N=8 BPS black holes preserving 1/8 supersymmetry

M. Bertolini\textsuperscript{a}, P. Frè\textsuperscript{b} and M. Trigiante\textsuperscript{c}

\textsuperscript{a}International School for Advanced Studies ISAS-SISSA and INFN
Sezione di Trieste, Via Beirut 2-4, 34013 Trieste, Italy

\textsuperscript{b}Dipartimento di Fisica Teorica, Università di Torino and INFN
Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy

\textsuperscript{c}Department of Physics, University of Wales Swansea, Singleton Park
Swansea SA2 8PP, United Kingdom

Abstract

In the context of $N=8$ supergravity we consider BPS black–holes that preserve 1/8 supersymmetry. It was shown in a previous paper that, modulo $U$–duality transformations of $E_7(7)$ the most general solution of this type can be reduced to a black-hole of the STU model. In this paper we analyze this solution in detail, considering in particular its embedding in one of the possible Special Kähler manifold compatible with the consistent truncations to $N=2$ supergravity, this manifold being the moduli space of the $T^6/\mathbb{Z}_3$ orbifold, that is: $SU(3,3)/SU(3) \times U(3)$. This construction requires a crucial use of the Solvable Lie Algebra formalism. Once the group-theoretical analysis is done, starting from a static, spherically symmetric ansatz, we find an exact solution for all the scalars (both dilaton and axion-like) and for gauge fields, together with their already known charge-dependent fixed values, which yield a $U$–duality invariant entropy. We give also a complete translation dictionary between the Solvable Lie Algebra and the Special Kähler formalisms in order to let comparison with other papers on similar issues being more immediate. Although the explicit solution is given in a simplified case where the equations turn out to be more manageable, it encodes all the features of the more general one, namely it has non-vanishing entropy and the scalar fields have a non-trivial radial dependence.

e-mail: teobert@sissa.it, fre@to.infn.it, m.trigiante@swansea.ac.uk

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1 Introduction

In the last three years there has been a renewed interest in the black-hole solutions of D=4 supergravity theories and, more in general, in black p-brane solutions of supergravity theories in higher dimensions. Among these solutions, of particular interest in the study of superstring dualities are those preserving a fraction of the original supersymmetries, which have been identified with the BPS saturated perturbative and non-perturbative states of superstring theory. This interpretation [1, 2] has found strong support with the advent of D-branes [3], allowing the direct construction of the BPS states. Indeed, although solutions of the classical low-energy supergravity theory, their masses, which saturate the Bogomolnyi bound (BPS saturated solutions), are protected from quantum corrections when the supersymmetry is high enough. This property promotes them to solutions of the whole quantum theory, and thus they represent an important tool in probing the non-perturbative regime of superstring theories.

This paper investigates the most general BPS saturated black-hole solution of $D = 4$ supergravity preserving 1/8 of the $N = 8$ supersymmetry, completing a programme started in [4, 5]. The basic result of [4] was to show that the most general 1/8 black-hole solution of $N = 8$ supergravity is a STU model solution, namely a solution where only 6 scalar fields (3 dilaton-like and 3 axion-like) and 8 charges (4 electric and 4 magnetic) are switched on. This solution is the most general modulo $U$–duality transformations. As it is well known ([6, 7]), the quantum $U$–duality group is the discrete version of the isometry group $U$ of the scalar coset manifold $U/H$ of $N = 8$ supergravity. Once a solution is found, acting on it with a $H = SU(8)$ transformation one generates the general charged black-hole and then acting with a $U = E_{7(7)}$ transformation one generates the most general solution, namely that with fully general asymptotic values of the scalar fields. In the context of $N = 8$ supergravity one of the results of [4] was the identification of the minimal content of dynamical fields and charges that a 1/8 black–hole solution should have, in order for its entropy to be non vanishing (regular solution). Nevertheless in that paper only a particular dilatonic solution was worked out explicitly. This solution had zero entropy since not only the dilatons but also the axions are part of the minimal set of fields necessary to describe a regular 1/8 black–hole. In [8] a very special solution of this kind was found, namely the double-extreme one, in which all scalar fields are taken to be constant and equal to the fixed values they anyhow must get at the horizon [9]. In the present paper we will consider a more general solution, namely a dynamical solution (i.e. not double-extreme) and with regular horizon (i.e. with non-vanishing entropy). The solution, corresponding to a specific configuration of scalar fields and charges, is obtained by performing $U$ and $H$ transformations in such a way that the quantized and central charges are put into the normal frame. Other regular solutions have been considered in various other papers like...
The aim of the present paper is, however, to consider the BPS generating solution that is the one depending on the least number of charges from which the all $U$-duality orbit may be reconstructed through the action of the $U$-duality group. A resumé of the essential properties of the generating BPS solutions in arbitrary dimensions $4 \leq D \leq 9$ can be found in [12]. In the context of toroidally compactified type II supergravity, the only regular black-hole solutions are the $1/8$ supersymmetry preserving ones while $1/2$ and $1/4$ black-holes, whose general form has been completely classified in [5], have zero horizon area. As it has been extensively explained in [4] and will be summarized in the following, a $1/8$ supersymmetry preserving $N = 8$ solution can be seen as a solution within a consistent truncation $N = 8 \rightarrow N = 2$ of the supergravity theory. In this truncation one needs specific choices of both the Hyperkähler and the Special Kähler manifold, describing the hyper and vector multiplets, respectively. Following the same lines of [4, 5] we will consider one of the possible non-trivial $N = 2$ embeddings of the STU model solution. This will be carried on with the essential aid of the Solvable Lie Algebra (SLA from now on) approach to supergravity theories, which is particularly useful to define a general method for the systematic study of BPS saturated black-hole solutions of supergravity. For a review on the solvable Lie algebra method see [13]. We give the details on our use of the Solvable Lie algebra in Appendix A.

