lambda_1 \int_{-\infty}^{\infty} dt\ \left(e^{i\beta\phi(0)/2}S_+ + S_-e^{-i\beta\phi(0)/2}\right) + \lambda_2 \int_{-\infty}^{\infty} dt\ \left(e^{-i\beta\phi(0)/2}S_+ + S_-e^{i\beta\phi(0)/2}\right), \tag{2.10}$$

where λ_1 and λ_2 are couplings which can be different. The MK model is obtained by setting $\lambda_2 = 0$ and $\lambda_1 = \lambda$. For the mMK model we need to take $\lambda_1 = \lambda e^{\hat{\phi}_0}$ and $\lambda_2 = \lambda e^{-\hat{\phi}_0}$. Using (2.4) and (2.5) the interaction term (2.10) can be written as

$$\begin{aligned} S_{\rm int} = -\frac{\lambda_1}{2} \int_{-\infty}^{\infty} & dt\big[(\psi_+a_- + \overline{\psi}_-a_+)S_+ + S_-(a_+\psi_- + a_-\overline{\psi}_+)\big] \\ &+ \frac{\lambda_2}{2} \int_{-\infty}^{\infty} dt\big[(a_+\psi_- + a_-\overline{\psi}_+)S_+ + S_-(\psi_+a_- + \overline{\psi}_-a_+)\big], \end{aligned} \tag{2.11}$$

where $a_\pm$ are the zero mode contributions

$$a_+ = e^{-i\sigma/2}, \quad a_- = (a_+)^\dagger = e^{i\sigma/2}, \tag{2.12}$$

and we have taken the average in fermionizing

$$e^{+i\sqrt{\pi}\phi(0)} = \frac{1}{2}(\psi_+ a_- + \overline{\psi}_- a_+), \quad e^{-i\sqrt{\pi}\phi(0)} = \frac{1}{2}(a_+\psi_- + a_-\overline{\psi}_+). \tag{2.13}$$

In order to ensure that the interaction is a bosonic scalar, the $a_\pm$ need to be treated as anticommuting with the ψ's

$$a_\pm \psi = -\psi a_\pm. \tag{2.14}$$

Aside from (2.14), they should be regarded as ordinary complex numbers with $a_\pm^2 = e^{\mp i\sigma}$, $a_+ a_- = a_- a_+ = 1$. We can rewrite (2.11) in a slightly different form. Rearranging we have

$$\begin{aligned} S_{\rm int} = -\frac{\lambda_1}{2}\int_{-\infty}^{\infty} dt \big[(\psi_+ e^{i\sigma} + \overline{\psi}_-)a_+ S_+ + S_- a_-(\psi_- e^{-i\sigma} + \overline{\psi}_+)\big] \\ -\frac{\lambda_2}{2}\int_{-\infty}^{\infty} dt \big[(\psi_- + \overline{\psi}_+ e^{i\sigma})a_+ S_+ + S_- a_-(\psi_+ + \overline{\psi}_- e^{-i\sigma})\big]. \end{aligned} \tag{2.15}$$

Define the operators d and $d^\dagger$

$$d = S_- a_-, \quad d^\dagger = a_+ S_+, \tag{2.16}$$

which are to be treated as anticommuting with the ψ's. The fermionized version of (2.10) becomes

$$\begin{aligned} S_{\rm int} = -\frac{\lambda_1}{2}\int_{-\infty}^{\infty} dt \big[(\psi_+ e^{i\sigma} + \overline{\psi}_-)d^\dagger + d(\psi_- e^{-i\sigma} + \overline{\psi}_+)\big] \\ -\frac{\lambda_2}{2}\int_{-\infty}^{\infty} dt \big[(\psi_- + \overline{\psi}_+ e^{i\sigma})d^\dagger + d(\psi_+ + \overline{\psi}_- e^{-i\sigma})\big]. \end{aligned} \tag{2.17}$$

This is the form of the interaction we will use to calculate the reflection S-matrices. We want to calculate the reflection S-matrices by varying the action with respect to all the fields, including d and $d^\dagger$, which are dynamic variables, $d = d(t)$, $d^\dagger = d^\dagger(t)$. To do so, we need to introduce a kinetic term for these operators. The appropriate kinetic term is

$$S_{\rm d-free} = i\int_{-\infty}^{\infty} dt\, d^\dagger(t)\partial_t d(t). \tag{2.18}$$

The action which we vary is thus

$$S = S_{\rm bulk} + S_{\rm bc} + S_{\rm d-free} + S_{\rm int}. \tag{2.19}$$

The term (2.18) needs some explanation. In writing (2.18) we have considered $d(t)$ and $d^\dagger(t)$ to be one dimensional "fermionic" fields. On the other hand, (2.16) would imply that d and $d^\dagger$ are $su(2)$ spin operators. In the isotropic Kondo model it is well known that there is a screening effect [27], whereby the spin j operators S_i act in the spin $j-1/2$ representation. We claim that a similar "screening" effect also occurs in the anisotropic massive models. This implies that for spin 1/2 the $su(2)$ structure is lost. The operators d and $d^\dagger$ are thus screened versions of the spin operators $S_\pm$, which for spin 1/2 can be effectively treated as one dimensional fermions satisfying

$$d^2 = {d^\dagger}^2 = 0, \quad \{d, d^\dagger\} = 1, \tag{2.20}$$

and due to (2.16) anticommuting with the ψ's

$$d\psi = -\psi d, \quad d^\dagger\psi = -\psi d^\dagger. \tag{2.21}$$

Equations (2.16) and (2.18) are statements of this screening. Alternatively, instead of introducing (2.16) and (2.18), we can work with the Heisenberg equations of motion for $S_\pm(t)$, namely $\partial_t S_\pm = i[H, S_\pm]$, as was done in [20]. This leads to the same equations of motion as the above formulation in terms of $d, d^\dagger$.

Varying (2.19) with respect to all the fields, $(\psi_\pm, \overline{\psi}_\pm, d, d^\dagger)$, and requiring the boundary terms to vanish, we obtain six equations (at $x = 0$)

$$\psi_+ - \overline{\psi}_- e^{-i\sigma} = \frac{i}{2}(\lambda_1 e^{-i\sigma} d - \lambda_2 d^\dagger) \tag{2.22a}$$

$$\psi_- - \overline{\psi}_+ e^{i\sigma} = -\frac{i}{2}(\lambda_1 e^{i\sigma} d^\dagger - \lambda_2 d) \tag{2.22b}$$

$$\psi_+ e^{i\sigma} - \overline{\psi}_- = \frac{i}{2}(\lambda_1 d - \lambda_2 e^{i\sigma} d^\dagger) \tag{2.23a}$$

$$\psi_- e^{-i\sigma} - \overline{\psi}_+ = -\frac{i}{2}(\lambda_1 d^\dagger - \lambda_2 e^{i\sigma} d) \tag{2.23b}$$

$$i\partial_t d = -\frac{\lambda_1}{2}(\psi_+ e^{i\sigma} + \overline{\psi}_-) - \frac{\lambda_2}{2}(\psi_- + \overline{\psi}_+ e^{i\sigma}) \tag{2.24a}$$

$$i\partial_t d^\dagger = \frac{\lambda_1}{2}(\psi_- e^{-i\sigma} + \overline{\psi}_+) + \frac{\lambda_2}{2}(\psi_+ + \overline{\psi}_- e^{-i\sigma}). \tag{2.24b}$$

We see that (2.22) and (2.23) are identical, and (2.22a) and (2.22b) are adjoints, as are (2.24a) and (2.24b). Taking time derivatives of (2.22) and using (2.24), we get the boundary equations of motion

$$\partial_t \left(\psi_+ - e^{-i\sigma}\overline{\psi}_-\right) = -\frac{1}{4}(\lambda_1^2 + \lambda_2^2)\left(\psi_+ + e^{-i\sigma}\overline{\psi}_-\right) - \frac{1}{2}\lambda_1\lambda_2\left(\psi_- e^{-i\sigma} + \overline{\psi}_+\right) \tag{2.25a}$$

$$\partial_t \left(\psi_- - e^{i\sigma}\overline{\psi}_+\right) = -\frac{1}{4}(\lambda_1^2 + \lambda_2^2)\left(\psi_- + e^{i\sigma}\overline{\psi}_+\right) - \frac{1}{2}\lambda_1\lambda_2\left(\psi_+ e^{i\sigma} + \overline{\psi}_-\right). \tag{2.25b}$$

The bulk variation of S_{bulk} gives the bulk equations of motion

$$(\partial_t - \partial_x)\psi_\pm + M\overline{\psi}_\pm = 0, \quad (\partial_t + \partial_x)\overline{\psi}_\pm - M\psi_\pm = 0. \tag{2.26}$$

Mode expansions satisfying (2.26) take the form

$$\psi_+ = \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta\, e^{-\theta/2}\left(\overline{\omega} A_-(\theta) e^{-i\vec{k}\cdot\vec{x}} + \omega A_+^\dagger(\theta) e^{i\vec{k}\cdot\vec{x}}\right) \tag{2.27a}$$

$$\overline{\psi}_+ = \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta\, e^{\theta/2}\left(\omega A_-(\theta) e^{-i\vec{k}\cdot\vec{x}} + \overline{\omega} A_+^\dagger(\theta) e^{i\vec{k}\cdot\vec{x}}\right) \tag{2.27b}$$

$$\psi_- = (\psi_+)^\dagger, \quad \overline{\psi}_- = (\overline{\psi}_+)^\dagger, \tag{2.27c}$$

where $\omega = \exp(i\pi/4)$, $\overline{\omega} = \exp(-i\pi/4)$ and $\vec{k}\cdot\vec{x} = Mt\cosh\theta - Mx\sinh\theta$. The variable θ is the rapidity and $A_\pm^\dagger(A_\pm)$ are fermion creation (annihilation) operators satisfying

$$\{A_\pm(\theta), A_\pm^\dagger(\theta')\} = \delta(\theta - \theta'). \tag{2.28}$$

