Partial breaking of global $D = 4$ supersymmetry, constrained superfields, and 3-brane actions

M. Roček*  
*Institute of Theoretical Physics, State University of New York  
Stony Brook, NY 11794-3840, USA  
and  
A.A. Tseytlin†  
†Theoretical Physics Group, Blackett Laboratory  
Imperial College, London SW7 2BZ, U.K.

Abstract  
We show that the connection between partial breaking of supersymmetry and nonlinear actions is not accidental and has to do with constraints that lead directly to nonlinear actions of the Born-Infeld type. We develop a constrained superfield approach that gives a universal way of deriving and using these actions. In particular, we find the manifestly supersymmetric form of the action of the 3-brane in 6-dimensional space in terms of $N = 1$ superfields by using the tensor multiplet as a tool. We explain the relation between the Born-Infeld action and the model of partial $N = 2$ supersymmetry breaking by a dual D-term. We represent the Born-Infeld action in a novel form quadratic in the gauge field strengths by introducing two auxiliary complex scalar fields; this makes duality covariance and the connection with the $N = 1$ supersymmetric extension of the action very transparent. We also suggest a general procedure for deriving manifestly duality symmetric actions, explaining in a systematic way relations between previously discussed Lorentz-covariant and noncovariant actions.

* e-mail address: rocek@insti.physics.sunysb.edu  
* e-mail address: tseytlin@ic.ac.uk  
† Also at Lebedev Physics Institute, Moscow.
April 1997
1. Introduction

Partial breaking of global $N = 2$ supersymmetry has been discussed from various points of view [1,2,3,4,5,6,7,8,9] (see also reviews in [10,11,12]). Several ways to construct models with partial supersymmetry breaking in various dimensions are known. One is a ‘solitonic’ realisation of partial supersymmetry breaking that uses a BPS soliton background in a higher-dimensional supersymmetric theory. Such a solution breaks some of translational invariances and thus some of supersymmetries. As a result, a lower-dimensional action for the corresponding collective coordinates has part of the supersymmetry realised linearly and part nonlinearly [2,3].

Another approach is to start with a generalised (nonrenormalisable) $N = 2$, $D = 4$ vector multiplet action and to add a dual (magnetic) D-term [5]. Here one works directly in four dimensions and the vacuum is translationally invariant but because of the nonlinear structure of the action, the ‘magnetic’ D-term can spontaneously break $N = 2$ supersymmetry to $N = 1$. This model can be also derived as a special limit of an $N = 2$ supergravity model [6].

One of the aims of the present paper is to achieve a better understanding of how to construct nonlinear actions of theories with partially broken supersymmetry in a closed, manifestly supersymmetric way. In [13], one of us showed how to describe the goldstino of spontaneously broken $N = 1$ supersymmetry in terms of a constrained chiral superfield. In this paper, we use the same approach to construct the $N = 1$ superfield goldstone multiplets of partially broken $N = 2$ supersymmetry. One example discussed previously in the literature is the $D = 4$, $N = 1$ supersymmetric Born-Infeld (BI) action [14,15] which can be interpreted [7] as the unique action corresponding to the situation when $N = 2$ supersymmetry is broken down to $N = 1$ in such a way that the vector multiplet remains massless (i.e., is the Goldstone multiplet). Our method gives a systematic derivation of the BI action of [7], as well as a supersymmetric membrane action in terms of a tensor multiplet described by a real linear superfield.

---

1 Starting with a higher-dimensional vector gauge theory leads only to scalar multiplet actions for the collective coordinates. To get a massless vector ($D \geq 4$) or tensor ($D \geq 6$) multiplet one must consider solitons in a theory containing higher tensors such as a tensor multiplet theory in $D = 6$ or supergravity.

2 While we were in the process of writing up our results, this action appeared in [16], but without a systematic derivation.
We also explain how the $N = 1$ super Born-Infeld action emerges from the model of [5] when one decouples (‘integrates out’) the massive chiral multiplet, and address the case of $N = 2 \rightarrow 1$ supersymmetry breaking with the scalar multiplet remaining massless. This case is related to the model of [3] and was previously discussed in [4] where, however, the full nonlinear form of the corresponding $N = 1$ chiral multiplet action was not determined. This action should be the manifestly $N = 1, D = 4$ supersymmetric form of the Nambu-type action for the 3-brane in six dimensions of ref. [3] in the static gauge. We propose a class of chiral multiplet actions. We discuss the superfield analog of the $D = 4$ scalar-tensor duality in the case of the nonlinear $D = 4$ Nambu action, explaining that one of the chiral multiplet actions should also have a second hidden supersymmetry.

Part of the motivation behind this work is to try to determine the complete $N = 1, D = 4$ superfield form of the static-gauge action for the D3-brane soliton of the type IIB supergravity (string theory) [17,18,19]. The bosonic part of this action is a ‘hybrid’ of the $D = 4$ BI and Nambu actions [20] (closely related – by T-duality – to the $D = 1$ 0B I action [21])

$$S = - \int d^4 x \sqrt{-\det(\eta_{ab} + \partial_a X^n \partial_b X^n + F_{ab})} , \quad n = 1, \ldots, 6 . \quad (1.1)$$

Here $a, b = 0, 1, 2, 3$ are the world-volume indices and $X^n$ are collective coordinates corresponding to the ‘transverse’ motion of a 3-brane in $D = 10$ space (we set the string tension to 1, $T^{-1} = 2\pi \alpha' = 1$, and ignore the overall factor of the D3-brane tension). The component form of the supersymmetric extension of this action with 4 linearly realised and 4 nonlinearly realised global supersymmetries was found in [22,23,24]. The leading term in this action is the $N = 4, D = 4$ super-Maxwell action. It would be interesting for several reasons (the quantum properties of D3-branes and their comparison with supergravity [25], possible hints about a non-abelian generalisation, etc.) to have a manifestly supersymmetric formulation of this action in terms of one vector and three chiral $N = 1, D = 4$ superfields. That would be a Born-Infeld generalisation of the corresponding unconstrained superfield form of the $N = 4, D = 4$ super-Maxwell action [26].

Since the action (1.1) contains both the vector and the scalars the knowledge of the $N = 1$ superfield form of the $D = 4$ BI action for a single vector field [14,15,7] is by far not sufficient to determine its superfield analogue. As a step towards understanding how to combine the vector and scalar dependencies one may try first to determine the superfield form of the action for the 3-brane of ref. [3], which does not contain a vector field. The model considered in [3] was the $N = 1, D = 6$ supersymmetric theory of
Maxwell multiplet coupled to two (charged) scalar multiplets. The 3-brane solution in this theory (the direct BPS analogue of the Abrikosov-Nielsen-Olesen string in $D = 4$) breaks translational invariance in 2 of 5 spatial directions and thus breaks half of $N = 1, D = 6$ or, equivalently, $N = 2, D = 4$, supersymmetry. The resulting static-gauge action for the 3-brane collective coordinates has the following bosonic part

$$S' = - \int d^4 x \sqrt{-\det(\eta_{ab} + \partial_a X^n \partial_b X^n)} , \quad n = 1, 2 .$$  \hspace{1cm} (1.2)$$

We thus seek a superspace extension of (1.2) with manifest $N = 1, D = 4$ supersymmetry as well as the broken ‘half’ of $N = 2$ supersymmetry realised nonlinearly.

Below we shall determine the supersymmetric form of (1.2) using two different approaches. One is a direct $N = 1$ chiral multiplet construction that has the right bosonic part matching (1.2). To demonstrate that this action has also hidden $N = 2$ supersymmetry we shall use the $D = 4$ duality transformation that ‘rotates’ one of the two scalars in (1.2) into an antisymmetric 2-tensor. It turns out to be simpler to construct the superfield action with partially broken $N = 2$ supersymmetry for a tensor multiplet rather than for a chiral multiplet.\(^3\)

In section 2 we discuss constrained superfields and how they may be used to express linearly transforming fields in terms of Goldstone fields. In particular, we apply this to partial supersymmetry breaking in the vector and tensor multiplets, and find the relation to the other examples of partial supersymmetry breaking described above. In section 3 we apply the lessons learned from the supersymmetric case to rewrite the Born-Infeld action in various forms that make duality transparent. We also consider scalar-tensor duality in the Born-Infeld-Nambu actions. In section 4 we give our conclusions and mention some open problems. In the Appendix we discuss in detail a manifestly duality invariant formalism for general systems, explaining relations between previously considered Lorentz-covariant and noncovariant actions.