The BPS saturated states are characterized by the property that they preserve a fraction of the original supersymmetries. This means that there is a suitable projection operator $\mathbb{P}^2_{BPS} = \mathbb{P}_{BPS}$ acting on the supersymmetry charge $Q_{SUSY}$, such that:

$$\langle \mathbb{P}_{BPS} Q_{SUSY} \rangle \mid_{BPS} = 0.$$  \hspace{1cm} (1.1)

Since the supersymmetry transformation rules are linear in the first derivatives of the fields, eq.(1.1) is actually a system of first order differential equations that must be combined with the second order field equations of supergravity. The solutions common to both system of equations are the classical BPS saturated states.

In terms of the gravitino and dilatino physical fields $\psi_{A\mu}$, $\chi_{ABC}$, $A, B, C = 1, \ldots, 8$, equation (1.1) is equivalent to

$$\delta_\epsilon \psi_{A\mu} = \delta_\epsilon \chi_{ABC} = 0 \hspace{1cm} (1.2)$$

whose solution is given in terms of the Killing spinor $\epsilon_A(x)$ subject to the supersymmetry preserving condition

$$\gamma^0 \epsilon_A = i \Omega_{AB} \epsilon^B \quad ; \quad A, B = 1, \ldots, n_{\text{max}}$$

$$\epsilon_A = 0 \quad ; \quad A = n_{\text{max}} + 1, \ldots, 8$$

where $n_{\text{max}}$ is twice the number of unbroken supersymmetries. Eq.(1.2) has the essential feature of breaking the original $SU(8)$ automorphism group of the supersymmetry algebra to the
subgroup $\hat{H} = Usp(n_{\text{max}}) \times SU(8 - n_{\text{max}}) \times U(1)$. Eqs.(1.2) will then provide different conditions on scalar fields transforming in different representations of $\hat{H}$. In other words the scalar manifold $E_7(7)/SU(8)$ of the original $N = 8$ theory will decompose into submanifolds spanned by scalar fields on which the Killing spinor equations impose different kind of conditions. This decomposition, as it was shown in [4] cannot be described as a decomposition of the isometry group $E_7(7)$ into the isometry groups of the submanifolds, but may be described by using the SLA formalism, i.e. expressing $E_7(7)/SU(8)$ and its submanifolds as group manifolds generated by suitable solvable Lie algebras. In this description the scalars are, as it will be explained more in detail in the sequel, parameters of the generating solvable algebra, and, according to the decomposition of $SU(8)$ into $\hat{H}$, the solvable algebra $Solv_7$ generating $E_7(7)$ will decompose into the direct sum of the solvable algebras generating the submanifolds whose scalar fields transform in representations of $\hat{H}$.