The boundary scattering of the particles at $x = 0$ can be formulated by introducing a formal boundary operator B, and interpreting the boundary equations (2.25) to hold when acting on B [18]. This will give us a set of equations, one equation for each type of particle $A_a^\dagger(\theta)$, of the form (sum over b)

$$A_a^\dagger(\theta)B = R_a^b(\theta) A_b^\dagger(-\theta)B, \tag{2.29}$$

where $R(\theta)$ is called the reflection S-matrix, with $R_a^b(\theta)$ being the amplitude for an incoming particle a to be reflected as an outgoing particle b. If the boundary has additional structure allowing it to exist in several stable degenerate states, then (2.29) needs to be generalized to

$$A_a^\dagger(\theta)B_\alpha = R_{a\alpha}^{b\beta}(\theta)A_b^\dagger(-\theta)B_\beta, \tag{2.30}$$

where $\{B_\alpha\}$ are the boundary states. The reflection S-matrix element $R_{a\alpha}^{b\beta}(\theta)$ gives the amplitude for the process $(a,\alpha)\to(b,\beta)$. Based on the "screening" discussed above, we will assume that (2.29) is sufficient to describe the boundary scattering involving a spin 1/2 impurity. It is known that this is sufficient in the massless Kondo case. However this does not entirely exclude the possible existence of boundary structure B_α, as it may turn out that certain identifications of the matrix elements $R_{a\alpha}^{b\beta}$ reduce the problem to (2.29). That is to say, the amplitudes R_a^b can describe the scattering process even if there is some (degenerate) boundary structure. Various symmetries of the system can effectively reduce the number of independent amplitudes $R_{a\alpha}^{b\beta}$ to a smaller set satisfying the same scattering constraints as R_a^b. Such a situation will occur if the soliton transition $a\to b$ restricts the boundary transition $\alpha\to\beta$ to be of a specific type, with all amplitudes having the same soliton transition being equal. In this case the indices (α,β) become redundant since the amplitudes for $(a,\alpha)\to(b,\beta)$ are uniquely classified by the soliton quantum numbers (a,b), i.e. $R_{a\alpha}^{b\beta}$ reduces to R_a^b, with any boundary structure being implicitly incorporated in R_a^b. Poles of $R_a^b(\theta)$ may then be indicative of this "hidden" boundary structure. As a specific example, suppose the boundary is described by two degenerate states $B_\pm$ with charge $(+,-)$ as for the solitons. Furthermore, let the symmetries (or other constraints) of the system be such that (i) every soliton charge (non-)conserving transition is accompanied by a boundary charge (non-)conserving transition and (ii) charge conjugation symmetry holds separately in the soliton and boundary sectors. Then $R_{a\alpha}^{b\beta}$ reduces to only two independent amplitudes of the form $R_{a\alpha}^{a\alpha}$ and $R_{a\alpha}^{\overline{a}\overline{\alpha}}$, where the bar denotes the charge conjugated state. The scattering process is thus effectively described by $R_a^a\equiv R_{a\alpha}^{a\alpha}$ and $R_a^{\overline{a}}\equiv R_{a\alpha}^{\overline{a}\overline{\alpha}}$. (A similar reduction occurs in the bulk S-matrix at the reflectionless points.) There is also the possibility of excited "boundary bound states," arising from poles in the reflection amplitudes $R_a^b(\theta)$ (see section 4). These issues will be considered in the sequel.

For the solitons (2.29) reads

$$A_\pm^\dagger(\theta)B = R_\pm^\pm(\theta)A_\pm^\dagger(\theta)B + R_\pm^\mp(\theta)A_\mp^\dagger(\theta)B. \tag{2.31}$$

In the case of higher spin, $j\geq 1$, it is likely that one needs to consider (2.30), with the α index taking into account any additional boundary structure due to the $S_\pm$ operators.

We now calculate the reflection amplitudes. Substituting the mode expansions (2.27) into (2.25) gives

$$\begin{aligned}
&\omega e^{-\theta/2}\left(iM\cosh\theta+\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{\theta}\right)A_+^\dagger(\theta)\\
&\qquad\qquad-\overline{\omega}e^{-i\sigma}e^{\theta/2}\left(iM\cosh\theta-\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{-\theta}\right)A_-^\dagger(\theta)\\
&=-\omega e^{\theta/2}\left(iM\cosh\theta+\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{-\theta}\right)A_+^\dagger(-\theta)\\
&\qquad\qquad+\overline{\omega}e^{-i\sigma}e^{-\theta/2}\left(iM\cosh\theta-\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{\theta}\right)A_-^\dagger(-\theta)
\end{aligned} \tag{2.32a}$$

$$\begin{aligned}
&\omega e^{-\theta/2}\left(iM\cosh\theta+\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{\theta}\right)A_-^\dagger(\theta)\\
&\qquad\qquad-\overline{\omega}e^{i\sigma}e^{\theta/2}\left(iM\cosh\theta-\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{-\theta}\right)A_+^\dagger(\theta)\\
&=-\omega e^{\theta/2}\left(iM\cosh\theta+\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{-\theta}\right)A_-^\dagger(-\theta)\\
&\qquad\qquad+\overline{\omega}e^{i\sigma}e^{-\theta/2}\left(iM\cosh\theta-\frac{1}{4}(\lambda_1^2+\lambda_2^2)-\frac{i}{2}\lambda_1\lambda_2e^{\theta}\right)A_+^\dagger(-\theta).
\end{aligned} \tag{2.32b}$$

(We did not explicitly write down the boundary operator B.) Solving for the reflection S-matrix according to (2.31) we find

$$R_+^+(\theta) = R_-^-(\theta) = -\frac{\left(e^{\rho(\theta)} + e^{-\rho(-\theta)}\right)}{2\cosh(\theta - \gamma(\theta))} \tag{2.33a}$$

$$R_+^-(\theta) = ie^{-i\sigma}\frac{1}{2\cosh(\theta-\gamma(\theta))}\left(e^{\theta-\gamma(\theta)+\rho(\theta)} - e^{-\theta+\gamma(\theta)-\rho(-\theta)}\right), \tag{2.33b}$$

$$R_-^+(\theta) = ie^{+i\sigma}\frac{1}{2\cosh(\theta-\gamma(\theta))}\left(e^{\theta-\gamma(\theta)+\rho(\theta)} - e^{-\theta+\gamma(\theta)-\rho(-\theta)}\right), \tag{2.33c}$$

where

$$e^{\gamma(\theta)} = \frac{\cosh\theta - \frac{i}{4M}(\lambda_1^2+\lambda_2^2) - \frac{1}{2M}\lambda_1\lambda_2 e^{\theta}}{\cosh\theta + \frac{i}{4M}(\lambda_1^2+\lambda_2^2) - \frac{1}{2M}\lambda_1\lambda_2 e^{-\theta}}, \quad e^{\rho(\theta)} = \frac{\cosh\theta - \frac{i}{4M}(\lambda_1^2+\lambda_2^2) - \frac{1}{2M}\lambda_1\lambda_2 e^{-\theta}}{\cosh\theta + \frac{i}{4M}(\lambda_1^2+\lambda_2^2) - \frac{1}{2M}\lambda_1\lambda_2 e^{-\theta}}. \tag{2.34}$$

Making use of the following relations

$$e^{\rho(\theta)}e^{\rho(-\theta)} = e^{\gamma(\theta)}e^{\gamma(-\theta)} \tag{2.35a}$$

$$e^{\gamma(i\pi/2\pm\theta)} = e^{-\gamma(i\pi/2\mp\theta)}, \quad e^{\rho(i\pi/2\pm\theta)} = e^{-\rho(-(i\pi/2\mp\theta))}, \tag{2.35b}$$

one can check that R satisfies the unitarity and crossing relations [18]

$$R_a^c(\theta)R_c^b(-\theta) = \delta_a^b, \quad R_{\overline{a}}^b\left(\frac{i\pi}{2}-\theta\right) = -R_{\overline{b}}^a\left(\frac{i\pi}{2}+\theta\right), \tag{2.36}$$

where $\overline{a} = -a$.

For either $\lambda_1 = 0$ or $\lambda_2 = 0$ we have

$$e^{\rho(\theta)} = e^{\gamma(\theta)} = e^{\rho(-\theta)} = e^{\gamma(-\theta)} \qquad \text{if } \lambda_1 = 0 \text{ or } \lambda_2 = 0. \tag{2.37}$$

The off-diagonal elements R_+^- and R_-^+ differ only by a phase. We will choose σ such that $R_+^- = R_-^+$. Thus from now on we take either $\sigma = 0$ or $\sigma = \pi$. A specific choice will be made below. Both cases can be obtained by applying a $U(1)$ transformation to the fermion operators [21]. For notational convenience define the following

$$P(\theta) \equiv R_+^+(\theta) = R_-^-(\theta), \quad Q(\theta) \equiv R_-^+(\theta) = R_+^-(\theta). \tag{2.38}$$

The free fermion reflection amplitudes (2.33) look quite complicated. However by making a change of basis, their structure becomes apparent. The complex fermions can be expressed in terms of real fermions $\psi_{1,2}$ and $\overline{\psi}_{1,2}$

$$\psi_\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2), \quad \overline{\psi}_\pm = \frac{1}{\sqrt{2}}(\overline{\psi}_1 \pm i\overline{\psi}_2). \tag{2.39}$$

In terms of these real fermions the action (2.19) becomes

$$\begin{aligned} S = i\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \sum_{i=1,2}\left[\psi_i(\partial_t - \partial_x)\psi_i + \overline{\psi}_i(\partial_t+\partial_x)\overline{\psi}_i + 2M\psi_i\overline{\psi}_i\right] \\ -i\int_{-\infty}^{\infty} dt\, e^{i\sigma}\left(\psi_1\overline{\psi}_1 - \psi_2\overline{\psi}_2\right) + \frac{i}{2}\int_{-\infty}^{\infty} dt\left(a_1\partial_t a_1 + a_2\partial_t a_2\right) \\ -i\frac{\tilde{\lambda}_1}{2}\int_{-\infty}^{\infty} dt\,(\psi_1 + e^{i\sigma}\overline{\psi}_1)a_1 - i\frac{\tilde{\lambda}_2}{2}\int_{-\infty}^{\infty} dt\,(\psi_2 - e^{i\sigma}\overline{\psi}_2)a_2, \end{aligned} \tag{2.40}$$

where

$$\tilde{\lambda}_1 = \lambda_1 e^{i\sigma} + \lambda_2, \quad \tilde{\lambda}_2 = \lambda_1 e^{i\sigma} - \lambda_2, \tag{2.41}$$

and we have defined the operators (not to be confused with the $a_\pm$)

$$a_1(t) = -\frac{i}{\sqrt{2}}(d^\dagger(t) - d(t)), \quad a_2(t) = \frac{1}{\sqrt{2}}(d^\dagger(t) + d(t)), \tag{2.42}$$

satisfying

$$\{a_1(t), a_2(t)\} = 0, \quad a_1(t)^2 = a_2(t)^2 = \frac{1}{2}. \tag{2.43}$$

The action (2.40) is simply that of two decoupled Ising models with boundary magnetic fields. The reflection S-matrix for the Ising model was calculated in [18]. Note that the boundary terms are different for $(\psi_1, \overline{\psi}_1)$ and $(\psi_2, \overline{\psi}_2)$, with different couplings in general. The bulk and boundary equations of motion are

$$(\partial_t - \partial_x)\psi_i + M\overline{\psi}_i = 0, \quad (\partial_t + \partial_x)\overline{\psi}_i - M\psi_i = 0 \quad (i = 1, 2) \tag{2.44}$$

$$-\partial_t(\psi_1 - e^{i\sigma}\overline{\psi}_1) = \frac{\tilde{\lambda}_1^2}{4}(\psi_1 + e^{i\sigma}\overline{\psi}_1), \quad -\partial_t(\psi_2 + e^{i\sigma}\overline{\psi}_2) = \frac{\tilde{\lambda}_2^2}{4}(\psi_2 - e^{i\sigma}\overline{\psi}_2), \quad (\text{at } x = 0), \tag{2.45}$$

which are just the Re and Im parts of (2.26) and (2.25). Mode expansions satisfying (2.44) are given by

$$\psi_i = \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta \, e^{-\theta/2} \big(\varpi A_i(\theta) e^{-i\vec{k}\cdot\vec{x}} + \omega A_i^\dagger(\theta) e^{i\vec{k}\cdot\vec{x}}\big) \tag{2.46a}$$

$$\overline{\psi}_i = \sqrt{\frac{M}{4\pi}} \int_{-\infty}^{\infty} d\theta \, e^{\theta/2} \big(\omega A_i(\theta) e^{-i\vec{k}\cdot\vec{x}} + \varpi A_i^\dagger(\theta) e^{i\vec{k}\cdot\vec{x}}\big), \tag{2.46b}$$

where $A_{1,2}^\dagger$ $(A_{1,2})$ are the real fermion creation (annihilation) operators satisfying (2.28). The complex fermion operators can be written as

$$A_\pm = \frac{1}{\sqrt{2}}(A_1 \mp iA_2), \quad A_\pm^\dagger = \frac{1}{\sqrt{2}}(A_1^\dagger \pm iA_2^\dagger). \tag{2.47}$$