\(^3\) This was previously considered in [4] using a different approach. The first two terms in the expansion of the supersymmetric analog of (1.2) are related to the action found in [4] by a field redefinition which eliminates terms in the action of [4] which are not invariant under $X^n \rightarrow X^n + a^n$, $a^n = \text{const}$. Our tensor multiplet results were also found in [16], where a dual chiral superfield action that agrees with our general form is also proposed; see also [27].
2. Constrained superfields and partial supersymmetry breaking

Spontaneously broken symmetries give nonlinear realizations of the broken symmetry group. A traditional way to find such realizations is to begin with a linear representation and impose a nonlinear constraint (e.g., spontaneously broken rotational symmetry may be non-linearly realized on a vector constrained to lie on the surface of a sphere). In [13], it was shown that an $N = 1$ chiral superfield obeying the nonlinear constraints

\[ (i) \quad \phi = \bar{D}^2(\phi \bar{\phi}) \; , \quad \phi^2 = 0 \; ; \quad (ii) \quad \langle D^2 \phi \rangle = 1 \]  

(2.1)

can be expressed in terms of a single fermionic field: the goldstino for the broken $N = 1$ supersymmetry. The two constraints can be understood as follows: the constraint on $\langle D^2 \phi \rangle$ implies that supersymmetry is spontaneously broken,\(^5\) whereas the nonlinear constraint removes the ‘radial’ degrees of freedom to leave only the goldstone field. We shall follow the same approach to find $N = 1$ superfield descriptions of the goldstone multiplets that arise when $N = 2$ supersymmetry is partially broken to $N = 1$. We consider different $N = 2$ superfields and find different $N = 1$ goldstone multiplets. The two examples that we have worked out below are the two simplest irreducible $N = 2$ multiplets: the vector and tensor multiplets.

2.1. The Vector Multiplet

The $N = 2$ vector multiplet is described by a constrained chiral field strength $W(x, \theta_1, \theta_2)$ that obeys the Bianchi identity [29] $(a, b = 1, 2)$

\[ D_{ab}^2 W = C_{ac} C_{bd} \bar{D}^{2cd} \bar{W} \, . \]  

(2.2)

It is convenient to define $D \equiv D_1$, $Q \equiv D_2$ and to rewrite (2.2) as:

\[ D^2 W = \bar{Q}^2 \bar{W} \; , \quad DQW = -\bar{DQ} \bar{W} \, . \]  

(2.3)

We break $N = 2$ supersymmetry to $N = 1$ by assuming that $W$ has a Lorentz-invariant condensate ($\langle W \rangle$):

\[ \langle W \rangle = -\theta_2^2 \; , \quad \langle Q^2 W \rangle = 1 \; , \quad D\langle W \rangle = 0 \, . \]  

(2.4)

---

\(^4\) We mostly use the conventions of [28]; in particular, $D^2 = \frac{1}{2} D^a D_a$, and the $2 \times 2$ charge conjugation matrix $C_{\alpha \beta}$ is $i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. However, the matrix $C_{ab}$ contracting the indices $a, b = 1, 2$ which label the two supersymmetries is defined without a factor of $i$.

\(^5\) We shall use $\langle \ldots \rangle$ to indicate that a (super)field has a classical vacuum expectation value.
Here we set the scale of the supersymmetry breaking to 1. We reduce the field content to a single \( N = 1 \) superfield by imposing

\[
W^2 = 0, \tag{2.5}
\]

where \( W \equiv W + \langle W \rangle \). Then \( Q^2(W + \langle W \rangle) = \bar{D}^2(W + \langle W \rangle) \) implies

\[
Q^2W = \bar{D}^2W - 1, \tag{2.6}
\]

and the constraint \( W^2 = 0 \) implies

\[
0 = \frac{1}{2}Q^2W = W(\bar{D}^2W - 1) + \frac{1}{2}Q\alpha WQ\alpha W. \tag{2.7}
\]

Projecting to \( N = 1 \) superspace by setting \( \theta_2 = 0 \) and defining the \( N = 1 \) superfields

\[
\phi \equiv W|_{\theta_2=0}, \quad W_\alpha \equiv -Q_\alpha W|_{\theta_2=0}, \tag{2.8}
\]

we find

\[
\phi = \phi \bar{D}^2\bar{\phi} + \frac{1}{2}W^{\alpha}W_\alpha. \tag{2.9}
\]

This is precisely the constraint of [7]. However, now we have an interpretation of the chiral object that they mysteriously introduced: it is the chiral superpartner of the vector multiplet in the \( N = 1 \) superspace description of the \( N = 2 \) vector multiplet. Note the close analogy between (2.1) and (2.4), (2.5), (2.9). Note also that (2.7) and (2.4) actually imply (2.5). The \( N = 2 \) supersymmetry transformation laws in superspace follow from the constraints and definitions of the \( N = 1 \) superfield components, and are given by

\[
\delta_2\phi \equiv (\eta^\alpha Q_\alpha + \bar{\eta}^\dot{\alpha} Q_\alpha)W|_{\theta_2=0} = -\eta^\alpha W_\alpha , \quad \delta_2W_\alpha = \eta_\alpha (\bar{D}^2\phi - 1) - i\bar{\eta}^\dot{\alpha} \partial_{\alpha\dot{\alpha}}\phi, \tag{2.10}
\]

which, up to conventions, agrees with [7].

2.2. Actions for the Vector Multiplet

Because of the constraints (2.4), (2.5), (2.6), there are many equivalent forms that all give the same action. In particular, a broad class of actions is proportional to the \( N = 2 \) Fayet-Iliopoulos term:

\[
L = \int d^2\theta_1 d^2\theta_2 \mathcal{F}(W)
\]
= \int d^2\theta_1 d^2\theta_2 \left[ \mathcal{F}(0) + \mathcal{F}'(0)(\langle W \rangle + W) + \frac{1}{2}\mathcal{F}''(0)(\langle W \rangle + W)^2 \right]
= \mathcal{F}''(0) \int d^2\theta_1 \phi. \tag{2.11}

We can also write first-order actions where we impose the constraints by a chiral restricted
\( N = 2 \) Lagrange multiplier \( \Lambda \)

\[
D^2 \Lambda = \bar{Q}^2 \Lambda, \quad DQ\Lambda = -\bar{D}Q\Lambda, \tag{2.12}
\]

namely,

\[
L_1 = \left( \int d^2\theta_1 d^2\theta_2 \frac{i}{2} \Lambda W^2 + \int d^2\theta_1 W \right) + h.c. , \tag{2.13}
\]

where we have scaled \( \mathcal{F}''(0) \to 1 \). In \( N = 1 \) superspace, this reduces to \( (\Lambda \to \Lambda, \chi_\alpha) \)

\[
L_1 = i \int d^2\theta d^2\bar{\theta} \left[ \frac{1}{2}(\bar{\Lambda}\phi^2 - \Lambda\bar{\phi}^2) + (\Lambda - \bar{\Lambda})\phi\bar{\phi} \right]
+ \left( \int d^2\theta \left[ i(-\Lambda\phi + \chi_\alpha W_\alpha \phi + \frac{1}{2}\Lambda W^\alpha W_\alpha) + \phi \right] + h.c. \right). \tag{2.14}
\]

Integrating over the \( N = 1 \) Lagrange multiplier superfields \( \Lambda \) and \( \chi_\alpha \) we get the constraint (2.7) (after using (2.5), which follows from the boundary condition (2.4)).

Another action that gives the same constraints and final \( N = 1 \) Born-Infeld action
is the standard free \( N = 2 \) vector action, \textit{i.e.}, the free \( N = 1 \) action for the vector \( (V) \) and chiral \( (\phi) \) superfields, plus a constraint term with a chiral \( N = 1 \) superfield Lagrange multiplier \( \Lambda \):

\[
S = \int d^4x \left( \int d^2\theta \left( \frac{1}{2}W^\alpha W_\alpha + \phi\bar{D}^2\bar{\phi} \right) + i\Lambda(\frac{1}{2}W^\alpha W_\alpha + \phi\bar{D}^2\bar{\phi} - \phi) \right) + h.c. \right), \tag{2.15}
\]

or, equivalently, after shifting \( \Lambda \to \Lambda + i \),

\[
S = \int d^4x \left[ \int d^2\theta \left( i\Lambda(\frac{1}{2}W^\alpha W_\alpha + \phi\bar{D}^2\bar{\phi} - \phi) + \phi \right) + h.c. \right]. \tag{2.16}
\]

The explicit solution of the constraint (2.9) is [7]

\[
\phi(W,\bar{W}) = \frac{1}{2}W^\alpha W_\alpha + \frac{1}{2}D^2 \left[ \frac{W^\alpha W_\alpha \bar{W}^\alpha \bar{W}_\alpha}{1 - \frac{1}{2}A + \sqrt{1 - A + \frac{1}{4}(D^2(W^\alpha W_\alpha) - \bar{D}^2(\bar{W}^\alpha \bar{W}_\alpha))^2}} \right], \tag{2.17}
\]
where
\[ A \equiv D^2(W^\alpha W_\alpha) + \bar{D}^2(\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}) . \]

Note that adding a D-term for \( V \) is not allowed here – this will break \( N = 1 \) supersymmetry as there are non-minimal terms in the action (the dependence of the bosonic part of the action on the auxiliary field D can be found by shifting \( F^2 \to F^2 - 2D^2, \ F \bar{F} \to F \bar{F} \) in the bosonic BI action, see below).