In the case at hand, namely a 1/8 supersymmetry preserving solution, we have $n_{\text{max}} = 2$ and $Solv_7$ must be decomposed according to the decomposition of the isotropy subgroup: $SU(8) \rightarrow SU(2) \times U(6)$. We showed in [4] that the corresponding decomposition of the solvable Lie algebra is the following one:

$$Solv_7 = Solv_3 \oplus Solv_4$$

(1.3)

where the rank three Lie algebra $Solv_3$ defined above describes the 30-dimensional scalar sector of $N = 6$ supergravity, while the rank four solvable Lie algebra $Solv_4$ contains the remaining forty scalars belonging to $N = 6$ spin $3/2$ multiplets. Both manifolds $\exp[Solv_3]$ and $\exp[Solv_4]$ have also an $N = 2$ interpretation since we have:

$$\exp[Solv_3] = \text{homogeneous special Kähler}$$

$$\exp[Solv_4] = \text{homogeneous quaternionic}$$

(1.4)

so that the first manifold can describe the interaction of 15 vector multiplets, while the second can describe the interaction of 10 hypermultiplets. Indeed if we decompose the $N = 8$ graviton multiplet in $N = 2$ representations we find:

$$N=8 \text{spin } 2 \overset{N=2}{\twoheadrightarrow} \text{spin } 2 + 6 \times \text{spin } 3/2 + 15 \times \text{vect. mult.} + 10 \times \text{hypermult.}$$

(1.5)

In order to end up with an $N = 2$ consistent truncation one has to consider $K \subset Solv_3$ and $Q \subset Solv_4$ such that $[K,Q] = 0$. The more simple case is to take $K = Solv_3$ and $Q = 0$ while the first non trivial one corresponds to take a one-dimensional quaternionic manifold for $Q$ and the corresponding compatible Special Kähler manifold for $K$ that it has been shown in [4] to be $SU(3,3)/SU(3) \times U(3)$. This is the case we will consider. In [4], via a group-theoretical
investigation of the structure of eq. (1.2) and of the above decomposition, it has been found the answer to the question of how many scalar fields are essentially dynamical, namely cannot be set to constants up to U–duality transformations. Introducing the decomposition (1.3) it has been found that the 40 scalars belonging to \( \text{Solv}_4 \) are constants independent of the radial variable \( r \). Only the 30 scalars in the Kähler algebra \( \text{Solv}_3 \) can be radially dependent. The result in this case is that 64 of the scalar fields are actually constant while 6 are dynamical. Moreover 48 charges are annihilated leaving 8 non-zero charges transforming in the representation \((2, 2)\) of \([\text{SL}(2, \IR)]^3\). Up to U–duality transformations the most general \( N = 8 \) black-hole is actually an \( N = 2 \) black–hole corresponding to a very specific choice of the special Kähler manifold, namely \( \exp[\text{Solv}_3] \) as in eq. (1.4). More precisely, the main result of [4] is that the most general 1/8 black–hole solution of \( N = 8 \) supergravity is related to the group \([\text{SL}(2, \IR)]^3\), namely the general solution is actually determined by the \( STU \) model studied in [8] and based on the solvable subalgebra:

\[
\text{Solv} \left( \frac{\text{SL}(2, \IR)^3}{U(1)^3} \right) \subset \text{Solv} \left( \frac{\text{SU}(3, 3)}{\text{SU}(3) \times U(3)} \right)
\]

(1.6)

The real parts of the 3 complex scalar fields parametrizing \([\text{SL}(2, \IR)]^3\) correspond to the three Cartan generators of \( \text{Solv}_3 \) and have the physical interpretation of radii of the torus compactification from \( D = 10 \) to \( D = 4 \). The imaginary parts of these complex fields are generalised theta angles.

The paper is organized as follows: in section 2 we give the general structure of the 1/8 SUSY preserving solution in the SLA context as an \( STU \) model solution embedded in \( SU(3, 3)/SU(3) \times U(3) \). In section 3 we write down in a algebraic way the killing spinor equation using the SLA formalism and we show how they match with those obtained via the more familiar Special Kähler formalism. In section 4 we discuss the structure and the main properties of the most general solution while in section 5, in order to make a concrete and more manageable example, we give the explicit solution in the simplified case \( S = T = U \). Although simpler, this solution encodes all non-trivial features of the most general one. Section 6 contains some conclusive remarks.