Substituting (2.46) into (2.45) we can obtain the reflection S-matrices for the real fermions and hence from (2.47) also for the complex fermions (solitons). We find

$$A_{1,2}^\dagger(\theta)B = R_{1,2}(\theta)A_{1,2}^\dagger(-\theta)B \tag{2.48}$$

$$R_{1,2}(\theta) = \widetilde{R}_{1,2}(\theta) f_{1,2}^{\mathrm{CDD}}(\theta) \tag{2.49}$$

$$\widetilde{R}_{1,2}(\theta) = i \tanh\left(i\frac{\pi}{4} \mp e^{i\sigma}\frac{\theta}{2}\right) \tag{2.50}$$

$$f_{1,2}^{\mathrm{CDD}}(\theta) = -\left(\frac{i\sinh\theta + (\Delta_{1,2} \mp e^{i\sigma})}{i\sinh\theta - (\Delta_{1,2} \mp e^{i\sigma})}\right), \quad \Delta_{1,2} = \frac{\tilde{\lambda}_{1,2}^2}{4M}. \tag{2.51}$$

The factor $\tilde{R}$ corresponds to a fixed boundary condition for one copy of Ising fermion and a free boundary condition for the other copy. In the absence of any interaction, the boundary equations are $(\sigma = 0)$

$$\psi_1 - \overline{\psi}_1 = 0, \quad \psi_2 + \overline{\psi}_2 = 0, \tag{2.52}$$

corresponding to a free boundary for $(\psi_1, \overline{\psi}_1)$ and a fixed boundary for $(\psi_2, \overline{\psi}_2)$. For infinite coupling $(|\tilde{\lambda}_{1,2}| \to \infty)$ the boundary conditions are

$$\psi_1 + \overline{\psi}_1 = 0, \quad \psi_2 - \overline{\psi}_2 = 0, \tag{2.53}$$

that is a fixed boundary for $(\psi_1, \overline{\psi}_1)$ and a free boundary for $(\psi_2, \overline{\psi}_2)$. (For $\sigma = \pi$ (2.52) and (2.53) are interchanged.) So we have a flow from free to fixed (fixed to free) for ψ_1 and $\overline{\psi}_1$ (ψ_2 and $\overline{\psi}_2$) as the interaction

varies from zero to infinity. In particular, the flow is different for the two Ising models. When expressed in terms of the complex fermions, (2.52) and (2.53) read

$$\psi_+ - \overline{\psi}_- = 0, \quad \psi_- - \overline{\psi}_+ = 0 \qquad (\tilde{\lambda}_{1,2} = 0) \tag{2.54}$$

$$\psi_+ + \overline{\psi}_- = 0, \quad \psi_- + \overline{\psi}_- = 0 \qquad (|\tilde{\lambda}_{1,2}| = \infty). \tag{2.55}$$

Both equations correspond to free boundary conditions for the Dirac fermions, and apart from a phase, give the same reflection amplitudes. Thus there is no flow between free and fixed boundary conditions for the solitons. In contrast, the MBSG model has a flow between free and fixed boundary conditions as the coupling λ goes from 0 to ∞.

The soliton reflection amplitudes are given by

$$P(\theta) = \frac{1}{2}(R_1(\theta) + R_2(\theta)), \quad Q(\theta) = \frac{1}{2}(R_1(\theta) - R_2(\theta)). \tag{2.56}$$

One can check that these are the same as (2.33). In particular we have

$$R_1(\theta, \sigma = \pi) = R_2(\theta, \sigma = 0), \quad R_2(\theta, \sigma = \pi) = R_1(\theta, \sigma = 0), \tag{2.57}$$

implying, as expected from (2.33), that $P(\theta)$ is independent of the choice for σ, whereas $Q(\theta)$ changes by an overall sign.

2.1. Free Fermion Reflection S-matrix for the MK model

We now specialize to the MK model by taking

$$\lambda_1 = \lambda, \quad \lambda_2 = 0. \tag{2.58}$$

Using (2.37) we get for the free fermion reflection S-matrix

$$P(\theta) = -\frac{\cosh\gamma(\theta)}{\cosh(\theta - \gamma(\theta))}, \quad Q(\theta) = ie^{i\sigma}\frac{\sinh\theta}{\cosh(\theta - \gamma(\theta))}, \tag{2.59}$$

with

$$e^{\gamma(\theta)} = \frac{\cosh\theta - i\Delta^{MK}}{\cosh\theta + i\Delta^{MK}}, \quad \Delta^{MK} = \frac{\lambda^2}{4M}, \tag{2.60}$$

recovering the results of [20]. From (2.56) alternative expressions are

$$P(\theta) = -\frac{i}{2}\tanh\left(i\frac{\pi}{4} - \frac{\theta}{2}\right)\left(\frac{i\sinh\theta + (\Delta^{MK} - 1)}{i\sinh\theta - (\Delta^{MK} - 1)}\right) - \frac{i}{2}\tanh\left(i\frac{\pi}{4} + \frac{\theta}{2}\right)\left(\frac{i\sinh\theta + (\Delta^{MK} + 1)}{i\sinh\theta - (\Delta^{MK} + 1)}\right) \tag{2.61a}$$

$$Q(\theta) = -e^{i\sigma}\frac{i}{2}\tanh\left(i\frac{\pi}{4} - \frac{\theta}{2}\right)\left(\frac{i\sinh\theta + (\Delta^{MK} - 1)}{i\sinh\theta - (\Delta^{MK} - 1)}\right) + e^{i\sigma}\frac{i}{2}\tanh\left(i\frac{\pi}{4} + \frac{\theta}{2}\right)\left(\frac{i\sinh\theta + (\Delta^{MK} + 1)}{i\sinh\theta - (\Delta^{MK} + 1)}\right). \tag{2.61b}$$

Note that the two CDD factors in (2.61) are different. The zero and infinite coupling limits are (the upper (lower) sign corresponds to $\lambda = 0$ (∞))

$$P(\theta) = -\mathrm{sech}\theta, \quad Q(\theta) = \pm ie^{i\sigma}\tanh\theta, \tag{2.62}$$

which are the same (apart from phases) as the MBSG reflection amplitudes with free boundary conditions ($\lambda = 0$ in (1.6)). These results confirm that in the soliton basis there is no flow between free and fixed boundary conditions for zero and infinite coupling. This can also be seen from the boundary equations (2.25) with $\lambda_2 = 0$.

To check if (2.61) agrees with the Kondo model we compute the massless limit. The massless limit for right-movers is obtained by letting $\theta \to \theta + \alpha$, and taking $\alpha \to \infty$, $M \to 0$ while keeping $M_0 \equiv Me^{\alpha}/2$ held fixed. This gives us the massless dispersion relation $E = p = M_0 e^{\theta}$. Taking the massless limit we find

$$\widetilde{R}_{1,2} \to \mp i e^{i\sigma}, \quad f_{1,2}^{\rm CDD} \to -\tanh\left(\frac{\theta - \theta_B^{MK}}{2} - \frac{i\pi}{4}\right) \tag{2.63}$$

$$P(\theta) = 0, \quad Q(\theta) = i e^{i\sigma} \tanh\left(\frac{\theta - \theta_B^{MK}}{2} - \frac{i\pi}{4}\right), \tag{2.64}$$

where we have defined

$$M_0 \exp(\theta_B^{MK}) = \frac{\lambda^2}{4}. \tag{2.65}$$

The massless reflection S-matrix is almost entirely due to the CDD factors $f_{1,2}^{\rm CDD}$. Apart from a phase, this is the spin 1/2 reflection S-matrix for the Kondo model [27]. The Kondo Hamiltonian (1.2) is also recovered in the massless limit. Thus the MK model reproduces the Kondo results at the free fermion point.

2.2. Free Fermion Reflection S-Matrix for the mMK model

The mMK model is obtained by setting

$$\lambda_1 = \sqrt{2M} e^{\hat{\phi}_0}, \quad \lambda_2 = \sqrt{2M} e^{-\hat{\phi}_0}. \tag{2.66}$$

For these values

$$e^{\gamma(\theta)} = -1, \quad e^{\rho(\theta)} = \frac{i\sinh\theta + \frac{1}{4M}(\lambda_1^2 + \lambda_2^2)}{i\sinh\theta - \frac{1}{4M}(\lambda_1^2 + \lambda_2^2)}, \tag{2.67}$$

giving the reflection S-matrix

$$P(\theta) = {\rm sech}\theta f^{mMK}(\theta), \quad Q(\theta) = i e^{i\sigma} \tanh\theta f^{mMK}(\theta), \tag{2.68}$$

where $f^{mMK}(\theta)$ is the CDD factor

$$f^{mMK}(\theta) = e^{\rho(\theta)} = \left(\frac{i\sinh\theta + \Delta^{mMK}}{i\sinh\theta - \Delta^{mMK}}\right), \quad \Delta^{mMK} = \frac{1}{4M}(\lambda_1^2 + \lambda_2^2) = \cosh 2\hat{\phi}_0. \tag{2.69}$$

The CDD factor $f^{mMK}(\theta)$ is due to the factors $f_{1,2}^{\rm CDD}(\theta)$ in (2.49). With (2.66) we have $f_1^{\rm CDD} = f_2^{\rm CDD}$, and f^{mMK} is simply equal to $-f_{1,2}^{\rm CDD}$. The hyperbolic factors in (2.68) come from the $\widetilde{R}_{1,2}(\theta)$ linear combinations

$$\frac{1}{2}(\widetilde{R}_1 + \widetilde{R}_2) = -{\rm sech}(\theta), \quad \frac{1}{2}(\widetilde{R}_1 - \widetilde{R}_2) = -i e^{i\sigma} \tanh\theta. \tag{2.70}$$

We see that the reflection amplitudes are the free MBSG amplitudes (at the free fermion point) multiplied by a CDD factor. The dependence on the boundary coupling is solely in the CDD factor. These elements are simpler than those of the MK model (2.61) where the two CDD factors are different. In contrast to the MK model, there is no zero coupling limit. From (2.66) it is clear that both λ_1 and λ_2 cannot vanish for a non-zero mass. We can take an infinite coupling limit by letting $\hat{\phi}_0 \to \infty$ or $\hat{\phi}_0 \to -\infty$, corresponding respectively to $(\lambda_1, \lambda_2) \to (\infty, 0)$ and $(\lambda_1, \lambda_2) \to (0, \infty)$. In either case we get

$$P(\theta) = -{\rm sech}\theta, \quad Q(\theta) = -i e^{i\sigma} \tanh\theta, \tag{2.71}$$

which are the same as (2.62).