The resulting action (2.11) is thus simply
\[ S = \int d^4x \left[ \int d^2\theta \phi(W, \bar{W}) + h.c. \right] , \tag{2.18} \]
i.e., \( \phi(W, \bar{W}) \) is nothing but the \( N = 1 \) supersymmetric BI action of [15]. We thus arrive at the same conclusion as [7]: \textit{the requirement of partially broken \( N = 2 \) supersymmetry uniquely fixes the action for the vector multiplet to be the supersymmetric Born-Infeld action.}

The Lagrange multiplier form of the supersymmetric BI action (2.16) dramatically simplifies the proof [7] of the \textit{duality covariance} of the \( N = 1 \) BI action (2.18),(2.17) (this is also true in the bosonic BI action case, see below). We relax the reality constraint on the chiral superfield \( W_\alpha \) and add the term with the dual field strength \( \bar{W}_\alpha \) as the Lagrange multiplier:
\[ \tilde{S} = \int d^4x \left( \int d^2\theta \left[ i\Lambda(\frac{1}{2}W^\alpha W_\alpha + \phi \bar{D}^2 \phi - \phi) + \phi - i\bar{W}^\alpha W_\alpha \right] + h.c. \right) . \tag{2.19} \]

Integrating out \( W_\alpha \) gives back the same supersymmetric BI action (2.16) with
\[ W \to \bar{W} , \quad \Lambda \to -\Lambda^{-1} , \quad \phi \to -i\Lambda \phi , \tag{2.20} \]
This is the direct superfield analogue of the transformation found in the bosonic case (see below). In \( N = 2 \) notation, this is precisely the duality transformation of [30] with \( \Lambda \) playing the role of \( \tau = \mathcal{F}'' \).

\textbf{2.3. Born-Infeld Action From the ‘Dual D-term’ Model}

In [5], partial supersymmetry breaking was induced by ‘magnetic’ Fayet-Iliopoulos (FI) terms. Consider the action
\[ S = -Im \int d^4x \left( \int d^2\theta \left[ \frac{1}{2} \mathcal{F}''(\phi) W^\alpha W_\alpha + m \mathcal{F}'(\phi) + (e - i\xi) \phi \right] + \int d^2\theta d^2\bar{\theta} \mathcal{F}'(\phi) \phi \right) . \tag{2.21} \]
We find it convenient to introduce the $N = 2$ FI terms in the $N = 1$ superpotential, and not as D-terms. As in [5], we choose the magnetic FI coefficient $m$ to be real; however, since we have rotated their D-term into the superpotential, we have a complex electric FI coefficient $(e - i\xi)$.

Expanding the field $\phi$ around its background value $\phi_{0}$ as $\phi = \phi_{0} + \varphi$, we find

$$S = -Im \int d^{4}x \int d^{2}\theta \left[ \tau(\phi_{0})(\frac{1}{2}W^{\alpha}W_{\alpha} + \varphi \tilde{D}^{2} \tilde{\varphi} + m\varphi) + (e - i\xi)\varphi + ... \right], \quad (2.22)$$

where $\tau(\phi) = F''(\phi)$ and the neglected terms are higher-order in the fluctuations (or independent of them). As in [5], the coupling is determined in terms of the FI terms by the condition that terms linear in $\phi$ cancel: $\tau(\phi_{0}) = -(e - i\xi)/m$.

Sending $\tau(\phi_{0})$ to infinity in a general complex direction sends the mass of the chiral multiplet to infinity and decouples it. At the same time, this produces the constraint we had before (see (2.16)). This both gives the rationale behind and makes precise the relation between the model of [5] and the supersymmetric BI action (expected on general grounds in [7]).

More precisely, one is expanding near the classical solution for which $\tau = \text{const}$ and is to integrate out the massive chiral multiplet $\varphi$. The leading (derivative-independent) part of the resulting low-energy effective action for the vector multiplet is then the $N = 1$ supersymmetric BI action. One may drop the derivative terms for $\varphi$ (as they would give derivatives of the vector field strength which may be ignored at low energies) and then solve for the scalar multiplet classically (i.e., integrate it out including only tree diagrams). This is similar to how the BI action is derived in string theory by integrating out massive string modes (see section 3.1 below). Sending $\tau(\phi_{0}) \to \infty$ corresponds to decoupling the propagating massive degrees of freedom of the chiral multiplet and thus effectively represents the procedure of integrating out the massive states.\(^6\)

We see the complete universality of the resulting BI action – nothing depends on the choice of $F$ since $\tau$ appears linearly in the action and plays the role of the Lagrange

---

\(^6\) The above short-cut $\tau \to \infty$ argument should apply after a field redefinition which effectively decouples the fluctuation field allowing one to ignore correction terms in (2.22). Simply dropping all massive mode contributions by sending their mass to infinity would give the Maxwell action instead of the BI action: one must first make a field redefinition that effectively accounts for the relevant coupling between the vector and scalar fluctuations, and then ignore irrelevant scalar multiplet couplings by sending its mass to infinity.
multiplier. But the effective action does depend on the parameters $m, e, \xi$ of the ‘microscopic’ theory: they appear in the overall (coupling constant) coefficient, $\theta$-term and the fundamental scale of the BI action:

$$S_{eff} = \int d^4x \left( \frac{m^2}{g^2} [1 - \sqrt{-\det(\eta_{ab} + m^{-1}F_{ab})}] + \frac{1}{4} \theta F^{ab}F_{ab}^* \right), \quad (2.23)$$

$$g^{-2} \equiv \frac{\xi}{m}, \quad \theta \equiv \frac{e}{m}. \quad (2.23)$$

The model (2.21) does not have a direct generalisation to the presence of other matter hypermultiplets minimally coupled to the vector multiplet [12]. This is not a problem in principle – the action we are interested in, such as (1.1), should contain non-minimal couplings. Indeed, there should exist an $N = 4$ supersymmetric 3-brane action with 6 scalars coupled to a $U(1)$ vector with non-linearly realised supersymmetry. Given that the pure BI action (with no scalars) follows from the model of [5] in the large mass limit, one should expect that such an action (where scalars are coupled to a vector in a non-minimal way and are actually neutral with respect to $U(1)$) should also follow from some generalisation of the ‘dual D-term’ model.

2.4. The Tensor Multiplet

We now consider the $N = 2$ tensor multiplet and find a Goldstone multiplet expressed in terms of an $N = 1$ real linear superfield. The procedure we follow is entirely parallel to the vector case: we partially break supersymmetry by choosing a particular background, and then eliminate the ‘radial’ fields by imposing a nilpotency constraint on the fluctuations. The $N = 2$ tensor multiplet is described by a pure imaginary isotriplet of scalar fields $L_{ab} \ (a, b = 1, 2)$ satisfying [31]

$$D_{\alpha(a} L_{bc)} = 0, \quad L_{ab} = -C_{ac}C_{bd} \bar{L}^{cd}, \quad (2.24)$$

or, equivalently,

$$D_{\alpha} L_{11} = 0, \quad Q_{\alpha} L_{11} = -2D_{\alpha} L_{12}, \quad 2Q_{\alpha} L_{12} = -D_{\alpha} L_{22}, \quad Q_{\alpha} L_{22} = 0. \quad (2.25)$$

In just the same way as for the vector multiplet, we impose the constraints:

$$Q^2 \langle L_{11} \rangle = -QD \langle L_{12} \rangle = -1, \quad \langle L_{11} - \langle L_{11} \rangle \rangle^2 = 0. \quad (2.26)$$
\[ \langle L_{11} \rangle = \frac{1}{2} \theta_1^2 \text{ and } \langle L_{12} \rangle = -\frac{1}{2} \theta_1 \theta_2. \]

We define the \( N = 1 \) component fields \( \phi, G \) as follows:

\[ \phi = L_{22}|_{\theta_2 = 0}, \quad G = -2L_{12}|_{\theta_2 = 0}, \quad \bar{\phi} = -L_{11}|_{\theta_2 = 0}. \quad (2.27) \]

