Hopf Solitons on $S^3$ and $R^3$.

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Abstract. The Skyrme-Faddeev system, a modified O(3) sigma model in three space dimensions, admits topological solitons with nonzero Hopf number. One may learn something about these solitons by considering the system on the 3-sphere of radius $R$. In particular, the Hopf map is a solution which is unstable if $R > \sqrt{2}$.

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1. Introduction

The simplest (3+1)-dimensional system admitting localized topological solitons is the O(3) nonlinear sigma model, in which a configuration is a map $\varphi : \mathbb{R}^3 \to S^2$. Given a suitable boundary condition at spatial infinity, $\varphi$ extends to a map from $S^3$ to $S^2$; and such maps are classified by their Hopf number $Q \in \pi_3(S^2) \cong \mathbb{Z}$. This raises the possibility of string-like topological solitons, which can be linked or knotted; and interest in such solitons has recently been rekindled by numerical investigations [1–5].

To define the $\sigma$-model, one needs to choose a Lagrangian. Let us concentrate here on static fields only: so one has to choose an energy functional $E[\varphi]$ for maps $\varphi : \mathbb{R}^3 \to S^2$. The simplest choice, namely $E_2[\varphi] = \int (\partial \varphi)^2$, does not admit static solitons (as the usual Derrick-Hobart scaling argument readily shows). In order to have stable solitons, one needs to prevent them from shrinking. There are various ways of achieving this; the one currently of most interest is the Skyrme method, namely adding a functional $E_4[\varphi]$ which is fourth-order in derivatives of $\varphi$. In this paper, we deal with the resulting Skyrme-Faddeev model [6]; so the energy functional is of the form $E = E_2 + E_4$.

Over the years, there have been several attempts to understand soliton solutions in this system. The task is hampered by the fact that the obvious ansatzes are incompatible with the equations of motion [7]. Related to this is the fact that topologically nontrivial configurations can admit at most an axial (one-parameter) symmetry [8]. So the situation is unlike that of the Skyrme model, with target space $S^3$, where the $Q = 1$ soliton is a spherically-symmetric “hedgehog”. In the Skyrme-Faddeev system, the minimal-energy $Q = 1$ soliton is believed to be toroidal in shape. Advances in computer power have now made numerical investigations feasible, and the recent work [1–5] involves numerical searches for static solitons.

In the case of the Skyrme model, one gains considerable insight from studying the system on the 3-sphere of radius $R$, rather than on flat 3-space $\mathbb{R}^3$ [9–11]. If $R = 1$, then the single Skyrmion is simply the identity map from $S^3$ to $S^3$, and it saturates the topological lower bound on the energy. As $R$ is increased, one reaches a point (in fact at $R = \sqrt{2}$) when this identity map becomes unstable, and there is a spontaneous breakdown of its symmetry. For $R > \sqrt{2}$, and in the flat-space limit $R \to \infty$, the single Skyrmion becomes localized around a particular point in space.

The purpose of this paper is to show that analogous results hold for Hopf solitons.
in the Skyrme-Faddeev model. In particular, we shall see that the standard Hopf map from $S^3_R$ to $S^2$ is unstable if $R > \sqrt{2}$; and we shall discuss the possibility of improving the topological lower bound on the energy, so that it can be attained if $R = 1$.

2. The Energy and its Topological Lower Bound

The system involves a scalar field $\varphi$ which takes values on the unit 2-sphere $S^2$. We are interested here in static configurations only: so $\varphi$ is defined on a positive-definite 3-space $(M, g_{jk})$, which we shall take to be either $\mathbb{R}^3$ (with standard Euclidean metric) or $S^3_R$ (the standard 3-sphere of radius $R$). In other words, we have a nonlinear $O(3)$ $\sigma$-model on $M$.

In the usual manner, $\varphi$ can be written as a unit 3-vector field $\varphi_a = \varphi_a(x^j)$, with $\varphi_a \varphi^a = 1$. [The indices $a, b, \ldots$ and $j, k, \ldots$ run over 1,2,3; the $x^j$ are local coordinates on $M$; and the Einstein summation convention applies throughout.]

