Embeddings of the Virasoro algebra and black hole entropy

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We consider embeddings of the Virasoro algebra into other Virasoro algebras with different central charges. A Virasoro algebra with central charge $c$ (assumed to be a positive integer) and zero mode operator $L_0$ can be embedded into another Virasoro algebra with central charge one and zero mode operator $c L_0$. We point out that this provides a new route to investigate the black hole entropy problem in $2+1$ dimensions.

Three-dimensional gravity was put forward by Deser, Jackiw and ’t Hooft [1] as an interesting toy model for gravitational physics. It was then argued in [2] (see also [3]) that it defines a soluble and finite quantum field theory. Questions such as which are the macroscopical degrees of freedom giving rise to the Bekenstein-Hawking entropy in three dimensions should then have an answer. However, despite interesting proposals [4,5] it is clear that the answer to this question is not yet available. Even more, the difficulties which arise from trying to provide a consistent quantum description of the black hole entropy have led to the suggestion that Einstein gravity represents a sort of thermodynamical description for the gravitational phenomena and it thus makes no sense to attempt to quantize it (see [6] and references therein). In the Loop representation approach to quantum gravity, successful computations for the black hole entropy, up to a numerical factor, have been achieved. See [7] and [8].

Our main tool in analysing the three-dimensional black hole entropy will be the discovery of Brown and Henneaux [9] that the asymptotic symmetry group of three dimensional anti-de Sitter (adS) space is generated by two copies of the Virasoro algebra with central charge $|c|$ [9]

\[ c = \frac{3l}{2G} \]  

and hence, is the two-dimensional conformal group. Here $G$ is Newton’s constant, we parametrize the negative cosmological constant as $\Lambda = -1/l^2$ and set $\hbar = 1$. In the semiclassical regime $G \to 0$, $c$ is large. The $2+1$ black hole [10] is asymptotically anti-de Sitter and thus the conformal group acts on it in a similar form. However, globally, $\text{adS}$ and the black hole differ since the latter is obtained from the former by identification of points. These identifications reduce the exact Killing symmetries from $SO(2,2)$ to $SO(2) \times \mathbb{R}$. For this reason, acting on the black hole, the Virasoro algebra reads [11],

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n^3 \delta_{n+m} \]  

having a one dimensional sub-algebra generated by $L_0$. The same holds for the other copy $\bar{L}_0$. We shall call (2) the Ramond Virasoro algebra. The more usual Neveu-Schwarz form of the Virasoro algebra is obtained from (2) by shifting the zero mode as $L_0 = L_0^{NS} - c/24$. The above convention, appropriated to periodic boundary conditions in the spinor field, is natural in the supergravity version of the super-conformal algebra [11,12]. The black hole mass and angular momenta are given in terms $L_0$ and $\bar{L}_0$ as,

\[ M l = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0 \]  

with no added constants. With these conventions, anti-de Sitter space has $J = 0$ and $M l = -c/12$. For $M \geq 0$ the associated metric represents a black hole [10], while the particle solutions studied in [13] have $-c/12 < l M < 0$.

An important open question in $2+1$ adS gravity is what is the conformal field theory (CFT) whose energy momentum tensor generates the $c = 3l/2G$ Virasoro algebra. At the classical level (and up to some global issues) this theory is described by a Liouville field [14] (see also [12]) which has two copies of the Virasoro algebra ($\bar{L}_0$, $L_0$) with the correct value of $c$. Since $2+1$ gravity has no other degrees of freedom, one would expect that the $2+1$ black hole entropy is associated to states in quantum Liouville theory with a given value of $\bar{L}_0$, $L_0$. The number of such states turns out to be proportional to the black hole area, but with a wrong power of Newton’s constant $G$. Note that this can happen because in adS gravity there are two length scales; the cosmological radius $l$, and the Planck length parameter $l_p = G$. See [15] for a first discussion on black hole entropy and the conformal algebra, and [16] for a recent review.