As for the vector multiplet, we apply \( \bar{Q}^2 \) to the nilpotency constraint to find:

\[ 0 = \bar{Q}^2(\tfrac{1}{2} \phi^2) = \phi \bar{Q}^2 \phi + \tfrac{1}{2} \bar{Q}A \phi \bar{Q} \phi = \phi - \phi D^2 \bar{\phi} + \tfrac{1}{2} \bar{D} \bar{\phi} G \bar{D} \bar{\phi} G. \quad (2.28) \]

This constraint can be solved straightforwardly to express \( \phi \) in terms of the linear superfield \( G \) that describes the tensor multiplet, leading to the following Lagrangian for \( G \)

\[ L_G = -\frac{1}{2} G^2 + \frac{\frac{1}{2} (D^\alpha G D_\alpha G) (\bar{D}^{\dot{\alpha}} G \bar{D}_{\dot{\alpha}} G)}{1 - \frac{1}{2} (G^2 + \bar{G}^2) + \sqrt{1 - (G^2 + \bar{G}^2) - \frac{1}{4} (G^2 - \bar{G}^2)^2}}, \quad (2.29) \]

where

\[ G^2 = (D^\alpha \bar{D}^{\dot{\alpha}} G) (D_\alpha \bar{D}_{\dot{\alpha}} G), \quad \bar{G}^2 = (\bar{D}^{\dot{\alpha}} D^\alpha G) (\bar{D}_{\dot{\alpha}} D_\alpha G). \]

This agrees with the results of [16]; however, our derivation explains why the final expression is unique.

### 2.5. Duality and the Chiral Multiplet

The action (2.29) may be dualized to replace the real linear superfield \( G \) that describes the tensor multiplet by a chiral superfield \( \varphi \) in the standard way: we relax the linearity constraint on \( G \), introduce a chiral Lagrange multiplier \( \varphi \) to re-impose the constraint, and eliminate \( G \). We start with

\[ L'_G = L_G + (\varphi + \bar{\varphi}) G, \quad (2.30) \]

and vary with respect to \( G \). Unfortunately, the resulting equations are difficult to solve in a closed form. Comparing directly to the bosonic action, once can easily deduce the form that the action must take:

\[ L_\varphi = \varphi \bar{\varphi} + \frac{\frac{1}{2} (D^\alpha \varphi D_\alpha \varphi) (\bar{D}^{\dot{\alpha}} \bar{\varphi} \bar{D}_{\dot{\alpha}} \bar{\varphi})}{1 + A + (D^2 \varphi D^2 \bar{\varphi}) f + \sqrt{(1 + A)^2 - B + (D^2 \varphi D^2 \bar{\varphi}) g}}; \quad (2.31) \]

where

\[ A = \partial^\alpha \varphi \partial_{\alpha \dot{\alpha}} \bar{\varphi}, \quad B = (\partial^\alpha \dot{\varphi} \partial_{\alpha \dot{\alpha}} \varphi) (\partial^{\alpha \dot{\alpha}} \varphi \partial_{\alpha \dot{\alpha}} \bar{\varphi}), \]

and \( f \) and \( g \) are unknown functions of \( A, B \) and \( D^2 \varphi D^2 \bar{\varphi} \) that do not change the bosonic part of the action.\(^7\) In a subsequent article [27], we have verified that the action (2.31) is

\(^7\) Actually, without loss of generality, we may drop \( g \) by a shift of \( f \).
indeed dual to (2.29), and have shown that the functions \( f, g \) are arbitrary, and may be chosen to vanish following suitable redefinitions of \( \varphi \). We observe that the action (2.31) has manifest translational symmetry and thus is indeed the action representing the 3-brane in 6 dimensions of ref. [3]. This contrasts with ref. [4], where the two leading terms in the corresponding chiral multiplet action were proposed. These terms are not invariant under a shift of the bosonic part of the chiral superfield \( \varphi \) by a constant and thus are not directly related to the supersymmetric form of the \( D = 6 \) 3-brane action. The relation, however, can be established by making a field redefinition that eliminates non-translationally invariant terms in the \( O(\varphi^2) + O(\varphi^4) \) action of [4].

3. ‘Quadratic’ form and duality transformations of Born-Infeld-Nambu actions

We now make some useful observations about the structure of the bosonic actions (1.1),(1.2) and their duality properties inspired in part by the component expansions of our supersymmetric results.

3.1. \( D = 4 \) Born-Infeld Action in Terms of Two Auxiliary Complex Scalar Fields

Let us start with the \( D = 4 \) BI action and present it in a simple form which is related to its supersymmetric generalisation. Introducing an auxiliary field \( V \), the BI Lagrangian can be written as

\[
8L_4 = -\sqrt{-\det(\eta_{ab} + F_{ab})} \rightarrow -\frac{1}{2} V \det(\eta_{ab} + F_{ab}) + \frac{1}{2} V^{-1}. \tag{3.1}
\]

Since in 4 dimensions

\[
-\det_4(\eta_{ab} + F_{ab}) = 1 + \frac{1}{2} F_{ab}F^{ab} - \frac{1}{16}(F_{ab}F^{*ab})^2, \quad F^{*ab} \equiv \frac{1}{2} \epsilon^{abcd} F_{cd}, \tag{3.2}
\]

we can put the action into a form quadratic in \( F_{ab} \) by introducing the second auxiliary field \( U \) to ‘split’ the quartic \((FF^*)^2\) term in (3.2)\(^9\)

\[
L_4 = \frac{1}{2} V + \frac{1}{2} V^{-1} + \frac{1}{2} V^{-1} U^2 + \frac{1}{4} V F_{ab}F^{ab} + \frac{1}{4} U F_{ab}F^{*ab}. \tag{3.3}
\]

\(^8\) We use Minkowski signature \((-+++)\) so that \(\epsilon^{abcd} \epsilon_{abcd} = -1\), etc. Complex conjugation is denoted by bar, Hodge duality by star \((F^{*ab})\), and fields of dual theory by tilde \(\tilde{A}_a\).

\(^9\) Similar representations exist for BI actions in \( D > 4 \) but are more complicated as they involve more auxiliary fields and more field strength invariants (three in \( D = 6 \), four in \( D = 8 \), etc.).
Finally, we can eliminate the terms with $V^{-1}$ by introducing a complex auxiliary scalar $a = a_1 + i a_2, \bar{a} = a_1 - i a_2$

$$L_4 = \frac{1}{2} V (1 - \bar{a} a + \frac{1}{2} F_{ab} F^{ab}) + \frac{1}{2} U [i(a - \bar{a}) + \frac{1}{2} F_{ab} F^{*ab}] - \frac{1}{2} (a + \bar{a}) . \quad (3.4)$$

Shifting $a \rightarrow a + 1$ and dropping the constant $-1$, (3.4) can be rewritten as

$$L_4 = -\frac{1}{2} V (a + \bar{a} + \bar{a} a - \frac{1}{2} F_{ab} F^{ab}) + \frac{1}{2} U [i(a - \bar{a}) + \frac{1}{2} F_{ab} F^{*ab}] - \frac{1}{2} (a + \bar{a}) , \quad (3.5)$$

or

$$L_4 = -Im \left( \left[ a + \frac{1}{2} \bar{a} a - \frac{1}{4} (F_{ab} F^{ab} + i F_{ab} F^{*ab}) \right] + ia \right) , \quad (3.6)$$

$$\lambda = \lambda_1 + i \lambda_2 \equiv U + i V . \quad \text{(3.6)}$$

Note that the constraint implied by $\lambda$ is solved by $a = a(F)$ with $Im a(F) = \frac{1}{4} F_{ab} F^{*ab}$ and the real part

$$Re a(F) = -1 + \sqrt{1 + \frac{1}{2} F^2 - \frac{1}{16} (FF^*)^2} , \quad (3.7)$$

which is (up to sign) the BI Lagrangian itself. This gives a natural ‘explanation’ for the square root structure of the BI action. One can thus view the BI action as resulting from a peculiar action for 2 complex non-propagating scalars ($\lambda, a$) coupled non-minimally to a vector.