The energy of $\varphi$ is defined by the functional

$$E := \int_M (c_2 E_2 + c_4 E_4) \, dV.$$  

Here $dV$ is the volume element $dV := \sqrt{g} \, dx^1 \wedge dx^2 \wedge dx^3$, where $g = \det(g_{jk})$; $c_2$ and $c_4$ are coupling constants; and

$$E_2 := g^{jk}(\partial_j \varphi^a)(\partial_k \varphi^a),$$
$$E_4 := \frac{1}{4} g^{jl} g^{km} F_{jk} F_{lm},$$

where $F_{jk} := \varepsilon_{abc} \varphi^a(\partial_j \varphi^b)(\partial_k \varphi^c)$. This is the Skyrme-Faddeev system. Note that $c_2$ and $c_4$ are dimensional quantities, with $\sqrt{c_4/c_2}$ having units of length and $\sqrt{c_4c_2}$ having units of energy.

If $M = \mathbb{R}^3$, then we impose the boundary condition that $\varphi$ is smoothly defined on the one-point compactification of $\mathbb{R}^3$ (in particular, $\varphi^a$ tends to a constant unit vector at spatial infinity). So for the purposes of differential topology, we may regard $\varphi$ as a smooth map from $S^3$ to $S^2$. Such maps are classified by an integer $Q$, namely the Hopf number. There is no local formula for $Q$ in terms of the field $\varphi$; a nonlocal expression is as follows.

Find $A_j$ such that $F_{jk} = \partial_j A_k - \partial_k A_j$; this can be done because the 2-form $F_{jk} \, dx^j \wedge dx^k$ is closed, and the deRahm cohomology group $H^2(S^3)$ is trivial. Then

$$Q = \frac{1}{32\pi^2} \int_M \eta^{jkl} F_{jk} A_l \, dV,$$
where $\eta^{jkl} = g^{-1/2} \varepsilon^{jkl}$ ($\varepsilon^{jkl}$ being the totally-skew symbol with $\varepsilon^{123} = 1$). The precise sign of $Q$, which has to do with orientations, need not concern us; in what follows, we take $Q \geq 0$.

It has been known for some time that there is a topological lower bound on the energy of configurations with $Q \neq 0$, at least on $M = \mathbb{R}^3$ [12]. The argument consists of a number of parts. First, one has a Sobolev-type inequality

\[
\left( \frac{1}{32\pi^2} \int_M \eta^{jkl} F_{jk} A_l dV \right)^{3/2} \leq C \left( \int_M \mathcal{E}_4 dV \right) \left( \int_M \sqrt{\mathcal{E}_4} dV \right),
\]

where $C$ is a (universal) constant. In other words, the integral of the Chern-Simons form is bounded above by an expression involving integrals of $F^2$ and $\sqrt{F^2}$. A value for $C$ for which this inequality holds, for $M = \mathbb{R}^3$, is [12, 8]

\[
C = \frac{1}{8\sqrt{2} \pi^4 3^{3/4}}.
\]

The second ingredient is the algebraic inequality

\[
\mathcal{E}_4 \leq \frac{1}{8} \mathcal{E}_2^2.
\]

This holds on $S^3_R$ as well as on $\mathbb{R}^3$. One way to establish it is to consider the $3 \times 3$ matrix $D^{ab} = g^{jk}(\partial_j \varphi^a)(\partial_k \varphi^b)$ (cf. [10]). This matrix has a zero eigenvalue (corresponding to the eigenvector $\varphi^a$); if we call the other two eigenvalues $\lambda_1$ and $\lambda_2$, then $\mathcal{E}_2 = \lambda_1 + \lambda_2$ and $\mathcal{E}_4 = \frac{1}{2} \lambda_1 \lambda_2$. The inequality follows immediately.

The final ingredient is the simplest, namely

\[
E \geq 2 \left( \int_M c_2 \mathcal{E}_2 dV \right)^{1/2} \left( \int_M c_4 \mathcal{E}_4 dV \right)^{1/2}.
\]

Putting the three inequalities together gives

\[
E \geq K Q^{3/4},
\]

where $K = 2^{7/4} C^{-1/2} (c_2 c_4)^{1/2}$.