The massless limit is taken as before with one important difference. Because the couplings are mass dependent, we must also let $\hat{\phi}_0 \to \infty$ while keeping λ_1 fixed. Of course λ_2 goes to zero. Explicitly for the massless limit of f^{mMK} we have

$$f^{mMK}(\theta) = \frac{iM\sinh\theta + M\cosh 2\hat{\phi}_0}{iM\sinh\theta - M\cosh 2\hat{\phi}_0} \longrightarrow \frac{iM_0 e^{\theta} + M_0 e^{\theta_B^{mMK}}}{iM_0 e^{\theta} - M_0 e^{\theta_B^{mMK}}} = \tanh\left(\frac{\theta - \theta_B^{mMK}}{2} - \frac{i\pi}{4}\right), \tag{2.72}$$

where θ_B^{mMK} is defined through

$$M_0 \exp(\theta_B^{mMK}) = \frac{M}{2} e^{2\hat{\phi}_0} = \frac{\lambda_1^2}{4}. \tag{2.73}$$

The resulting reflection amplitudes are identical to (2.64)

$$P(\theta) = 0, \quad Q(\theta) = ie^{i\sigma}\tanh\left(\frac{\theta - \theta_B^{mMK}}{2} - \frac{i\pi}{4}\right). \tag{2.74}$$

Again we recover the Kondo results. Since $\lambda_2 \to 0$, the mMK Hamiltonian (1.5) goes over to the Kondo Hamiltonian (1.2).

3. Reflection S-Matrices away from the Free Fermion point and Integrability

In an integrable boundary field theory, knowledge of the particle spectrum (both the bulk spectrum and the boundary states) can allow one to determine the reflection S-matrix, up to CDD factors, by imposing the constraints of boundary factorizability (i.e. boundary Yang-Baxter (BYB) equation), unitarity and crossing symmetry [18],[28]. For both spin 1/2 massive Kondo models, the bulk spectrum is the sine-Gordon spectrum. Also as mentioned above, we will not explicitly consider any boundary state structure as in (2.30). Thus if the massive Kondo models are integrable their reflection S-matrices must satisfy the same constraints as for the MBSG model.

The general solution to the boundary scattering constraints for a sine-Gordon bulk spectrum, at arbitrary β, was described in GZ [18]. The GZ solution depends on two formal parameters. The difference between the MBSG, MK and mMK reflection amplitudes will then be in the relation of these formal parameters to the physical parameters (mass, couplings and β). If the results of section 2 are extendible to arbitrary β, then the free fermion matrices should agree with those of GZ at $\beta = \sqrt{4\pi}$.

Before making this comparison let us review GZ's solution. Define

$$\Lambda = \frac{8\pi}{\beta^2} - 1. \tag{3.1}$$

For arbitrary $\Lambda > 0$, GZ's solution to the scattering equations is

$$P_+^{\rm GZ}(\theta) \equiv R_+^+(\theta) = \cos(\xi - i\Lambda\theta) r(\theta), \tag{3.2a}$$

$$P_-^{\rm GZ}(\theta) \equiv R_-^-(\theta) = \cos(\xi + i\Lambda\theta) r(\theta), \tag{3.2b}$$

$$Q_+^{\rm GZ}(\theta) \equiv R_+^-(\theta) = Q_-^{\rm GZ}(\theta) \equiv R_-^+(\theta) \equiv Q^{\rm GZ}(\theta) = -\frac{k}{2}\sin(2i\Lambda\theta) r(\theta). \tag{3.2c}$$

The function $r(\theta)$ can be written as

$$r(\theta) = r_0(\theta) r_1(\theta). \tag{3.3}$$

The factor $r_0(\theta)$ is independent of the boundary couplings and ensures that the crossing symmetry constraint is satisfied. The boundary dependence is contained in $r_1(\theta)$, which takes the form

$$r_1(\theta) = \frac{1}{\cos\xi}\sigma(\eta,\theta)\sigma(i\vartheta,\theta). \tag{3.4}$$

Here $\sigma(x,\theta)$ is a function not to be confused with the phase σ. The variables k,ξ,η and ϑ are formal parameters with no θ dependence and satisfy

$$\cos\eta\cosh\vartheta = -\frac{1}{k}\cos\xi, \quad \cos^2\eta + \cosh^2\vartheta = 1 + \frac{1}{k^2}. \tag{3.5}$$

Thus only two of k,ξ,η and ϑ are independent. These parameters have to be related to the physical parameters, which is often the hardest part. The functions $r_0(\theta)$ and $\sigma(x,\theta)$ can be expressed as infinite products of Γ-functions (see [18]). Making use of [29]

$$\ln\Gamma(z) = \int_0^{+\infty}\frac{dt}{t}\left[(z-1)e^{-t} - \frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right], \quad \mathrm{Re}(z)>0, \tag{3.6}$$

the infinite Γ products can be shown to have integral representations, giving for r_0 and σ

$$\sigma(x,\theta) = \frac{\cos x}{\cosh(\Lambda\theta + ix)}\exp\left[i\int_{-\infty}^{\infty} dy\,\sin(2\Lambda\theta y/\pi)\frac{\sinh\left(\Lambda+\frac{2}{\pi}x\right)y}{2y\cosh\Lambda y\sinh y}\right] \tag{3.7}$$

$$r_0(\theta) = \exp\left[-i\int_{-\infty}^{\infty} dy\,\sin(2\Lambda\theta y/\pi)\frac{\sinh\frac{3}{2}\Lambda y\sinh\left(\frac{\Lambda-1}{2}\right)y}{y\sinh\frac{y}{2}\sinh 2\Lambda y}\right]. \tag{3.8}$$

We will take $\xi = n\pi$ and write for the diagonal amplitude

$$P^{\mathrm{GZ}}(\theta) \equiv P_+^{\mathrm{GZ}}(\theta) = P_-^{\mathrm{GZ}}(\theta) = \cos(\xi)\cos(i\Lambda\theta)r(\theta). \tag{3.9}$$

At the free fermion point, $\Lambda = 1$, the expressions for the amplitudes simplify greatly

$$P^{\mathrm{GZ}}(\theta) = -\frac{(\cos\xi\cosh\theta)/k}{(-i\sinh\theta+\cos\eta)(-i\sinh\theta+\cosh\vartheta)} = \frac{\cos\eta\cosh\vartheta\cosh\theta}{(-i\sinh\theta+\cos\eta)(-i\sinh\theta+\cosh\vartheta)} \tag{3.10}$$

$$Q^{\mathrm{GZ}}(\theta) = \frac{i\sinh\theta\cosh\theta}{(-i\sinh\theta+\cos\eta)(-i\sinh\theta+\cosh\vartheta)}. \tag{3.11}$$

In particular, if

$$\cos\eta = -\cosh\vartheta = \pm 1, \tag{3.12}$$

or equivalently

$$k = \cos\xi = \pm 1, \tag{3.13}$$

we have (for either sign)

$$P_{\mathrm{free}}^{\mathrm{GZ}}(\theta) = \mathrm{sech}\theta, \quad Q_{\mathrm{free}}^{\mathrm{GZ}}(\theta) = -i\tanh\theta, \tag{3.14}$$

which are the MBSG amplitudes for free boundary conditions.

3.1. The MK Reflection S-Matrix at the Reflectionless points

In this subsection we propose a natural extension of the MK reflection S-matrix away from the free fermion point. The MK reflection S-matrix (2.59) at the free fermion point can be rewritten as

$$P(\theta) = \frac{\cosh\gamma(\theta)\cosh(\theta)}{(-i\sinh\theta+1)(-i\sinh\theta+\cosh\gamma(\theta))}\left(\frac{i\sinh+(\Delta^{MK}-1)}{i\sinh-(\Delta^{MK}-1)}\right) \tag{3.15a}$$

$$Q(\theta) = \frac{-ie^{i\sigma}\sinh(\theta)\cosh(\theta)}{(-i\sinh\theta+1)(-i\sinh\theta+\cosh\gamma(\theta))}\left(\frac{i\sinh+(\Delta^{MK}-1)}{i\sinh-(\Delta^{MK}-1)}\right). \tag{3.15b}$$

Comparing these with (3.10) and (3.11) we see that

$$P(\theta) = P^{\rm GZ}(\theta, \eta = 0, \vartheta = \gamma(\theta)) \times f^{\rm CDD}(\theta), \quad Q(\theta) = -e^{i\sigma} Q^{\rm GZ}(\theta, \eta = 0, \vartheta = \gamma(\theta)) \times f^{\rm CDD}(\theta), \tag{3.16}$$

where

$$f^{\rm CDD} = \frac{i \sinh +(\Delta^{MK} - 1)}{i \sinh -(\Delta^{MK} - 1)}. \tag{3.17}$$

The phase difference between $P(\theta)$ and $Q(\theta)$ is irrelevant, since if (P, Q) satisfy the scattering equations then so do $(P, -Q)$. We will set $\sigma = \pi$ for the MK model. Up to a CDD factor, we see that the MK reflection amplitudes agree with those of GZ provided we take $\eta = 0$ and more importantly, allow ϑ to have the θ dependence $\vartheta = \gamma(\theta)$. However, such a θ dependence is not allowed by the BYB constraint [18]. Unitarity and crossing symmetry alone do not rule out the possibility of a θ dependence for the parameters $(\xi, k, \eta, \vartheta)$. If there is no BYB constraint, as is the case at the free fermion point, and more generally at the reflectionless points, then (3.15) is a possible solution to the scattering equations. Away from the reflectionless points, the BYB equation forces the scattering amplitudes to take the general form of GZ's solution (3.2) (up to CDD factors), with the formal parameters θ independent. This means that (3.15) cannot be extended to arbitrary values of Λ in a manner consistent with (3.2). Since GZ's solution is the unique solution to the boundary scattering equations for a sine-Gordon spectrum at arbitrary Λ, we conclude that assuming there are no boundary degrees of freedom, the MK model is not integrable (i.e. there is no reflection S-matrix consistent with BYB, unitarity and crossing symmetry) for generic Λ. Again we would like to emphasize that for higher spin $j \geq 1$, one probably needs to solve a more general BYB equation with boundary spin states and amplitudes $R^{b\beta}_{a\alpha}(\theta)$. In which case a comparison with the MBSG model cannot be made.