The bosonic part of the supersymmetric action (2.16) is exactly the BI action represented in the form with two auxiliary complex scalar fields (3.6), with the scalar fields $a$ and $\lambda$ in (3.6) being the corresponding scalar components of the chiral superfields $\phi$ and $\Lambda$ in (2.16) (note that $D^2 W^2 = -\frac{1}{4} F^{ab} F_{ab} - \frac{i}{4} F^{ab} F^{*ab} + \frac{1}{2} D^2$). It is thus guaranteed that once we solve for $\Lambda, \phi$ the action (2.16) should become the $N = 1$ supersymmetric extension [15,7] of the BI action. It is clear from (2.16) that the bosonic part of the action is quadratic in the auxiliary field $D$ of $V$ so that $D = 0$ is always a solution; adding the FI term breaks $N = 1$ supersymmetry giving a solution with $D = \xi + O(\xi, F)$.

Shifting $\lambda$ by $i$ the action (3.6) can be put into a form that does not contain terms linear in the fields

$$L_4 = -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \bar{a} a - Im \left( \left[ a + \frac{1}{2} \bar{a} a - \frac{1}{4} (F_{ab} F^{ab} + i F_{ab} F^{*ab}) \right] \right) . \quad (3.8)$$

This may be viewed as a special case of the following action for a vector coupled non-minimally to massive scalars

$$L_4 = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} (\partial_a \varphi_n)^2 - \frac{1}{2} m_n^2 \varphi_n^2 + g_{nkm} \varphi_n \varphi_m \varphi_k + \varphi_n (\alpha_n F_{ab} F^{ab} + \beta_n F_{ab} F^{*ab}) . \quad (3.9)$$
In the limit when the masses of scalars are much larger than their gradients so that the \((\partial_a \varphi_n)^2\) terms may be ignored, (3.9) reduces to (3.8) with the scalars \(\varphi_n\) being linear combinations of \(\lambda_1, \lambda_2, a_1, a_2\) in (3.8). This action may be viewed as a truncation of the cubic open string field theory action which reproduces the BI action as an effective action upon integrating out (at the string tree level) massive string modes \(\varphi_n\) [21,32]. The kinetic term \((\partial_a \varphi_n)^2\) may be dropped since it leads to the derivative \(\partial_c F_{ab}\)-dependent terms which, by definition, are not included in the leading part of the low-energy effective action action.

Let us note in passing that this quadratic in \(F_{ab}\) form of the BI action has an obvious nonabelian generalisation. As suggested by the open string field theory analogy, one replaces, e.g., for \(SO(N)\) case, \(F_{ab}\) by an antisymmetric matrix in the fundamental representation and replaces the scalars \(\varphi_n\) (i.e., \(\lambda\) and \(a\)) by symmetric matrices. The result is

\[
(L_4)_{nonab.} = \text{tr} \left[ -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \bar{a} a - Im \left( \lambda \left[ a + \frac{1}{2} \bar{a} a - \frac{1}{4} (F_{ab} F^{ab} + i F_{ab} F^{*ab}) \right] \right) \right].
\] (3.10)

Integrating out the matrices \(\lambda\) and \(a\) one finds a nonabelian version of the \(D = 4\) BI action in which all corrections to the standard YM \(\text{tr} F^2\) term depend only on the two symmetric matrices \((F_{ab} F^{ab})^{pq}\) and \((F_{ab} F^{*ab})^{pq}\). It should be straightforward to write down the supersymmetric generalisation of (3.10). Like the symmetrized trace action of [33] this non-abelian action has two required features: the standard BI action as its abelian limit, and the single trace structure. However, it is different from the symmetrized trace action (by some commutator terms) already at the \(F^4\) level. Indeed, the latter action

\[
\text{Str}[I - \sqrt{-\det(\eta_{ab} + F_{ab})}] = -\frac{1}{4} \text{tr} FF + \frac{1}{32} \text{Str}[(FF)^2 + (FF^*)^2] + ...
\] (3.11)

contains the terms like \(\text{tr}(E_i E_j B_i B_j)\), \(\text{tr}(E_i E_j E_i E_j)\) and \(\text{tr}(B_i B_j B_i B_j)\) (where \(E_i = F_{0i}\), \(B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}\)) which cannot be written in terms of \((F_{ab} F^{ab})^{pq} = 2(B_i B_i - E_i E_i)^{pq}\) and \((F_{ab} F^{*ab})^{pq} = 4(E_i B_i)^{pq}\) only for generic internal symmetry group. In contrast to (3.11) it is not immediately clear how to generalise the action (3.10) to dimensions higher than 4.

The BI action (3.1),(3.2) is obviously invariant under \(F_{ab} \to F_{ab}^*\). In addition, it is covariant under the vector-vector duality transformation [34,35]. Since, in this form, the BI action is quadratic in the vector field, it is very simple to demonstrate its covariance under the duality. Adding the Lagrange multiplier term \(\frac{1}{2} \tilde{F}^{*ab} F_{ab}\), where \(\tilde{F}_{ab}\) is the strength of
the dual vector field, and integrating out $F_{ab}$ we find that the dual action has the same form as (3.6) with (cf. (2.20))\(^{10}\)
\[
F_{ab} \rightarrow \tilde{F}_{ab}, \quad \lambda \rightarrow -\lambda^{-1}, \quad a \rightarrow -i\lambda a.
\]
As in the Maxwell theory case, the action (3.6) is not invariant under this duality. There exists, however, an equivalent action containing one extra vector field variable which is manifestly duality-symmetric (similar duality-symmetric actions were constructed in $D = 2$ [37,38] and $D \geq 4$ [38]). A systematic way of deriving such duality-symmetric actions is explained in the Appendix. Given an action (e.g., for a $D = 4$ vector) that depends only on the field strength, one puts it in a special first-order form by gauging the symmetry $A_a \rightarrow A_a + c_a$, $c_a = \text{const}$ as follows. One introduces a gauge field $V_{ab}$; then minimal coupling implies that $F_{ab}$ is replaced by $F_{ab} + V_{ab}$. One imposes the constraint $dV = 0$ with a Lagrange multiplier (dual field) $\tilde{A}_a$ (i.e., one adds the term $\frac{1}{2} \tilde{F}^{*ab}V_{ab}$). Choosing the ‘axial’ gauge $V_{ij} = 0$ ($i, j = 1, 2, 3$) and integrating out the lead to an action for $A_i$ and $\tilde{A}_i$ that is manifestly duality-invariant, i.e., invariant under the interchange of $A_i$ and $\tilde{A}_i$. The lack of manifest Lorentz invariance is not a problem, as it is merely a consequence of a noncovariant gauge choice.

The ‘quadratic’ form of the BI action (3.6) makes it easy to obtain this duality symmetric version of the action. One finds ($I, J = 1, 2$; $i, j, k = 1, 2, 3$)
\[
\hat{L}_4 (A, \tilde{A}) = -\frac{1}{2} [\mathcal{E}_I^b \mathcal{L}_{IJ} \mathcal{B}_{J}^I + \mathcal{B}_I^b (\mathcal{L}^T \mathcal{M} \mathcal{L})_{IJ} \mathcal{B}_{J}^I] - \text{Im} [\lambda (a + \frac{1}{2} \bar{a}a) + ia],
\]
\[
\mathcal{E}_I^b = \partial_0 A_i^I = (E_i, \tilde{E}_i), \quad \mathcal{B}_I^b = \epsilon_{ijk} \partial_j A_k^I = (B_i, \tilde{B}_i), \quad \mathcal{A}_I^I = (A_i, \tilde{A}_i),
\]
\[
\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{M} = \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix}.
\]
The vector field terms in this action are the same as in the case of the standard abelian vector theory coupled to scalars [39] and are invariant under the $SL(2, R)$ duality transformations
\[
A_i \rightarrow \omega^T A_i, \quad \mathcal{M} \rightarrow \omega^T \mathcal{M} \omega, \quad \omega \mathcal{L} \omega^T = \mathcal{L}, \quad \omega \in SL(2, R).
\]
Because the last term in (3.13) is linear in $a$, it can absorb any variation, i.e. we can choose its transformation to extend the invariance to the complete action.

\(^{10}\) The equations of motion derived from the vector terms in the action (3.6) have the full $SL(2, R)$ invariance (cf. [36]): $\lambda \rightarrow \frac{m\lambda + n}{k\lambda + l}$, $F_{ab} \rightarrow (kU + l) F_{ab} + kVF_{ab}$, $ml - nk = 1$. 