It seems likely that the value of $C$ given by (2) can be improved (i.e. reduced). The conjecture here, which is motivated by the Hopf map (see section 3), is that (1) is true for

\[
C = \frac{1}{64\sqrt{2} \pi^4}.
\]
It would follow that the value of $K$ in (3) is $K_0 = 32 \pi^2 (c_2 c_4)^{1/2}$; and this “reference” value is used in the discussion that follows. Note that the value of $K$ obtained from (2) is about half of $K_0$.

The recent numerical investigations have provided some data concerning stable solutions (minimum-energy configurations) in the $Q = 1$, $Q = 2$ and $Q = 3$ sectors on $M = \mathbb{R}^3$. One of these [2] assumed axial symmetry, and obtained energies

$$E_{Q=1} = 1.25 K_0,$$
$$E_{Q=2} = 1.19 K_0 2^{3/4}.$$

The other [3] was a fully three-dimensional simulation, and this yielded

$$E_{Q=1} = 1.13 K_0,$$
$$E_{Q=2} = 1.11 K_0 2^{3/4},$$
$$E_{Q=3} = 1.14 K_0 3^{3/4},$$
$$E_{Q=4} = 1.18 K_0 4^{3/4},$$
$$:$$
$$E_{Q=8} = 1.15 K_0 8^{3/4}.$$

The latter figures probably underestimate the true energies, owing to the effect of working in a finite-volume subset of $\mathbb{R}^3$. So the current evidence suggests that solitons have an energy around 20% above our conjectured lower bound.


This section deals with the standard Hopf map from $S^3_R$ to $S^2$, which has topological charge $Q = 1$. In particular, its energy is $E = 16\pi^2 (R + R^{-1})$, which is bounded below by $32\pi^2$ (this bound being attained when $R = 1$). For $R > \sqrt{2}$, by contrast, we shall see that the Hopf map is unstable. (From now on, $c_2$ and $c_4$ are set equal to 1; this sets the length and energy scales.) These results are analogous to those for Skyrmions [10].

The Hopf map $\Phi$ may be described as follows. A point of $S^3_R$ corresponds to a pair $(Z^0, Z^1)$ of complex numbers satisfying $Z^A \bar{Z}_A = 1$, where $\bar{Z}_0 = \overline{Z^0}$ and $\bar{Z}_1 = \overline{Z^1}$. The Hopf map sends this point to the point on $CP^1 = S^2$ which has homogeneous coordinates $[Z^0, Z^1]$. The energy density of this map $\Phi$ is constant on $S^3_R$; in fact it has $\mathcal{E}_2 = 8/R^2$ and $\mathcal{E}_4 = 8/R^4$, so that $E = 16\pi^2 (R + R^{-1})$ as stated above.
The gauge potential $A^j$ turns out to be a right-invariant vector field on $S^3$; in other words, it corresponds to (a generator of) $SU(2)$ acting on $SU(2) \cong S^3$ by left multiplication. The inequality (1) is a statement about vector fields on $S^3$ (note that each side is independent of the radius $R$). For right-invariant vector fields, one has equality in (1) with $C$ given by (4); and this suggests that (1) holds with that that value of $C$. But proving this would seem to require some delicate global analysis, which will not be attempted here.

Each of $\int \mathcal{E}_2 \, dV$ and $\int \mathcal{E}_4 \, dV$ is stationary with respect to variations of the field: in other words, $\Phi$ is a solution of the Euler-Lagrange equations for all values of $R$. Clearly one expects this solution to be unstable for large $R$, and we shall now see this in more detail.

The energy of a perturbation $\Phi + \delta \Phi$ of the Hopf map has the form

$$E[\Phi + \delta \Phi] = E[\Phi] + \int_M G[\delta \Phi] \, dV,$$

where $G[\delta \Phi]$ is a quadratic function of $\delta \Phi$ and $\partial_j (\delta \Phi)$. We are interested in the eigenvalues (particularly the negative eigenvalues) of the quadratic form $\delta E = \int G[\delta \Phi] \, dV$.