The reflectionless points occur at integer Λ, for which the BYB constraint is trivially satisfied. The reason for this being that if $\Lambda = n \geq 1$, the bulk scattering matrix takes on a simple form with only one independent scattering amplitude. In general the bulk sine-Gordon scattering matrix is

$$a(\theta) \equiv S^{++}_{++}(\theta) = S^{--}_{--}(\theta) = \sin[\Lambda(\pi + i\theta)]Z(\theta) \tag{3.18a}$$

$$b(\theta) \equiv S^{+-}_{+-}(\theta) = S^{-+}_{-+}(\theta) = -\sin(i\Lambda\theta)Z(\theta) \tag{3.18b}$$

$$c(\theta) \equiv S^{-+}_{+-}(\theta) = S^{+-}_{-+}(\theta) = \sin(\Lambda\pi)Z(\theta), \tag{3.18c}$$

where the function $Z(\theta)$ can be found in [30]. For integer Λ we have

$$b(\theta) = a(\theta) \ (\Lambda \text{ odd}) \ \text{ or } \ b(\theta) = -a(\theta) \ (\Lambda \text{ even}), \quad \text{and} \ \ c(\theta) = 0. \tag{3.19}$$

The expression for $a(\theta)$ reduces to

$$a(\theta) = (-1)^{\Lambda} \prod_{n=1}^{\Lambda-1} f_{n/\Lambda}(\theta), \tag{3.20}$$

where

$$f_\alpha = \frac{\sinh\left(\frac{1}{2}[\theta + i\pi\alpha]\right)}{\sinh\left(\frac{1}{2}[\theta - i\pi\alpha]\right)}. \tag{3.21}$$

One can check that for (3.19) there is no constraint from the BYB equation. Since the off-diagonal bulk amplitude $c(\theta)$ vanishes, these integer points are referred to as the reflectionless points.

With no BYB constraint, it should be possible to extend (3.15) to the reflectionless points. In terms of $R_{1,2}$, the unitarity and crossing relations for integer Λ become

Unitarity:

$$R_1(\theta)R_1(-\theta) = 1, \quad R_2(\theta)R_2(-\theta) = 1. \tag{3.22}$$

Crossing:

Odd Λ

$$R_1(i\pi/2 - \theta) = a(2\theta)\, R_1(i\pi/2 + \theta), \quad R_2(i\pi/2 - \theta) = a(2\theta)\, R_2(i\pi/2 + \theta) \tag{3.23a}$$

Even Λ

$$R_1(i\pi/2-\theta) = -a(2\theta)\, R_2(i\pi/2+\theta), \quad R_2(i\pi/2-\theta) = -a(2\theta)\, R_1(i\pi/2+\theta). \tag{3.23b}$$

Since $a(-\theta) = 1/a(\theta)$, the two crossing equations for even Λ are identical.

Now recall the free fermion reflection amplitudes in the Ising notation (2.56), where ($\sigma = \pi$)

$$R_{1,2}(\theta) = \widetilde{R}_{1,2}(\theta) f_{1,2}^{\rm CDD}(\theta) \tag{3.24}$$

$$\widetilde{R}_{1,2}(\theta) = i\tanh\big(i\frac{\pi}{4} \pm \frac{\theta}{2}\big), \quad f_{1,2}^{\rm CDD} = -\left(\frac{i\sinh\theta + (\Delta^{MK} \pm 1)}{i\sinh\theta - (\Delta^{MK} \pm 1)}\right). \tag{3.25}$$

Considering the structure of the reflection matrices (3.24) for $\Lambda = 1$, we will look for solutions of (3.22) and (3.23) of the following form

$$R_1(\theta) = \tilde{r}_1(\theta) f_1^{\rm CDD}(\theta), \quad R_2(\theta) = \tilde{r}_2(\theta) f_2^{\rm CDD}(\theta), \tag{3.26}$$

where $\tilde{r}_{1,2}(\theta)$ are independent of the boundary coupling λ. The boundary dependence being contained in the CDD factors. For odd Λ, the CDD factors can be different for R_1 and R_2, as they are for $\Lambda = 1$. However for even Λ, since $a(2\theta)$ is independent of λ, the crossing relation implies that the CDD factors must be the same. Requiring the CDD factors to agree with (3.25) for $\Lambda = 1$ suggests we take the general form

$$f_{1,2}^{\rm CDD} = -\frac{i\sinh\theta + \Gamma_{1,2}(\Lambda, \lambda^2/G)}{i\sinh\theta - \Gamma_{1,2}(\Lambda, \lambda^2/G)}, \tag{3.27}$$

where $\Gamma_{1,2}(\Lambda, \lambda^2/G)$ are functions of the dimensionless parameters Λ and λ^2/G, satisfying

$$\Gamma_{1,2}\left(\Lambda = 1, \frac{\lambda^2}{G}\right) = \Delta^{MK} \pm 1 \tag{3.28}$$

$$\Gamma_1\left(\Lambda = 2n, \frac{\lambda^2}{G}\right) = \Gamma_2\left(\Lambda = 2n, \frac{\lambda^2}{G}\right). \tag{3.29}$$

Note that (3.28) makes sense since at the free fermion point $M \propto G$.

The crossing relations that need to be satisfied can now be written as

$$a(2\theta)\, \tilde{r}_{1,2}(\theta + i\pi/2)\tilde{r}_{1,2}(\theta - i\pi/2) = 1 \ (\Lambda \text{ odd}) \tag{3.30a}$$

$$-a(2\theta)\, \tilde{r}_1(\theta + i\pi/2)\tilde{r}_2(\theta - i\pi/2) = 1 \ (\Lambda \text{ even}) \tag{3.30b}$$

To solve these equations we make use of the following expression for $a(2\theta)$

$$a(2\theta) = \prod_{n=1}^{\Lambda} f_{-n/2\Lambda}(\theta - i\pi/2) f_{-n/2\Lambda}(\theta + i\pi/2), \tag{3.31}$$

where the f_α's are given by (3.21). Since

$$f_{1/2}(\theta) = \widetilde{R}_1(\theta), \tag{3.32}$$

we take $\tilde{r}_1(\theta)$ to be

$$\tilde{r}_1(\theta) = \prod_{n=1}^{\Lambda} f_{n/2\Lambda}(\theta). \tag{3.33}$$

This will satisfy the odd Λ crossing relation for $\tilde{r}_1(\theta)$. Similarly, knowing that

$$-f_{1/2}(\theta - i\pi) = \widetilde{R}_2(\theta), \tag{3.34}$$

we take for $\tilde{r}_2(\theta)$

$$\tilde{r}_2(\theta) = -\prod_{n=1}^{\Lambda} f_{n/2\Lambda}(\theta - i\pi). \tag{3.35}$$

Using

$$f_\alpha(\theta + i2\pi m) = f_{\alpha+2m}(\theta) = f_\alpha(\theta), \tag{3.36}$$

where m is any integer, we see that (3.35) satisfies the odd Λ crossing relation for $\tilde{r}_2(\theta)$. Unitarity for $\tilde{r}_{1,2}$ is also easily checked with the relations

$$f_\alpha(\theta)f_\alpha(-\theta) = 1, \quad f_\alpha(-\theta) = f_{-\alpha}(\theta). \tag{3.37}$$

The solutions (3.33) and (3.35) are not valid for even Λ. These expressions imply

$$\tilde{r}_1(\theta + i\pi/2)\tilde{r}_2(\theta - i\pi/2) = -\prod_{n=1}^{\Lambda} f^2_{n/2\Lambda}(\theta + i\pi/2), \tag{3.38}$$

which does not satisfy the crossing relation (3.30b). Possible solutions for (3.30b) are

$$\tilde{r}_1(\theta) = -\tilde{r}_2(\theta) = \prod_{n=1}^{\Lambda} f_{n/2\Lambda}(\theta), \tag{3.39}$$

and

$$\tilde{r}_1(\theta) = -\tilde{r}_2(\theta) = \prod_{n=1}^{\Lambda} f_{n/2\Lambda}(\theta - i\pi). \tag{3.40}$$

We can now combine the odd and even Λ solutions into a single expression. For any integer Λ we have

$$\tilde{r}_1(\theta) = \prod_{n=1}^{\Lambda} f_{n/2\Lambda}\left(\theta - i\pi\rho_1(\Lambda)\right) \tag{3.41}$$

$$\tilde{r}_2(\theta) = -\prod_{n=1}^{\Lambda} f_{n/2\Lambda}\left(\theta - i\pi\rho_2(\lambda)\right) \tag{3.42}$$

where for odd Λ

$$\rho_1(\Lambda = 2n+1) = 0, \quad \rho_2(\Lambda = 2n+1) = 1, \tag{3.43}$$

and for even Λ we can have two choices

$$\rho_1(\Lambda = 2n) = 0, \quad \rho_2(\Lambda = 2n) = 0, \tag{3.44a}$$

or

$$\rho_1(\Lambda = 2n) = 1, \quad \rho_2(\Lambda = 2n) = 1. \tag{3.44b}$$

These expressions for $\rho_{1,2}(\Lambda)$ are mod 2 because of (3.36). If it were not for the specific forms (3.25) for $\Lambda = 1$, we would also have two choices for odd Λ. As an example, we can take for (3.44a)

$$\rho_1(\Lambda) = 0, \quad \rho_2(\Lambda) = \sin^2(\Lambda\pi/2). \tag{3.45}$$

Putting everything together we have for any integer Λ

$$R_{1,2}(\theta) = \mp\left(\frac{i\sinh\theta + \Gamma_{1,2}(\Lambda, \lambda^2/G)}{i\sinh\theta - \Gamma_{1,2}(\Lambda, \lambda^2/G)}\right)\prod_{n=1}^{\Lambda} f_{n/2\Lambda}\left(\theta - i\pi\rho_{1,2}(\Lambda)\right), \tag{3.46}$$

giving for the soliton reflection S-matrix at the reflectionless points

$$P(\theta) = \frac{1}{2}\sum_{j=1,2}(-1)^j\left(\frac{i\sinh\theta + \Gamma_j(\Lambda,\lambda^2/G)}{i\sinh\theta - \Gamma_j(\Lambda,\lambda^2/G)}\right)\prod_{n=1}^{\Lambda} f_{n/2\Lambda}\left(\theta - i\pi\rho_j(\Lambda)\right) \tag{3.47a}$$

$$Q(\theta) = -\frac{1}{2}\sum_{j=1,2}\left(\frac{i\sinh\theta + \Gamma_j(\Lambda,\lambda^2/G)}{i\sinh\theta - \Gamma_j(\Lambda,\lambda^2/G)}\right)\prod_{n=1}^{\Lambda} f_{n/2\Lambda}\left(\theta - i\pi\rho_j(\Lambda)\right). \tag{3.47b}$$

Note that for even Λ

$$P(\theta) = 0, \quad \Lambda \text{ even}. \tag{3.48}$$

Further restrictions on $\Gamma_{1,2}(\Lambda,\lambda^2/G)$ and $\rho_{1,2}(\Lambda)$ can be obtained by analyzing the pole structure (see section 4), and in addition $\Gamma_{1,2}(\Lambda,\lambda^2/G)$ is also constrained by the massless limit.