15
The ‘doubled’ form of the BI action depending only on $A_i, \tilde{A}_i$ can be obtained either by eliminating the scalar fields from (3.13) or, directly, from the original action (3.1),(3.2) by using the procedure of gauging ($F_{ab} \rightarrow F_{ab} + V_{ab}, \ L_4 \rightarrow L_4 + \frac{1}{2} \tilde{F}^{*ab}V_{ab}$), choosing the ‘axial’ gauge ($V_{ij} = 0$) and integrating out $V_{0i}$ as explained above and in the Appendix. The gauge-fixed action is found to be

$$\hat{L}_4 = -v_i \tilde{B}_i - \sqrt{1 - (E_i + v_i)^2 + B_i^2 - [(E_i + v_i)B_i]^2}, \quad v_i \equiv V_{0i}. \quad (3.16)$$

Shifting $v_i \rightarrow v_i' - E_i$ and eliminating $v_i'$ from the action by solving its equation of motion gives (after symmetrising the first term using integration by parts)

$$\hat{L}_4 = \frac{1}{2}(E \cdot \tilde{B} - \tilde{E} \cdot B) - \sqrt{1 + B^2 + \tilde{B}^2 + B^2\tilde{B}^2 - (B \cdot \tilde{B})^2}, \quad (3.17)$$

which can be represented also in the form (cf. (3.13))

$$\hat{L}_4 = \frac{1}{2}\varepsilon_i^j \varepsilon_{Ij}B_i^j - \sqrt{\det(\delta_{ij} + B_i^j B_j^i)} \quad (3.18)$$

This is obviously invariant under the $O(2)$ duality rotations in $(A, \tilde{A})$ plane, in particular, under $A_i \leftrightarrow -\tilde{A}_i$, i.e.,

$$A \rightarrow \mathcal{L}A, \quad \mathcal{E}_i \rightarrow \mathcal{L}\mathcal{E}_i, \quad B_i \rightarrow \mathcal{L}B_i. \quad (3.19)$$

### 3.2. Scalar-Tensor Duality and Born-Infeld-Nambu Actions

Starting with the Dp-brane action like (1.1), i.e.,

$$S = -\int d^{p+1}x \sqrt{-\det(g_{ab} + F_{ab})}, \quad g_{ab} = \eta_{ab} + \partial_a X^n \partial_b X^n, \quad (3.20)$$

and performing a vector duality transformation by adding a Lagrange multiplier term $\frac{1}{2}H^{*ab}F_{ab}$ and integrating out $F_{ab}$ one finishes (in $D \leq 5$) with the same action with $F_{ab} \rightarrow H_{ab}$ [35]. Here we study what happens if instead we dualise one of the scalars, e.g., $Y \equiv X^1$. Since for general $D$

$$\det(M_{ab} + P_a P_b) = [1 + (M^{-1})^{ab}P_a P_b] \det M, \quad (3.21)$$

\[11\] In $D = 4$ this is true for an arbitrary 4-d metric $g_{ab}$. In $D = 5$ where $H^{*ab} = \frac{1}{8}\varepsilon^{abcde}H_{cde}$ one finds the dual Lagrangian in the form $\sqrt{-\det(g_{ab} + \frac{2\varepsilon^abg_{ab}H^{*cd}}{\sqrt{5}}H_{cd})}$. 

16
where, in the present case \( P_a = \partial_a Y \) and \( s \neq 1 \)

\[
M_{ab} = \hat{g}_{ab} + F_{ab}, \quad \hat{g}_{ab} \equiv \eta_{ab} + \partial_a X^s \partial_b X^s, \tag{3.22}
\]

we can write the action in the form similar to (3.1)\(^{12}\)

\[
L_D = -\frac{1}{2} V [1 + (M^{-1})^{ab} \partial_a Y \partial_b Y] \det M + \frac{1}{2} V^{-1}. \tag{3.23}
\]

Replacing \( \partial_a Y \) by \( P_a \) and adding the Lagrange multiplier term \( H^a P_a \), \( H^a \equiv \frac{1}{p!} \epsilon^{ab_1...b_p} H_{b_1...b_p} \equiv p \partial_{[b_1} \tilde{Y}_{b_2...b_p]} \) one can integrate over \( P_a \) to get

\[
\tilde{L}_D = -\frac{1}{2} V \det M + \frac{1}{2} V^{-1} \det M^{-1} M_{ab} H^a H^b + \frac{1}{2} V^{-1}, \tag{3.24}
\]

or, equivalently,

\[
\tilde{L}_D = \frac{1}{2} \hat{V}(\det M + M_{ab} H^a H^b) - \frac{1}{2} \hat{V}^{-1}, \quad \hat{V} \equiv V^{-1} \det M^{-1}. \tag{3.25}
\]

Using (3.21) this can be put back into the ‘determinant’ form

\[
\tilde{L}_D = -\sqrt{-\det(M + M_{ab} H^a H^b)} = -\det M \sqrt{-\det[(M^{-1})^{ab} + H^a H^b]}
\]

\[
= -\sqrt{-\det(M_{ab} + \det M_{ac} M_{bd} H^c H^d)}. \tag{3.26}
\]

In \( D = 2 \), the action takes the same form when written in terms of \( \partial_a \tilde{Y}, \quad H^a = \epsilon^{ab} \partial_a \tilde{Y}, \)

\( i.e., \) the \( D = 2 \) Born-Infeld-Nambu action is invariant under the scalar-scalar duality (note that \( \det_2 M_{ab} = \det \hat{g} - \frac{1}{4} (\epsilon^{ab} F_{ab})^2 \))

\[
\tilde{L}_2 = -\sqrt{-\det(\eta_{ab} + \partial_a X^s \partial_b X^s + \partial_a \tilde{Y} \partial_b \tilde{Y} + F_{ab})}. \tag{3.27}
\]

In \( D = 3 \), we get an action with an extra vector \( \tilde{Y}_a \) instead of the scalar \( Y \) \( (H^a = \frac{1}{2} \epsilon^{abc} \tilde{F}_{bc}, \quad \hat{g} \equiv \det \hat{g}_{ab}) \)

\[
\tilde{L}_3 = -\sqrt{-\hat{g}(1 + \frac{1}{2} \hat{g}^{ac} \hat{g}^{bd} F_{ab} F_{cd} + \frac{1}{2} \hat{g}_{ab} H^a H^b]}
\]

\[
= -\sqrt{-\hat{g} \sqrt{1 + \frac{1}{2} \hat{g}^{ac} \hat{g}^{bd} (F_{ab} F_{cd} + \tilde{F}_{ab} \tilde{F}_{cd})}. \tag{3.28}
\]

\(^{12}\) Note that in contrast to the vector case, one auxiliary field suffices to make the action quadratic in the scalar field derivatives.
Note that dualising a scalar in $D = 3$ we do get an action with two vectors but not simply the usual $D = 3$ BI-Nambu action with $F_{ab} \rightarrow F_{ab} + \tilde{F}_{ab}$ (the determinant in such an action would contain the cross-term $F^{ab}\tilde{F}_{ab}$). The inverse to this $D = 3$ transformation, i.e., ‘vector → scalar’ duality (previously discussed in [40,35]) gives the membrane action with one extra scalar and no vectors.

In $D = 4$, the scalar $Y$ is traded for the antisymmetric tensor $B_{ab} \equiv \tilde{Y}_{ab}$

$$\tilde{L}_4 = -\sqrt{-\hat{g}} \left[ \det(\delta^C_b + \hat{g}^{ca}F_{ab}) + \frac{1}{2}\hat{g}^{ab}\hat{g}^{cd}\hat{g}^{ef}H_{ace}H_{bdf} \right]$$

$$= -\sqrt{-\det(\hat{g}_{ab} + F_{ab}) + \hat{g}_{ab}H^{*a}H^{*b}} .$$

These actions can be simplified when $F_{ab} = 0$ and $\hat{g}_{ab} = \eta_{ab} + \partial_aX\partial_bX$, which, in particular, is the case of the 3-brane ($D = 4$) in 6 dimensions (1.2). For arbitrary dimension $D$,

$$(\eta_{ab} + Q_aQ_b)^{-1} = \eta_{ab} - \frac{Q_aQ_b}{1 + Q^2} ,$$

$$-\det(\eta_{ab} + Q_aQ_b + P_aP_b) = 1 + Q^2 + P^2 + Q^2P^2 - (QP)^2 ,$$

so that (here $Q_a = \partial_aX$, $P_a = \partial_aY$; we use the flat Minkowski metric $\eta_{ab}$ to contract the indices)

$$L_D = -\sqrt{1 + (\partial X)^2 + (\partial Y)^2 + (\partial X)^2(\partial Y)^2 - (\partial X\partial Y)^2} .$$

(3.30)

The dual action

$$\tilde{L}_D = -\sqrt{1 + \partial_aX\partial_aX - H^{*a}H^{*a} - (H^*a\partial_aX)^2} ,$$

(3.31)

can be expressed in terms of a complex vector $G_a$

$$\tilde{L}_D = -\sqrt{1 + \frac{1}{2}(G_aG_a + \bar{G}_a\bar{G}_a) + \frac{1}{16}(G_aG_a - \bar{G}_a\bar{G}_a)^2} , \quad G_a \equiv \partial_aX + iH^*_a .$$

(3.32)

In $D = 4$, $H^{*a} = \frac{1}{6}\epsilon^{abcd}H_{bcd}$. This form of the action can be compared with the bosonic part of the tensor multiplet action (2.29).