Let us think of $Z^A$ as transforming under the fundamental representation of $U(2)$. The Hopf map is invariant under this $U(2)$, in the sense that $\Phi(Z^A)$ and $\Phi(U^A \bar{Z}^B)$ are related by an $SO(3)$ transformation on the target space $S^2$. Consequently, $U(2)$ also acts on the tangent space at $\Phi$, i.e. on the space of perturbations about $\Phi$. So we can decompose this perturbation space into irreducible representations of $U(2)$.

Hence we consider perturbations as follows: the perturbed field maps $Z^A$ to the point on $\mathbb{CP}^1$ with homogeneous coordinates $Z^A + \Theta^A$, where

$$\Theta^A = T^A_{\bar{F} \cdots R} \bar{Z}^B \cdots \bar{Z}^D \bar{Z}^P \cdots \bar{Z}^R$$

($T^A_{\cdots}$ being a constant infinitesimal tensor). In terms of the unit 3-vector field $\varphi^a(Z^B, \bar{Z}^B)$, the Hopf map is given by

$$\varphi^1 + i \varphi^2 = 2Z^0 \bar{Z}_1,$$

$$\varphi^3 = Z^1 \bar{Z}_1 - Z^0 \bar{Z}_0;$$

and the perturbation described above is $\delta \varphi^a(Z^B, \bar{Z}^B)$, where

$$\delta \varphi^1 + i \delta \varphi^2 = 2(\bar{Z}_1)^2 \Omega - 2(Z^0)^2 \bar{\Omega},$$

$$\delta \varphi^3 = -2\bar{Z}_0 \bar{Z}_1 \Omega - 2Z^0 Z^1 \bar{\Omega},$$
with $\Omega = Z^1\Theta^0 - Z^0\Theta^1$. For the first few of these modes, the change $\delta E$ in energy is as follows.

(i) If $\Theta^A = T^A$ constant, then $\delta E = 4\pi^2(T^A\tilde{T}_A)(2/R - R)$; in other words, these modes are positive if $R < \sqrt{2}$ and negative if $R > \sqrt{2}$. So the Hopf map is indeed unstable for $R > \sqrt{2}$.

(ii) If $\Theta^A = T^{AB}\tilde{Z}_B$ with $T^{AB}$ skew, then $\delta E = 0$: this is a zero-mode.

(iii) If $\Theta^A = T^A_Z\tilde{Z}^D$ with $T^A_Z = 0$ (the trace part does not contribute to $\delta \varphi^a$) and $\tilde{T}^A_B = T^A_B$, then $\delta E = 128\pi^2(T^B_A\tilde{T}_B^A)/(3R)$: these modes are positive for all $R$.

In conclusion, we can identify a perturbation of the Hopf map which becomes a negative mode for $R > \sqrt{2}$. It seems likely that there are no negative modes for $R < \sqrt{2}$, i.e. the Hopf map is then stable; but this has not been proved.

4. Approximate Solutions on $S^3_R$ and $\mathbb{R}^3$.

To begin with, let us investigate a one-parameter family of $Q = 1$ configurations, which contains the Hopf map $\Phi : S^3_R \to S^2$, and also contains one of the negative modes identified above. This family of fields $\Phi_\lambda$, with $\lambda$ a positive parameter, may be described geometrically as follows.

Let $P$ denote a stereographic projection $P : S^3_R \to \mathbb{R}^3$, and let $D_\lambda$ denote a dilation on $\mathbb{R}^3$ with (constant) scale-factor $\lambda > 0$. Then define $\Phi_\lambda$ by

$$\Phi_\lambda := \Phi \circ P^{-1} \circ D_\lambda \circ P.$$ 

The energy of this field turns out to be

$$E_\lambda := E[\Phi_\lambda] = \frac{64\pi^2\lambda R}{(\lambda + 1)^2} + \frac{8\pi^2(\lambda^2 + 1)}{\lambda R}.$$ 

Note that $E_1 = 16\pi^2(R + R^{-1})$, as must be the case (since $\Phi_1 = \Phi$).