A striking observation made in [5] (see also [17]) is that if the boundary CFT satisfies the following two conditions: (i) $\text{Tr} q^{L_0} = \text{modular invariant}$ (with $q = \exp(2\pi i r)$ and $\text{Tr}$ denotes both, the character for a given representation and the sum over representations), and (ii) $L_0 \geq -c/24$, $\bar{L}_0 \geq -c/24$ (note that $\bar{L}_0$ is the Ramond Virasoro operator satisfying (2)), then the degeneracy of states for a given value of $2\bar{L}_0 = l M + J$ and $2\bar{L}_0 = l M - J$ give rise exactly to the Bekenstein-Hawking entropy of a black hole of mass $M$ and angular momenta $J$. An example of a CFT satisfying these two conditions is a set of $c$ free bosons whose diagonal energy momentum tensor has a central charge equal to $c$ and satisfy both (i) and (ii). On the contrary, Liouville theory fails to have the right lower bound on $\bar{L}_0$, $L_0$ [18] and does not give the right degeneracy [16].

The failure of Liouville theory to provide the right counting has lead many authors to conclude that the microscopic origin of the black hole entropy needs an additional structure (probably string theory) not seen by the
gravitational field, which would represent only an expectation value for the true quantum fields. Specifically, in the context of the adS/CFT conjecture [19,21,20], it has been suggested in [6] that the quantum CFT is related to Liouville theory by an expression of the form 
\[ \langle T_{CFT} \rangle = T_{\text{Liouville}}. \]
Incidentally, it is worth mentioning here that the adS/CFT conjecture has been used to compute the central charge of \( T_{CFT} \) [22,23] and agreement with the Brown-Henneaux value (1) is found.

Further evidence for a string theory description of the black hole was presented in [24] where a string propagating in the adS background was studied. A formula for the spacetime conformal generators was given in terms of the string currents. The spacetime central charge in this approach is associated to the winding of the worldsheet current in spacetime. Since the string theory is unitary (for \( SL(2,\mathbb{R}) \) string theories see [25] and references therein), the corresponding spacetime CFT is expected to give rise to the right degeneracy. This, to our knowledge, has not not been carried out in detail.

Whether string theory is the only solution to this problem or not is not yet clear. However, the recent developments in the subject have made it clear that the microscopical description of the three-dimensional black hole entropy requires more degrees of freedom than those arising from a naïve analysis of the classical boundary dynamics. (A counterexample to this statement is Carlip’s [4] original calculation which requires only an affine \( SO(2,2) \) algebra, arising in a natural way in \( 2+1 \) gravity [26], plus some boundary conditions. The main problems with that proposal seems to be the large number of negative-norm states being counted and the physical meaning of the boundary conditions.)

In this paper we present a new route to attack this problem. We shall add a set of new degrees of freedom to the classical dynamics which upon quantization will account correctly for the Bekenstein-Hawking entropy. These new degrees of freedom will have a simple geometrical interpretation although their fundamental quantum origin is not yet known to us. We shall first show our main results and then discuss their significance and possible interpretation.

Let \( Q_n (n \in \mathbb{Z}) \) be a set of operators satisfying the (Ramond) Virasoro algebra with central charge equal to 1,
\[ [Q_n, Q_m] = (n-m)Q_{n+m} + \frac{1}{12} n^3 \delta_{n,m}. \]
Irreducible and unitary representations for this algebra are known and are uniquely classified by a highest weight state \( |h> \) with conformal weight \( \langle Q_0 + 1/24| h> = h|h> \). The shift 1/24 appears here because \( Q_0 \) is the Ramond Virasoro operator entering in (4).