The scalar action (3.30) may be put into a first-order form similar to (3.4) in the case of BI action (3.1). Introducing $\varphi = X + iY$, $\bar{\varphi} = X - iY$ and the real auxiliary field $V$ we may replace (3.30) by (cf. (3.1))

$$L_D = -\frac{1}{2}V^{-1}\left[ (1 + \frac{1}{2}\partial\varphi\partial\bar{\varphi})^2 - \frac{1}{4}(\partial\varphi)^2(\partial\bar{\varphi})^2 \right] - \frac{1}{2}V .$$

(3.33)
Using one real ($\alpha$) and one complex ($\beta$) auxiliary fields we can put (3.33) into the form quadratic in $\varphi, \bar{\varphi}$

$$L_D = -\frac{1}{2} V(1 + \bar{\beta}\beta - \alpha^2) + \alpha(1 + \frac{1}{2} \partial \varphi \partial \bar{\varphi}) + \frac{1}{4} \beta(\partial \bar{\varphi})^2 + \frac{1}{4} \bar{\beta}(\partial \varphi)^2.$$ (3.34)

The field $V$ thus plays the role of a Lagrange multiplier which restricts $\alpha, \beta$ to a 2-dimensional hyperboloid. This form of the action may be useful for trying to construct a $N = 1$ supersymmetric extension of (3.30) based on chiral multiplet. Let us note also that (3.30) can be also written in the form

$$L_D = -\sqrt{1 + \frac{1}{2} \partial \varphi \partial \bar{\varphi} + \frac{1}{4} (\partial \varphi \partial \bar{\varphi})^2} - \frac{1}{4} (\partial \varphi)^2 (\partial \bar{\varphi})^2$$

(3.35)

which can be compared to the supersymmetric action (2.31).

### 4. Conclusions

There exists a remarkable connection between (i) partial supersymmetry breaking, (ii) nonlinear realisations of extended supersymmetry, (iii) BPS solitons, and (iv) nonlinear Born-Infeld-Nambu type actions. We have shown that the connection between partial breaking of supersymmetry and nonlinear actions is not accidental and has to do with constraints that lead directly to nonlinear actions of BI type. We believe that this constrained superfield approach is the simplest and most transparent way of deriving and using these actions.

$N > 1$ susy can be partially broken either by a non-translationally invariant background (soliton) in a second-derivative higher-dimensional theory or by a translationally invariant vacuum in a nonrenormalisable theory in four dimensions containing non-minimal interactions. We have seen how to determine the resulting actions in a model-independent way.

Inspired by our supersymmetric results, we have found a particularly simple way of demonstrating the self-duality of the BI action. This led us to a discussion of the general form of scalar-tensor duality in $D = 4$ actions, and duality in other dimensions.

We found a closed form of the action for the tensor multiplet which after duality becomes the full nonlinear action of the $D = 6$ 3-brane. A direct derivation of this action
using the methods of [3] would be complicated, whereas our approach gives it in a universal way. Unfortunately, we have not found the $D = 4$, $N = 4$ supersymmetric extension (with $N = 2$ supersymmetry realized nonlinearly) of the BI action.\textsuperscript{13} To do this, we would need to couple the $N = 1$ vector and chiral (or tensor) multiplets.\textsuperscript{14} In [12] it was noted that there is no obvious generalisation of the model of [5] to the case of an $N = 2$ vector multiplet coupled to a charged hypermultiplet. This should not be a problem since we need a non-minimal coupling – as is evident from our results, scalars and vectors should couple non-minimally via field strengths in BI actions. So some version of constrained superfield approach should work.

It is not clear if the above constrained approach can be generalised to the nonabelian case, where $\phi$ in (2.16) will be in the adjoint representation.

Acknowledgements

We would like to thank I. Antoniadis, W. Siegel, F. Gonzalez-Rey, I.Y. Park and P. van Nieuwenhuizen for useful discussions of related questions. The work of M.R. was supported in part by NSF grant No. PHY 97221101. A.A.T. is grateful to the Institute of Theoretical Physics of SUNY at Stony Brook for hospitality during his stay in April 1997 while most of this work was completed and acknowledges also the support of PPARC and the European Commission TMR programme grant ERBFMRX-CT96-0045.

\textsuperscript{13} The $N = 2$ supersymmetric form of the BI action was recently considered in [41] (see also [42,43] in connection with $N = 2$ actions). However, the action proposed there has higher derivative terms at the component level, and is not invariant under constant shifts of the physical scalar components.

\textsuperscript{14} It may be of some use to note that BI action in $D = 5$ is dual to the antisymmetric tensor action for $H_{mnk}$, \emph{i.e.}, $\sqrt{-\det(\eta_{mn} + H_{mn})}$ [35]; upon dimensional reduction to $D = 4$ it becomes an action for a vector $B_{a5}$ and a tensor $B_{ab}$ instead of a vector and a scalar as before duality in $D = 5$. That suggests a possibility of getting a manifestly supersymmetric action for a vector multiplet and a tensor multiplet by starting with a duality-rotated action for a vector multiplet in $D = 5$. 

20
Appendix: Duality-symmetric actions – the ‘gauging’ approach

One particular definition of a duality relation between two quantum field theories in $D$ dimensions is that it gives a map between correlators of certain operators in one theory and correlators of corresponding subset of operators in the dual theory. At the level of the partition function or the generating functional for the special operators examples of such duality relations can be understood as a series of formal transformations of gaussian path integrals [44], but not as a local change of variables in the original action.

At the same, it is possible to define manifestly duality invariant actions where duality becomes just a local field transformation accompanied by a transformation of the external coupling parameters (e.g., $R \rightarrow \alpha'/R$). To achieve this one doubles the number of field variables by introducing the dual fields on the same footing as the original fields. This was suggested in the context of scalar $D = 2$ theories in [37] and generalized to the heterotic string type $D = 2$ actions in [38] and to $D = 4$ abelian vector actions in [39] (see also [45] for earlier work). The price for having duality as a symmetry of the action is the lack of manifest Lorentz invariance (the off-shell Lorentz invariance appears only after one integrates out one of the dual fields recovering the original or dual Lorentz-invariant actions, see also [47,48]).

In what follows we demonstrate that such non-Lorentz invariant ‘doubled’ actions are, in fact, gauge fixed versions of duality and Lorentz invariant ‘extended’ actions containing extra degrees of freedom. An application of this observation to the nontrivial example of the bosonic Born-Infeld action was considered above in section 3.1 (see (3.13),(3.18)).

These ‘extended’ actions are closely related to first-order actions (see, e.g., [44,49,50,51]) usually discussed in the context of path integral demonstration of duality. Given an action that depends only on a field strength $F = dA$, one can treat $F$ as an independent field by adding a constraint that imposes the Bianchi identity, thus relating it to the original field. An equivalent first-order action is obtained by gauging the symmetry $A \rightarrow A + c$, where $c$ is a constant [50]. The corresponding gauge field will be denoted by $V$. Its field strength is set equal to zero by a Lagrange multiplier that plays the role of the dual field.

---

15 The dual theories may not be completely equivalent, e.g., in $S$-matrix sense: only certain types of observables may be related.

16 Such actions (which are of first order in time derivatives) can be also interpreted as phase space actions (i.e., as original actions expressed in terms of phase space variables) with the dual fields playing the role of the integrated canonical momenta; for a related canonical approach to duality see [46].
A. Gauge fixing the original field and integrating out the auxiliary field \( V \) gives the dual action. If, instead, one chooses a non-Lorentz-covariant ‘axial’ gauge on \( V \) and integrates out the remaining components of \( V \), one obtains precisely the duality symmetric ‘doubled’ action for \( A \) and \( \tilde{A} \).

An advantage of considering the generalised actions depending on \( A, \tilde{A} \) and \( V \) is that in addition to be duality invariant (i.e., invariant under \( A \leftrightarrow \tilde{A} \) and a gauge transformation of \( V \)), it is also Lorentz invariant. Thus it may be used as a starting point for constructing manifestly Lorentz and duality invariant world sheet and space-time actions in string theory. The additional variables \( V \) may eventually find their place in a more fundamental formulation of the theory.