Now let us find the value of $\lambda$ for which $E_\lambda$ is a minimum. A straightforward calculation shows that if $R < \sqrt{2}$, then the minimum occurs at $\lambda = 1$ (i.e. for the Hopf map); whereas if $R > \sqrt{2}$, then the minimum occurs when $\lambda$ is either of the roots of $\lambda^2 + 2(1 - \sqrt{2}R)\lambda + 1 = 0$. In the latter case, the minimum value is $E_\lambda(\text{min}) = 32\sqrt{2}\pi^2 - 16\pi^2/R$. The soliton, instead of being spread out over the whole of $S^3$, is then localized around a particular point (the base-point of the stereographic projection).
In the limit as \( R \to \infty \), one gets a \( Q = 1 \) configuration on \( \mathbb{R}^3 \) which simply consists of an inverse stereographic projection \( \mathbb{R}^3 \to S^3 \) followed by the Hopf map (this kind of approximation has long been considered). Its energy is \( 32\sqrt{2}\pi^2 \), which is about 13% higher than that of the numerical solution.

As an example of higher-charge configurations, consider the field \( \Psi_{m,n} \) which maps \((Z^0, Z^1)\) to the point with homogeneous coordinates \([ (Z^0)^m/|Z^0|^{m-1}, (Z^1)^n/|Z^1|^{n-1} ] \). So \( m = n = 1 \) gives the Hopf map. For \( m > 1 \) or \( n > 1 \), the field \( \Psi_{m,n} \) is continuous but not smooth; however, its partial derivatives are continuous and bounded on the complement of a set of measure zero in \( S^3 \), and so in particular its energy is well-defined. In fact, the charge and energy turn out to be

\[
Q = mn, \\
E[\Psi_{m,n}] = 8\pi^2 R + 4\pi^2 (m^2 + n^2)(R + 2/R). \tag{5}
\]

If we minimize (5) with respect to the radius \( R \), then we get

\[
E_{\min} = 8\pi^2 \sqrt{2(m^2 + n^2)(2 + m^2 + n^2)}. \tag{6}
\]

(The idea here is that higher-charge configurations prefer slightly more living-space, and (6) corresponds to the sphere-size in which they are most comfortable.) The first few cases are as follows.

(i) If \( m = 2 \) and \( n = 1 \), then \( Q = 2 \) and \( E_{\min} = 32\pi^2 2^{3/4} \times 1.24 \), i.e. 24% greater than our reference value.

(ii) If \( m = 3 \) and \( n = 1 \), then \( Q = 3 \) and \( E_{\min} = 32\pi^2 3^{3/4} \times 1.70 \), which is rather high.

(iii) If \( m = n = 2 \), then \( Q = 4 \) and \( E_{\min} = 32\pi^2 4^{3/4} \times 1.12 \), quite close to the reference value.

So the configuration \( \Psi_{m,n} \) is a reasonable one. But there are several other simple expressions for higher-charge fields, for example related to the “rational-map” ansatz for Skyrmions (cf. [3]), and these deserve investigation as well.

5. Concluding Remarks.

Most recent work on Hopf solitons in the Skyrme-Faddeev system has involved intensive numerical simulations. The results reported in this paper are aimed at complementing those studies. One particular theme is the way in which simple explicit solutions on \( S^3_R \)
become unstable as $R$ increases, and collapse into localized structures. Clearly there is much scope for further investigation of such configurations.

Although the Hopf map is indeed a solution of the field equations, it is not, strictly speaking, known to be stable for $R < \sqrt{2}$. The arguments for stability of the identity map in the Skyrme model [9, 10, 13] appear not to adapt to this case, because of the nonlocality of the topological charge density (and the absence of a Bogomolny-type lower bound on the energy). It would be useful if the inequality (1), (4) could be established; in addition to providing a good lower bound on the energy, this would prove that the Hopf map is stable for $R = 1$.

References.

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