For integer values of the central charge \( c \), the Virasoro algebra (2) is a sub-algebra of (4) (see [27] and references therein for an extensive discussion on this point). Define the generators \( L_n \) by
\[ L_n = \frac{1}{c} Q_{cn}, \quad n \in \mathbb{Z}, \quad (c \in \mathbb{N}). \]
For \( c > 1 \) the \( L_n \)'s are a subset of the \( Q_n \)'s. Computing the commutator of two \( L_n \)'s we obtain,
\[ [L_n, L_m] = \frac{1}{c^2} [Q_{cn}, Q_{cm}] = \frac{1}{c^2} \left( c(n-m)Q_{c(n+m)} + \frac{c^3 n^3}{12} \delta_{c(n+m)} \right) = (n-m)L_{n+m} + \frac{c}{12} n^3 \delta_{n+m}. \]
It is interesting to see how the above mechanism applies in the Euclidean canonical formalism. The Euclidean black hole has the topology of a solid torus [15] whose boundary is a torus with a modular parameter
\[ \tau = \frac{\beta}{2\pi} \left( \Omega + \frac{i}{\bar{\Omega}} \right), \]
where \( \beta \) and \( \Omega \) are, respectively, the black hole temperature and angular velocity [29,32]. (This definition of \( \tau \)

\[ Q_n, Q_m = (n-m)Q_{n+m} + \frac{1}{12} n^3 \delta_{n,m}. \]
differ from the one used in [29] in the factor $2\pi$.) The gravitational partition function, under some appropriated boundary conditions and considering only a boundary at infinity, can then be expressed in terms of the character [14,29]

$$Z[\tau] = \text{Tr} \exp(2\pi i \tau L_0 - 2\pi i \tilde{\tau} \tilde{L}_0).$$  \hspace{1cm} (9)

where $L_0$ is the zero mode Virasoro operator in (2).

The expected behaviour in the semiclassical Gibbons-Hawking (GH) approximation for $Z$ is (see [33] for a recent discussion)

$$\ln Z_{GH}(\beta) = \frac{\pi^2 \beta^2}{2G \beta},$$  \hspace{1cm} (10)

where we have set $\Omega = 0$ (no angular momentum) for simplicity. This follows from evaluating the black hole free energy $-\beta F = -\beta M + S$ with $M = \tau^2/\alpha(8G^2)$, $S = 2\pi \alpha/(4G)$ and $\beta = 2\pi \beta^2/\alpha$. 

The question of the degeneracy of states can now be reformulated as whether the partition function (9) reproduces or not this semiclassical behaviour.

If $Z[\tau]$ defined in (9) was modular invariant with $L_0^c \geq -c/24$ and $c$ given in (1), then it is direct to see that $Z[\tau]$ would behave exactly as (10) [30]. This is nothing but the canonical version of the results obtained in [5]. The trouble is that for $c > 1$, either looking at representations of the Liouville theory or the Virasoro algebra (2) itself, it is not possible to fulfill both conditions. Indeed, (9) does not show the behaviour (10). The trace in (9) needs to be taken on a bigger Hilbert space; more degrees of freedom are necessary.

Let us take the full algebra generated by the $Q$'s, of which (2) is a sub-algebra, and compute the trace over representations of (4) with central charge one. First we use (5) and write

$$Z_Q[\hat{\tau}] = \text{Tr}_{Q} \exp(2\pi i \tau Q_0 - 2\pi i \tilde{\tau} \tilde{Q}_0).$$  \hspace{1cm} (11)

with $\hat{\tau} = \tau/c$. We have thus replaced the Virasoro character with central charge $c$ and modular parameter $\tau$, with a different character with central charge one and modular parameter $\tau/c$. In symbols,

$$ch(c, \tau) \rightarrow ch(1, \hat{\tau} = \tau/c).$$  \hspace{1cm} (12)