To illustrate this procedure let us start with an action for a scalar in \( D = 2 \) or a vector in \( D = 4 \) that depends on the field \( A \) only through the field strength, \( S = \int d^D x \, L(F) \), \( F = dA \), where \( L \) may depend also on other fields. For example, in \( D = 2 \), \( L_2 = -\frac{1}{2} F_a F^a \), \( F_a = \partial_a A \). In \( D = 4 \) one may consider an action for several vector fields \( A^a \) coupled to scalar fields \( \varphi \):

\[
L_4(A, \varphi) = G_{pq}(\varphi) F_{ab}^{pq} F^{qab} + B_{pq}(\varphi) F_{ab}^{pq} F_{cd}^{qab} + J^p_q(\varphi) F_{ab}^{pq} + L'_4(\varphi),
\]

where \( a, b = 0, 1, 2, 3 \). The ‘gauged’ forms of the extended ‘first-order’ Lagrangians are\(^{17}\)

\[
\hat{L}_2(A, \tilde{A}, V) = L_2(V_a + \partial_a A) + \epsilon^{ab} \partial_b \tilde{A} V_a,
\]

\[
\hat{L}_4(A, \tilde{A}, V) = L_4(V_{ab} + \partial_a A_b - \partial_b A_a) + \frac{1}{2} \epsilon^{abcd} \partial_a \tilde{A}_b V_{cd}.
\]

The corresponding actions are invariant under the gauge transformations

\[
A' = A + c, \quad V' = V - dc,
\]

as well as under the duality transformations that interchange \( A \) and \( \tilde{A} \) and act on \( V \) in a nonlocal way.

Integration over \( \tilde{A} \) in (A1),(A2) gives back the original actions \( \int L_D(dA) \) (after a redefinition of \( A \) or gauge-fixing the remaining longitudinal part of \( V \) to zero). Gauge-fixing \( A = 0 \) and integrating out \( V \) gives the dual Lagrangian \( \hat{L}_D(\tilde{A}) \). Classically, \( V \) is eliminated by solving its equations of motion. The resulting classically equivalent Lagrangian is local provided \( L_D \) is an algebraic function of \( F = dA \), as, e.g., in the case of the Born-Infeld action discussed in [34,35] and above.

\(^{17}\) For simplicity here we shall consider the case of a single scalar (\( D = 2 \)) or vector (\( D = 4 \)). Various possible generalisations (e.g., to several scalar or vector fields, curved \( D \)-dimensional space-time, other \( p = \frac{1}{2}(D - 2) \)-form dualities, etc.) are straightforward.
Fixing instead the ‘axial’ gauge on $V$, namely, $V_1 = 0$ in $D = 2$ and $\epsilon^{ijk}V_{jk} = 0$ ($i, j = 1, 2, 3$) in $D = 4$ and integrating out the remaining gauge field components (i.e., $V_0$ and $V_{0i}$) we get from (A1),(A2) in the simplest $L_2 = -\frac{1}{2}F_{ab}^2$, $L_4 = -\frac{1}{4}F_{ab}^2$ cases (we use the freedom to add a total derivative to put the Lagrangians in a more symmetric form)

$$\hat{L}_2(A, \tilde{A}) = -\frac{1}{2} \left[ -\partial_0 A \partial_1 \tilde{A} - \partial_0 \tilde{A} \partial_1 A + (\partial_1 A)^2 + (\partial_1 \tilde{A})^2 \right], \quad (A4)$$

$$\hat{L}_4(A, \tilde{A}) = -\frac{1}{4} \left[ - F_{0k} \epsilon^{ijk} \hat{F}_{ij} + \hat{F}_{0k} \epsilon^{ijk} F_{ij} + F_{ij} F^{ij} + \hat{F}_{ij} \hat{F}^{ij} \right] \quad (A5)$$

or, in a compact symbolic form,

$$L_D(A) = \frac{1}{2} (\partial_0 A L \partial A - \partial A M \partial A), \quad A = (A, \tilde{A}), \quad (A6)$$

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{M} \mathcal{A} \\ -I \mathcal{A}^2 & 0 \end{pmatrix} \quad \mathcal{M} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (A7)$$

The actions (A4) [37] and (A5) [39] are clearly invariant under the discrete duality transformations $A \rightarrow L A$, i.e.,

$$D = 2: \quad A \rightarrow \tilde{A}, \quad \tilde{A} \rightarrow A; \quad D = 4: \quad A_i \rightarrow -\tilde{A}_i, \quad \tilde{A}_i \rightarrow -A_i. \quad (A7)$$

In more general cases they are invariant under (A7) accompanied by a redefinition of the ‘external’ coupling constants (scalar fields) which parametrize the matrix $\mathcal{M}$. Similar action is found in the BI case (see (3.13),(3.17)).

The action (A2) is the ‘master action’ which unifies various other actions with less number of field variables but the same number of physical degrees of freedom. The ‘axial’ gauge choice we used is not the only possible one. For example, one may choose $V_{ab}$ in the form

$$V_{ab} = u_a v_b - u_b v_a, \quad u_a v_a = 0, \quad u_a = \partial_a \psi, \quad (A8)$$

where $v_a$ is an arbitrary vector field which is supposed to be integrated out while $\psi$ is kept along with $A_i, \tilde{A}_i$. In the case when $\psi = t$, i.e., $u_a = (1, 0, 0, 0)$, we are back to the axial gauge, $V_{ij} = 0$, $V_{0i} = v_i$, and integrating out $v_i$ leads to (A5). Keeping $\psi$ arbitrary one obtains the ‘covariantized’ form of the action (A5) suggested in [52] (which does not contain extra dynamical degrees of freedom present in the earlier proposal of [53])

$$L_4(A, \tilde{A}, u) = -\frac{1}{4} F_{ab} F_{ab} + \frac{1}{4} (F_{ab} - \hat{F}_{*ab})(F_{ac} - \hat{F}_{ac}) \frac{u_b u^c}{u^2} \quad (A9)$$

23
The residual gauge invariance of this action allowing one to choose a gauge $u_a = \delta_a^0$ [52] (in which (A9) reduces back to (A5)) is now understood as being just a remaining part of the original gauge invariance (A3) of the ‘master action’ (A2).

We would like to emphasize that it is the ‘gauged’ first-order action (A2) depending on $A_a, \tilde{A}_a$ and $V_{ab}$ that is the genuine Lorentz-covariant action behind both the duality symmetric ‘doubled’ action (A5) of [39] and the action with an extra vector $u_a$ of [52].

Let us also note that to obtain the corresponding actions describing self-dual fields one just sets $A = L A$ in the above ‘doubled’ actions. In particular, this leads to a simple ‘self-dual vector’ action in the BI case: as follows from (3.17),

$$L_4^{(+)} = E_i B_i - \sqrt{1 + 2B_i B_i},$$

(A10)

which generalises the action of the self-dual analogue $(E_i B_i - B_i B_i)$ [48] of the Maxwell action.

The approach discussed above can be readily applied to $D = 6$ antisymmetric tensor theories, e.g., to the 5-brane theory containing 2-tensor with self-dual field strength (see, e.g., [54]). The ‘master’ action in this case depends on $B_{ab}, \tilde{B}_{ab}$ and $V_{abc}$; self-duality is imposed by identifying the spatial components of $B_{ab}$ and $\tilde{B}_{ab}$ after (partial) integrating out $V_{abc}$.

An alternative but equivalent procedure is to start with the (in general, nonpolynomial) action $\int L(dB)$ for a p-form field (e.g., $p = 2, D = 6$), write down the corresponding Lagrangian in terms of the phase space variables, i.e., the fields $B_{ij}$ and $p^{ij} = \partial L/\partial (\partial_0 B_{ij})$, and replace the momentum $p^{ij}$ by a new field $\tilde{B}_{ij}, \quad p^{ij} = \frac{1}{2} \epsilon^{ijkl} \frac{\partial}{\partial \psi} \tilde{B}_{kl}$. The result is the duality-symmetric Lagrangian which generalises (A6) to the case of an arbitrary non-quadratic function $L(dB)$, i.e., $\tilde{L} = \epsilon_5 \partial_0 \tilde{B} \tilde{B} - L(H_{ijk}, \tilde{H}_{0ij}(p(\tilde{B})))$, where $H_{abc} = 3\partial_{[a} B_{bc]}$.

To obtain the action for a self-dual field one sets $B = \tilde{B}$. The ‘Lorentz-covariant’ version of the resulting action (generalising that of [55] in the case of the quadratic action (A6)) is found by repeating the above steps in the ‘covariant canonical formalism’ set-up, i.e., with $dt \rightarrow \partial_\psi dx^a$, etc., thus introducing the dependence on $u_a = \partial_a \psi$ as in (A9).
References