The massless limit for arbitrary Λ is obtained by taking $G \to 0$, along with $\theta \to \theta + \alpha$ and $\alpha \to \infty$, while keeping $G_0 = G^{(\Lambda+1)/2\Lambda}e^{\alpha}/2$ held fixed. In general $M \propto G^{(\Lambda+1)/2\Lambda}$, thus $G^{(\Lambda+1)/2\Lambda}$ acts as a mass, with G_0 replacing M_0 away from the free fermion point. This gives our "massless limit prescription". In order for the massless limit to make sense and be of the form (2.64), the limits $\lim_{G\to 0} G^{(\Lambda+1)/2\Lambda}\Gamma_{1,2}(\Lambda,\lambda^2/G)$ must be well-defined and equal. Considering (3.28), a reasonable choice for $\Gamma_{1,2}$ is to take

$$\Gamma_{1,2}\left(\Lambda,\frac{\lambda^2}{G}\right) = c(\Lambda)\left(\frac{\lambda^2}{G}\right)^{(\Lambda+1)/2\Lambda} + \tau_{1,2}(\Lambda), \tag{3.49}$$

where $c(\Lambda)$ and $\tau_{1,2}(\Lambda)$ are functions chosen to satisfy (3.28) and (3.29). The values of $\tau_{1,2}(\Lambda)$ are irrelevant in the massless limit. Following this procedure we find

$$\tilde{r}_{1,2}(\theta) \longrightarrow \pm i e^{i\frac{\pi}{4}(\Lambda-1)}, \quad f^{\mathrm{CDD}}_{1,2}(\theta) \longrightarrow -\tanh\left(\frac{\theta-\theta_B^{MK}}{2} - \frac{i\pi}{4}\right) \tag{3.50}$$

$$P(\theta) = 0, \quad Q(\theta) = -ie^{i\frac{\pi}{4}(\Lambda-1)}\tanh\left(\frac{\theta-\theta_B^{MK}}{2} - \frac{i\pi}{4}\right), \tag{3.51}$$

where θ_B^{MK} is Λ dependent, defined to satisfy

$$G_0 \exp\theta_B^{MK} = \lim_{G\to 0} G^{(\Lambda+1)/2\Lambda}\Gamma_1\left(\Lambda,\frac{\lambda^2}{G}\right) = \lim_{G\to 0} G^{(\Lambda+1)/2\Lambda}\Gamma_2\left(\Lambda,\frac{\lambda^2}{G}\right). \tag{3.52}$$

We recover the Kondo reflection S-matrix for all integer values of Λ. Because $\tilde{r}_{1,2}(\theta)$ is independent of the boundary, it only contributes a phase. This phase is actually necessary to satisfy the massless crossing relation. In the massless unitarity relation the phase cancels since it appears conjugated in $Q(-\theta)$. Massless unitarity and crossing relations are discussed in [31].

3.2. The mMK Reflection S-Matrix for general β

The mMK scattering amplitudes at the free fermion point are easily related to GZ's general solution. Comparing (2.68) and (3.14) we find

$$P(\theta) = P^{\mathrm{GZ}}_{\mathrm{free}}(\theta) f^{mMK}(\theta), \quad Q(\theta) = Q^{\mathrm{GZ}}_{\mathrm{free}}(\theta) f^{mMK}(\theta), \tag{3.53}$$

where we have set the phase $\sigma = \pi$. These are just the MBSG amplitudes with the free boundary condition multiplied by a CDD factor, with the formal parameters satisfying (3.12) and (3.13). Without loss of generality, we will fix ξ to be 0 for all Λ. For real ϑ, (3.12) gives the values

$$\eta(\Lambda = 1) = \pi \Longrightarrow k(\Lambda = 1) = 1, \quad \vartheta(\Lambda = 1) = 0. \tag{3.54}$$

Unlike the MK model, (3.53) and (3.54) are consistent with the generic result (3.2), thus allowing for an extension to arbitrary Λ. Taking into account the structure of the free fermion amplitudes (3.53), we propose the following soliton reflection S-matrix for the mMK model

$$P(\theta) = P_{\text{free}}(\theta) f^{mMK}(\theta), \quad Q(\theta) = Q_{\text{free}}(\theta) f^{mMK}(\theta), \tag{3.55}$$

where $P_{\text{free}}(\theta)$ and $Q_{\text{free}}(\theta)$ are the MBSG amplitudes for free boundary conditions

$$P_{\text{free}}(\theta) = \cosh(\Lambda\theta) r_0(\theta) \sigma\left(\frac{4\pi^2}{\beta^2}, \theta\right) \sigma(0, \theta) \tag{3.56a}$$

$$Q_{\text{free}}(\theta) = -i \sinh(\Lambda\theta) \frac{\cosh(\Lambda\theta)}{\sin(\Lambda\pi/2)} r_0(\theta) \sigma\left(\frac{4\pi^2}{\beta^2}, \theta\right) \sigma(0, \theta), \tag{3.56b}$$

and

$$f^{mMK} = \frac{i \sinh\theta + \Gamma(\Lambda, \hat{\phi}_0)}{i \sinh\theta - \Gamma(\Lambda, \hat{\phi}_0)}, \tag{3.57}$$

with $\Gamma(\Lambda, \hat{\phi}_0)$ some function satisfying

$$\Gamma(\Lambda = 1, \hat{\phi}_0) = \cosh 2\hat{\phi}_0. \tag{3.58}$$

The CDD factor (3.57) is a generalization of (2.69) to arbitrary values of Λ. In (3.56) the parameter η has been given a Λ dependence

$$\eta(\Lambda) = \frac{\pi}{2}(\Lambda + 1) = \frac{4\pi^2}{\beta^2} \Longrightarrow k(\Lambda) = \frac{1}{\sin(\Lambda\pi/2)}, \tag{3.59}$$

while $\vartheta(\Lambda) = \vartheta(\Lambda = 1) = 0$. This serves to produce the correct pole structure, namely a simple pole at $\theta = i\pi/2$ for all Λ [18]. It appears that Q_{free} is singular for even Λ because of the $\sin(\Lambda\pi/2)$ factor. But an identical factor appears in the numerator of $\sigma(4\pi^2/\beta^2, \theta)$ (3.7), making (3.56b) well-defined for all Λ. Since at the free fermion point $\lambda = \sqrt{2M} \propto \sqrt{G}$, and G is the only available bulk parameter, we must have $\lambda^2 \propto G$ for arbitrary Λ, thus explaining the relation (1.5c).

The main feature of (3.55) is that the boundary dependence occurs only in the CDD factor, just as for MK model. The minimal (non-CDD) parts P_{free} and Q_{free} are independent of $\hat{\phi}_0$. In contrast, the boundary dependence for the MBSG model is in general more complicated, with the formal parameters η and ϑ (and hence the factor $r_1(\theta)$) being functions of the physical parameters. With the boundary dependence isolated, the massless limit of the minimal factors follows by taking $\theta \to \infty$. The exact dependence of η and ϑ on Λ is not important in the massless limit as long as they are independent of the boundary coupling. The massless limit for the CDD factor is taken in the same manner as for the MK model, with the difference that $\hat{\phi}_0$ also has to be appropriately scaled such that $\sqrt{G}e^{\hat{\phi}_0}$ is held fixed. Holding $\sqrt{G}e^{\hat{\phi}_0}$ fixed amounts to keeping the coupling λ_1 constant as $G \to 0$. The massless limit restricts $\Gamma(\Lambda, \hat{\phi}_0)$ to the extent that $\lim_{G\to 0, \hat{\phi}_0 \to \infty} G^{(\Lambda+1)/2\Lambda} \Gamma(\Lambda, \hat{\phi}_0)$, with $\sqrt{G}e^{\hat{\phi}_0}$ fixed, must be well-defined. Making use of the integral representations (3.7) and (3.8) we find for the functions $r_0(\theta)$ and $\sigma(x, \theta)$

$$\lim_{\theta\to\infty} r_0(\theta) = 1, \quad \lim_{\theta\to\infty} \sigma(x, \theta) = 2\cos(x) e^{-ix} \lim_{\theta\to\infty} e^{-\Lambda\theta}, \tag{3.60}$$

which gives for the massless amplitudes

$$P(\theta) = 0, \quad Q(\theta) = i e^{-i\frac{\pi}{2}(\Lambda+1)} \tanh\left(\frac{\theta - \theta_B^{mMK}}{2} - \frac{i\pi}{4}\right), \tag{3.61}$$

where θ_B^{mMK} as a function of Λ is defined through

$$G_0 \exp\theta_B^{mMK} = \lim_{G\to 0, \hat{\phi}_0\to\infty} G^{(\Lambda+1)/2\Lambda} \Gamma(\Lambda, \hat{\phi}_0)\Big|_{\sqrt{G}e^{\hat{\phi}_0}\ \text{fixed}}. \tag{3.62}$$

So up to some phase we obtain the correct Kondo results for all Λ. This suggests that the integrable massive generalization of Kondo is given by the mMK model (1.5). We again point out that the situation is different here as compared with the MBSG model. For MBSG, η and ϑ are boundary scale dependent. To get the massless limit, these parameters also have to be rescaled such that $g(\theta, \eta, \vartheta) \to \theta - \theta_B$ remains finite, where g is some function giving the massless limit prescription [16]. In our case the prescription (for the minimal part) is very simple, let $\theta \to \infty$.

4. Boundary Bound States

Important information on the spectrum of a theory can be obtained from the poles of the reflection amplitudes. Poles in the reflection S-matrix R can be of two types. If there are bulk bound states interacting with the boundary, then poles will be found at $\theta = i\pi/2$ and $\theta = i\pi/2 - \theta_b$, where θ_b is the bound state pole in the bulk S-matrix. These bulk states will give zero-momentum one-particle contributions to the boundary state $|B\rangle$. Secondly, poles associated with boundary bound states can also occur. These are excitations of the boundary ground state appearing in the region $0 \leq \theta \leq i\pi/2$. If the pole occurs at $\theta = i\pi/2$, then there will be a degeneracy in the ground state and again a zero-momentum state will be found in $|B\rangle$. At the free fermion point we only expect boundary bound states since there are no bulk bound states (the bulk S-matrix is -1).