The character (11) can be computed exactly. After an appropriated sum over zero modes (see below) we find [31]

$$Z_Q[\hat{\tau}] = \frac{A}{\text{Im}(\hat{\tau})^{1/2} |\eta(\hat{\tau})|^2}$$  \hspace{1cm} (13)

which is invariant under modular mappings acting on $\hat{\tau}$. $A$ is a constant which does not depend on $\tau$. The asymptotic behaviour of (13) can be determined either by using the well known asymptotic expansions for the Dedekind function, or by looking at (11) and using modular invariance, as done in [30]. In any case, one finds

$$\ln Z_Q \sim \frac{i\pi}{6\tau},$$  \hspace{1cm} (14)

which, in terms of $\tau$, reproduces exactly the Gibbons-Hawking approximation (10). We then see a complete analogy between the relations $Q_0 = cL_0$ (microcanonical) and $\hat{\tau} = \tau/c$ (canonical) which encode the addition of the new degrees of freedom. In particular, note that the semiclassical approximations $Q_0$ large and $\hat{\tau}$ small are controlled by the central charge $c$, without imposing any conditions over $L_0$ or $\tau$. This means that the asymptotic behaviours (7) and (10) are actually universal for all values of $M$ and $J$ since the semiclassical condition $c >> 1$ ensures both $Q_0$ large and $\hat{\tau}$ small [35].

It is important to mention here that the exact result (13) for the partition function arose after an integration over zero modes (see [31]). This integration is actually not necessary to have the right semiclassical limit. Indeed, for each representation with conformal weight $h$, the character approaches (10) for small $\beta$. We have chosen to perform the integration over $h$ in order to find a modular invariant partition function, which is likely to be an important property of the black hole dynamics. Making the integral over the conformal dimensions is also an statement on the spectrum of the associated CFT. From the geometrical point of view, we know that positive values of $L_0 + \tilde{L}_0$ represent black holes, while $-c/12 < L_0 + \tilde{L}_0 < 0$ give rise to conical singularities. The states considered in the computation of (13) have $Q_0 + 1/24 = h + N$ and we have integrated over all positive values of $h$. This means $Q_0 \geq -1/24$. In terms of $L_0$ this implies $L_0 \geq -1/(24c)$. Thus, the modular invariant partition function (13) does contain states corresponding to the particles studied in [13]. However, curiously, not all particle masses are allowed but only the small region $-1/(12c) < L_0 + \tilde{L}_0 < 0$. In particular, anti-de Sitter space with $L_0 + \tilde{L}_0 = -c/12$ is not included.

The issue of modular invariance brings into the scene another important point. The boundary of the black hole is a torus with a modular parameter $\tau$, and one could have expected the partition function to be modular invariant under modular mappings acting on $\tau$. On the contrary, we have found that $\hat{Z}$ is invariant under the modular group acting on $\hat{\tau} = \tau/c$. This is not surprising since all we have done is to use the identity $\tau L_0 = \hat{\tau} Q_0$ and compute the trace over representations of $Q_0$. (This is summarized in (12).) It correct, this scaling of the modular parameter should have a precise meaning to be uncovered. In particular, one would like to know whether the addition of the new generators, giving rise to (4) and $\hat{\tau}$ could be understood in terms of the gravitational variables themselves, up to some rescaling or duality transformations, or a more sophisticated mechanism like introducing string degrees of freedom is necessary.

To summarize, we have shown that a Virasoro algebra with central charge $c$ ($c$ integer) can be understood as a sub-algebra of another Virasoro algebra with central charge one. We have thus imposed the quantization
condition $c \in \mathbb{N}$ where $c$ is the central charge (1), and have extended the Brown-Henneaux conformal algebra by adding new generators. The new conformal algebra reproduces in a natural way the semiclassical aspects of the 2+1 black hole thermodynamics. An important open question not addressed here is the uniqueness of this approach. The embeddings of the Virasoro algebra studied here are of course not unique, although it is not clear to us that other embeddings of (2) will give rise to the right entropy. At any rate, a more detailed calculation of other semiclassical quantities such as decay rates [36] in the $Q$-theory should test its uniqueness and correctness.

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