4.1. Boundary bound states for the MK model

At the free fermion point we can map the MK model to the two Ising models, called Ising1 for $(\psi_1, \overline{\psi}_1)$ and Ising2 for $(\psi_2, \overline{\psi}_2)$, to study the poles. These Ising models differ in their boundary conditions at zero and infinite coupling. The poles of the MK reflection S-matrix occur at

$$\theta_1 = i\frac{\pi}{2}, \quad \text{and} \quad \sin v_2 = 1 - \frac{\lambda^2}{4M} \;\; (\theta_2 = iv_2), \tag{4.1}$$

where θ_1 is due to Ising1 and $\theta_2 = iv_2$, which only exists in the physical region if $\lambda^2 < 4M$, is due to Ising2. As in [18] these have the following interpretation. Each Ising model has two degenerate boundary ground states $|0, \pm\rangle_B^i$ $\{i = 1, 2\}$, giving two different expectation values for the spin field $\sigma(x)$, $\langle\sigma(x)\rangle_\pm \propto \pm 1$. For the Ising2 piece, the degeneracy is broken by the boundary magnetic field

$$|0, \pm\rangle_B^2 \longrightarrow |0\rangle_B^2, |1\rangle_B^2, \tag{4.2}$$

and the pole at θ_2 corresponds to the boundary bound state $|1\rangle_B^2$ with energy

$$E_1^{(2)} - E_0^{(2)} = M \cos v_2. \tag{4.3}$$

In the Ising1 case the degeneracy persists for a non-zero boundary interaction ($\lambda \neq 0$). Both states have the same energy

$$E_+^{(1)} = E_-^{(1)}. \tag{4.4}$$

Thus for $\lambda^2 < 4M$ we have two sets of degenerate states

$$\{|0, +\rangle_B^1 |0\rangle_B^2, |0, -\rangle_B^1 |0\rangle_B^2\} \quad \text{and} \quad \{|0, +\rangle_B^1 |1\rangle_B^2, |0, -\rangle_B^1 |1\rangle_B^2\}, \tag{4.5}$$

associated with the poles at θ_1 and θ_2 respectively. This suggests that one should actually regard the boundary as having structure B_α, where $\alpha = \pm$ corresponds to the above degenerate states. Because of the pole at $i\pi/2$, there will be a zero-momentum contribution to $|B\rangle$ of the form $A_1^\dagger(0) = \frac{1}{\sqrt{2}}(A_+^\dagger(0) + A_-^\dagger(0)$. If $\lambda^2 > 4M$, the pole at θ_2 leaves the physical strip and we are left with the degenerate ground states.

It is not clear what this pole structure implies for the states of ϕ and the spin operators $S_\pm$. We make the following remark. For a spin 1/2 impurity along with the screening effect, the boundary is described by the Hilbert space

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1. \tag{4.6}$$

For an infinite interaction ($\lambda = \infty$) the boundary would exist in the $|j,m\rangle = |0,0\rangle$ state. With a finite interaction ($\lambda^2 < 4M$), the space (4.6) decomposes into two sets of degenerate states, which are some linear combinations of $|0,0\rangle$, $|1,0\rangle$ and $|1,\pm 1\rangle$. These have to be related to the Ising states.

Away from the free fermion point the bulk S-matrix has soliton bound states. These are the charge neutral breathers with masses

$$m_n = 2M \sin\left(\frac{n\pi}{2\Lambda}\right), \quad n = 1, 2, \ldots < \Lambda. \tag{4.7}$$

The bound states should give rise to poles in the reflection amplitudes. For odd $\Lambda > 1$, the poles in $P(\theta)$ (3.47a) at

$$\theta = i\frac{n\pi}{2\Lambda}, \quad n = 1, 2 \ldots < \Lambda, \tag{4.8}$$

are associated with the breathers. These poles also occur in $Q(\theta)$, implying that the boundary should carry a $(+,-)$ "topological charge" since the breathers are charge neutral. This would agree with a description of the boundary in terms of $B_\pm$, with the degenerate states (4.5) having different charges. To account for the breathers at even $\Lambda > 1$ we must take (3.44a) for $\rho_i(\Lambda)$, which will reproduce the poles (4.8). The diagonal amplitude $P(\theta)$ vanishes for even Λ. This would make sense if at even Λ the boundary charge is not conserved, leading to a change in the soliton charge.

With the choice (3.44a), the pole at $\theta_1 = i\pi/2$ is present for all integer Λ and the ground state is twofold degenerate. The existence of a boundary bound state for $\Lambda > 1$ depends on $\Gamma_{1,2}(\Lambda, \lambda^2/G)$. If we have $-1 < \Gamma_{1,2}(\Lambda, \lambda^2/G) < 0$, then a pole will exist in the physical strip at

$$\theta_2 = iv_2, \quad \sin v_2 = -\Gamma_{1,2}(\Lambda, \lambda^2/G), \tag{4.9}$$

leading to a bound state. However for $\Lambda > 1$, a map to the Ising picture does not exist and the boundary states cannot be explained as in (4.5). For odd Λ we can have $-1 < \Gamma_{1,2}(\Lambda = 2n+1, \lambda^2/G) < 0$ and $\Gamma_1 \neq \Gamma_2$, which will lead to two distinct bound states.

4.2. Boundary bound states for the mMK model

The pole structure for the mMK model is identical to that of the MBSG model with free boundary conditions, apart from any poles due to the CDD factor. We will only comment on the boundary bound states. At the free fermion point the mMK model can be written as a sum of two Ising models, Ising1 and Ising2, with couplings $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. Consider for the moment the more general system (2.40). The first Ising copy leads to the pole at $\theta_1 = i\pi/2$ as for the MK model. The second copy contains the CDD factor

$$f_2^{\mathrm{CDD}} = -\left(\frac{i\sinh\theta + (\Delta_2 - 1)}{i\sinh\theta - (\Delta_2 - 1)}\right), \tag{4.10}$$

which gives a boundary bound state provided the "magnetic field" is not too strong

$$\tilde{\lambda}_2^2 < 4M. \tag{4.11}$$

In terms of λ_1 and λ_2 ($\sigma = \pi$)

$$\tilde{\lambda}_2^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_2\lambda_2. \tag{4.12}$$

Now specialize to the mMK system with

$$\lambda_{1,2} = \sqrt{2M} e^{\pm\hat{\phi}_0}, \tag{4.13}$$

giving

$$\tilde{\lambda}_2^2 = 4M(\cosh 2\hat{\phi}_0 + 1) > 4M. \tag{4.14}$$

Thus there is no boundary bound state because the coupling $|\tilde{\lambda}_2|$ is too large for all values of $\hat{\phi}_0$. The poles for the CDD factor $f^{mMK}(\theta)$

$$f^{mMK}(\theta) = -f_2^{\rm CDD}(\theta) = \left(\frac{i \sinh\theta + \cosh 2\hat{\phi}_0}{i \sinh\theta - \cosh 2\hat{\phi}_0} \right), \tag{4.15}$$

occur outside the physical strip at

$$\theta_0 = \pm 2\hat{\phi}_0 - i\frac{\pi}{2}. \tag{4.16}$$

So the boundary states at $\Lambda = 1$ consist of two ground states

$$\{|0,+\rangle_B^1 |0\rangle_B^2, |0,-\rangle_B^1 |0\rangle_B^2\}, \tag{4.17}$$

corresponding to the pole at $i\pi/2$.

By choosing $\eta = \frac{\pi}{2}(\Lambda+1)$, we maintain the pole at $i\pi/2$ for all Λ, though its interpretation in terms of the Ising states is lost for $\Lambda > 1$. Away from the free fermion point $f^{mMK}(\theta)$ has a pole at

$$\sinh\theta_2 = -i\Gamma(\Lambda, \hat{\phi}_0). \tag{4.18}$$

If $-1 < \Gamma(\Lambda, \hat{\phi}_0) < 0$, this pole will lie in the physical strip implying a bound state.

5. Conclusions

We have presented two massive versions of the anisotropic spin 1/2 Kondo model and discussed their integrability. Both models correspond to a sine-Gordon theory in the bulk, but differ slightly in their boundary interactions. Our arguments for integrability were based on the assumption that there are no explicit boundary degrees of freedom in the scattering description, which is supported by the screening at the free fermion point, on constraints from the boundary Yang-Baxter equation, and comparison with the massive boundary sine-Gordon model at the free fermion point. We argued that the most natural massive extension of the Kondo model (MK) is not integrable except perhaps at the reflectionless points, and we proposed a reflection S-matrix for these points. The modified massive Kondo model (mMK), on the other hand, we argued is integrable for all β and proposed an exact reflection S-matrix, which is a CDD factor times the massive boundary sine-Gordon solution corresponding to a free boundary condition.

Since we did not prove the integrability of our models by studying integrals of motion, our proposed S-matrices remain conjectures. It would be interesting to develop conformal perturbation theory to study integrability in the presence of boundary operators such as $S_\pm$.

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Appendix

We briefly discuss how the anisotropic spin 1/2 Kondo model can be mapped onto the one dimensional bosonic field theory

$$H^K = \frac{1}{2}\int_{-\infty}^{0} dx\ \left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) + \lambda\left(S_+ e^{i\beta\phi(0)/2} + S_- e^{-i\beta\phi(0)/2}\right), \tag{A1}$$

where λ and β are free parameters explained below. A more complete derivation can be found in [32] (see also [33]), which we will follow closely. The map involves the technique of abelian bosonization, as was first applied to the Kondo model by Schotte [34]. For further information on bosonization see [35]-[38] and references therein.

We begin with the anisotropic spin 1/2 Kondo Hamiltonian

$$H^K = H_0^F + H_{\rm int} \tag{A2}$$

$$H_0^F = \sum_{\vec{k}\sigma} \epsilon(\vec{k}) c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} \tag{A3}$$

$$H_{\rm int} = \frac{J_z}{2} s_z \sum_{\vec{k}\vec{k}'\sigma\sigma'} c^\dagger_{\vec{k}\sigma}(\sigma^z)_{\sigma\sigma'} c_{\vec{k}'\sigma'} + \frac{J_\perp}{2} \sum_{\vec{k}\vec{k}'\sigma\sigma'} \left(s_x c^\dagger_{\vec{k}\sigma}(\sigma^x)_{\sigma\sigma'} c_{\vec{k}'\sigma'} + s_y c^\dagger_{\vec{k}\sigma}(\sigma^y)_{\sigma\sigma'} c_{\vec{k}'\sigma'}\right). \tag{A4}$$

The interaction is taken to be pointlike with only s-wave scattering. Thus the plane wave states can be expanded in spherical waves about the impurity, with only the angular momentum $l = m = 0$ modes being retained. Also since we are usually interested in low energy excitations near the fermi surface, $\epsilon(\vec{k})$ can be expanded as

$$\epsilon(k) = \epsilon_F + v_F(k - k_F), \tag{A5}$$

where $k = |\vec{k}|$. With these approximations, and measuring the energy and momentum relative to the fermi surface, the free part becomes (with $v_F = 1$)

$$H_0^F = \sum_{p\sigma} p c^\dagger_{p\sigma} c_{p\sigma}, \tag{A6}$$

where $c^\dagger_{p\sigma}$ create s-wave electrons of momentum $k = k_F + p$. As is common for bosonization, in fact necessary [35], we allow p to be unbounded in H_0^F and regularize by introducing an exponential cutoff in the interaction. The ground state consists of all states with $p < 0$ filled. Using the fermion fields

$$\tilde{\psi}^\dagger_\sigma(x) = \frac{1}{\sqrt{L}} \sum_p c^\dagger_{p\sigma} e^{-ipx}, \tag{A7}$$

where L is the normalization length and $p = 2\pi n/L, n = 0, \pm1, \pm2, \ldots$, combined with a redefinition of the coupling constants, the interaction takes the form

$$H_{\rm int} = \frac{J_z}{4} S_z \left(\tilde{\psi}^\dagger_\uparrow\tilde{\psi}_\uparrow(0) - \tilde{\psi}^\dagger_\downarrow\tilde{\psi}_\downarrow(0)\right) + \frac{J_\perp}{2}\left(S_+\tilde{\psi}^\dagger_\downarrow\tilde{\psi}_\uparrow(0) + S_-\tilde{\psi}^\dagger_\uparrow\tilde{\psi}_\downarrow(0)\right), \tag{A8}$$

where $S_z = \sigma^z$ and $S_\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. The coupling constants are now dimensionless.

To bosonize, it is convenient to separate the so-called fermion charge and spin degrees of freedom. Define the following charge and spin density operators for $k > 0$

$$\rho_c(k) = \sum_p (c^\dagger_{p+k\uparrow} c_{p\uparrow} + c^\dagger_{p+k\downarrow} c_{p\downarrow}), \quad \rho_c(-k) = \rho_c^\dagger(k) \tag{A9}$$

$$\rho_s(k) = \sum_p (c^\dagger_{p+k\uparrow} c_{p\uparrow} - c^\dagger_{p+k\downarrow} c_{p\downarrow}), \quad \rho_s(-k) = \rho_s^\dagger(k). \tag{A10}$$

Then the operators

$$a_k^c = \sqrt{\frac{\pi}{kL}}\rho_c(-k), \quad a_k^s = \sqrt{\frac{\pi}{kL}}\rho_s(-k), \tag{A11}$$

obey canonical boson commutation relations

$$[a_k^c, a_{k'}^{c\dagger}] = [a_k^s, a_{k'}^{s\dagger}] = \delta_{kk'}, \tag{A12}$$

when acting on the ground state. Furthermore, instead of H_0^F we can use the boson Hamiltonian

$$H_0^B = \sum_{k>0} k(a_k^{c\dagger}a_k^c + a_k^{s\dagger}a_k^s), \tag{A13}$$

to obtain the same dynamics. The charge and spin sectors separate in the linear approximation. The J_z term can now be expressed as

$$\frac{J_z}{4}S_z\left(\tilde\psi_\uparrow^\dagger\tilde\psi_\uparrow(0) - \tilde\psi_\downarrow^\dagger\tilde\psi_\downarrow(0)\right) = \frac{J_z}{4}S_z\sum_{k>0}e^{-ak/2}\sqrt{\frac{k}{\pi L}}(a_k^s + a_k^{s\dagger}). \tag{A14}$$

Here we have introduced a cutoff which eliminates values of $k > a^{-1} \sim k_F$, thus regularizing divergent momentum sums. To write the $J_\perp$ interaction in terms of the boson degrees of freedom, we make use of the "bosonization" rule

$$\tilde\psi_{\uparrow,\downarrow}(x) = \frac{1}{\sqrt{2\pi a}}e^{-i\tilde\phi_{\uparrow,\downarrow}(x)}, \tag{A15}$$

where the fields $\tilde\phi_{\uparrow,\downarrow}$ are

$$\tilde\phi_{\uparrow,\downarrow} = i\sum_{k>0}e^{-ak/2}\sqrt{\frac{2\pi}{kL}}\left(b_{k\uparrow,\downarrow}e^{-ikx} - b_{k\uparrow,\downarrow}^\dagger e^{ikx}\right), \quad \tilde\phi_{\uparrow,\downarrow}^\dagger = \tilde\phi_{\uparrow,\downarrow} \tag{A16}$$

with the b operators related to the charge and spin operators

$$b_{k\uparrow} = \frac{1}{\sqrt2}(a_k^c + a_k^s), \quad b_{k\downarrow} = \frac{1}{\sqrt2}(a_k^c - a_k^s). \tag{A17}$$

For the same spin index σ $(\uparrow,\downarrow)$, the exponentials in (A15) behave as fermion fields. The only problem with $\exp(\pm i\tilde\phi_\sigma)$ is that for different spins $\sigma \neq \sigma'$ these commute. This can be fixed by using more complex representations, such as including Klein factors [35] (see also [38],[24]). The Klein factors serve to produce the correct commutation relations as well as create and destroy electrons, just as the fields $\tilde\psi^\dagger$ and $\tilde\psi$. The exponentials alone cannot change the number of electrons. We will not consider these important issues here. With (A15) the $J_\perp$ term can be written as

$$\frac{J_\perp}{2}\left(S_+\tilde\psi_\downarrow^\dagger\tilde\psi_\uparrow(0) + S_-\tilde\psi_\uparrow^\dagger\tilde\psi_\downarrow(0)\right) = \frac{J_\perp}{4\pi a}\left(S_+e^{i\sqrt{2\pi}\tilde\phi(0)} + S_-e^{-i\sqrt{2\pi}\tilde\phi(0)}\right), \tag{A18}$$

where

$$\tilde\phi(x) = \frac{1}{\sqrt{2\pi}}\left(\tilde\phi_\downarrow(x) - \tilde\phi_\uparrow(x)\right) = -i\sum_{k>0}e^{-ak/2}\sqrt{\frac{2}{kL}}\left(a_k^s e^{-ikx} - a_k^{s\dagger}e^{ikx}\right) \tag{A19}$$

We see that the exchange interaction involves only the spin sector, the charge sector is completely decoupled. We will only be interested in the spin sector and drop the s superscript. Combining everything we have

$$H = \sum_{k>0}ka_k^\dagger a_k - \frac{J_z}{4}\frac{1}{\sqrt{2\pi}}S_z\partial_x\tilde\phi(0) + \frac{J_\perp}{4\pi a}\left(S_+e^{i\sqrt{2\pi}\tilde\phi(0)} + S_-e^{-i\sqrt{2\pi}\tilde\phi(0)}\right). \tag{A20}$$

Now we can write down the boson field theory Hamiltonian associated with (A20). Consider the following action on the half line

$$S = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 - (\partial_x\phi)^2\right). \tag{A21}$$

Varying the action and requiring the boundary terms to vanish gives the boundary condition

$$\partial_x\phi = 0, \quad \text{at } x = 0. \tag{A22}$$

A mode expansion satisfying the bulk equation of motion, $({\partial_t}^2 - {\partial_x}^2)\phi = 0$, can be written as

$$\phi(t,x) = -i\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} \left(a(k)e^{-i\vec{k}\vec{x}} - a^\dagger(k)e^{i\vec{k}\vec{x}}\right), \tag{A23}$$

where $\vec{k}\vec{x} = |k|t - kx$. The boundary condition (A22) requires that $a(k) = a(-k)$. This allows us to separate ϕ into its left and right moving parts as follows

$$\phi = \phi_L + \phi_R, \quad \phi_{L,R}(t,x) = -i\int_0^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} \left(a(k)e^{-ik(t\pm x)} - a^\dagger(k)e^{ik(t\pm x)}\right). \tag{A24}$$

Since $\phi_L(x) = \phi_R(-x)$, the free Hamiltonian obtained from (A21) is

$$H_0^K = \int_0^{\infty} dk\, k a^\dagger(k)a(k), \tag{A25}$$

which is just the continuum limit of the free part of (A20). Also taking the continuum limit of $\tilde{\phi}$ we find that

$$\tilde{\phi}(t, x=0) = \phi(t,0) = 2\phi_L(t,0) = 2\phi_R(t,0), \quad \partial_x\tilde{\phi}(t,x=0) = \partial_t\phi(t,0) = \pi(t,0). \tag{A26}$$

Of course we should include the exponential cutoff in (A23). This shows that the anisotropic Kondo model (to be exact, the spin sector) is equivalent to the scalar boundary field theory

$$H^K = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) - \frac{J_z}{4}\frac{1}{\sqrt{2\pi}} S_z\pi(0) + \frac{J_\perp}{4\pi a}\left(S_+ e^{i\sqrt{2\pi}\phi(0)} + S_- e^{-i\sqrt{2\pi}\phi(0)}\right). \tag{A27}$$

Finally to obtain (A1) we make the unitary transformation

$$H \to UHU^\dagger, \quad U = e^{i\mu S_z\phi(0)}, \tag{A28}$$

where μ is a constant. The transformed Hamiltonian, apart from constants, is found to be

$$H^K = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) - \left(\frac{J_z}{4}\frac{1}{\sqrt{2\pi}} + \mu\right) S_z\pi(0) + \frac{J_\perp}{4\pi a}\left(S_+ e^{i(\sqrt{2\pi}+2\mu)\phi(0)} + S_- e^{-i(\sqrt{2\pi}+2\mu)\phi(0)}\right). \tag{A29}$$

Choosing μ to be

$$\mu = -\frac{J_z}{4}\frac{1}{\sqrt{2\pi}}, \tag{A30}$$

the J_z term cancels, leaving the boundary field theory Hamiltonian (A1)

$$H^K = \frac{1}{2}\int_{-\infty}^{0} dx\, \left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) + \lambda\left(S_+ e^{i\beta\phi(0)/2} + S_- e^{-i\beta\phi(0)/2}\right), \tag{A31}$$

where

$$\lambda = \frac{J_\perp}{4\pi a}, \quad \beta = \sqrt{2\pi}\left(2 - \frac{J_z}{2\pi}\right). \tag{A32}$$

The coupling λ has a canonical dimension of mass whereas β is dimensionless. For an antiferromagnetic J_z, $4\pi > J_z > 0$, we have $0 < \beta < \sqrt{8\pi}$. If we take $J_\perp$ to be small, then the isotropic Kondo model corresponds to J_z being small $J_z \approx J_\perp$. Thus the isotropic point occurs approximately at $\beta = \sqrt{8\pi}$. We have shown that the anisotropic Kondo model can be described by the bosonic boundary field theory (A31).

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