2 Embedding in the \( N = 8 \) theory and solvable Lie algebras

As previously emphasized, the most general 1/8 black–hole solution of \( N = 8 \) supergravity is, up to U–duality transformations, a solution of an \( STU \) model suitably embedded in the original \( N = 8 \) theory. Therefore, in dealing with the \( STU \) model we would like to keep trace of this embedding. To this end, we shall use, as anticipated, the mathematical tool of SLA which in general provides a suitable and simple description of the embedding of a supergravity theory in
a larger one. The SLA formalism is very useful in order to give a geometrical and a quasi easy characterization of the different dynamical scalar fields belonging to the solution. Secondly, it enables one to write down the somewhat heavy first order differential system of equations for all the fields and to compute all the geometrical quantities appearing in the effective supergravity theory in a clear and direct way. Instead of considering the STU model embedded in the whole $N = 8$ theory with scalar manifold $\mathcal{M} = E_7(7)/SU(8)$, it suffices to focus on its $N = 2$ truncation with scalar manifold $\mathcal{M}_{T_6/Z_3} = [SU(3,3)/SU(3) \times U(3)] \times \mathcal{M}_{\text{Quat}}$ which describes the classical limit of type $IIA$ Supergravity compactified on $T_6/Z_3$. $\mathcal{M}_{\text{Quat}}$ being the quaternionic manifold $SO(4,1)/SO(4)$ describing 1 hyperscalar. Within this latter simpler model we are going to construct the $N = 2$ STU model as a consistent truncation. The embedding of the STU scalar manifold $\mathcal{M}_{\text{STU}} = (SL(2,\mathbb{R})/U(1))^3$ inside $\mathcal{M}_{T_6/Z_3}$ and the latter within $\mathcal{M}$ is described in detail in terms of SLA in [4]. In this paper it was shown that up to $H = SU(8)$ transformations, the $N = 8$ central charge which is a $8 \times 8$ antisymmetric complex matrix can always be brought to its normal form in which it is skewdiagonal with complex eigenvalues $Z, Z_i$, $i = 1, 2, 3$ ($|Z| > |Z_i|$). In order to do this one needs to make a suitable 48–parameter $SU(8)$ transformation on the central charge. This transformation may be seen as the result of a 48–parameter $E_7(7)$ duality transformation on the 56 dimensional charge vector and on the 70 scalars which, in the expression of the central charge, sets to zero 48 scalars (24 vector scalars and 24 hyperscalars from the $N = 2$ point of view) and 48 charges. Taking into account that there are 16 scalars parametrizing the submanifold $SO(4,4)/SO(4) \times SO(4)$, $SO(4,4)$ being the centralizer of the normal form, on which the eigenvalues of the central charge do not depend at all, the central charge, in its normal form will depend only on the 6 scalars and 8 charges defining an STU model. The isometry group of $\mathcal{M}_{\text{STU}}$ is $[SL(2,\mathbb{R})]^3$, which is the normalizer of the normal form, i.e. the residual $U$–duality which can still act non trivially on the 6 scalars and 8 charges while keeping the central charge skew diagonalized. As we shall see, the 6 scalars of the STU model consist of 3 axions $a_i$ and 3 dilatons $p_i$, whose exponential $\exp p_i$ will be denoted by $-b_i$. 

In the framework of the STU model, the central charge eigenvalues $Z(a_i, b_i, p^\Lambda, q_\Lambda)$ and $Z_i(a_i, b_i, p^\Lambda, q_\Lambda)$ are, respectively the local realization on moduli space of the $N = 2$ supersymmetry algebra central charge and of the 3 matter central charges associated with the 3 matter vector fields. The BPS condition for a 1/8 black–hole is that the ADM mass should equal the modulus of the central charge:

$$M_{\text{ADM}} = |Z(a_i, b_i, p^\Lambda, q_\Lambda)|. \quad (2.1)$$

At the horizon the field dependent central charge $|Z|$ flows to its minimum value:

$$|Z|_{\text{min}}(p^\Lambda, q_\Lambda) = |Z(a_{i, \text{fix}}^f, b_{i, \text{fix}}^f, p^\Lambda, q_\Lambda)|$$
which is obtained by extremizing it with respect to the 6 moduli \( a_i, b_i \). At the horizon the other eigenvalues \( Z_i \) vanish. The value \( |Z|_{\text{min}} \) is related to the Bekenstein Hawking entropy of the solution and it is expressed in terms of the quartic invariant of the 56-representation of \( E_{7(7)} \), which in principle depends on all the 8 charges of the STU model. Nevertheless there is a residual \([U(1)]^3 \in [SL(2, \mathbb{R})]^3 \) acting on the \( N = 8 \) central charge matrix in its normal form. These three gauge parameters can be used to reduce the number of charges appearing in the quartic invariant (entropy) from 8 to 5. We shall see how these 3 conditions may be implemented on the 8 charges at the level of the first order BPS equations in order to obtain the 5 parameter generating solution for the most general \( 1/8 \) black holes in \( N = 8 \) supergravity.

This generating solution coincides with the solution generating the orbit of \( 1/2 \) BPS black-holes in the truncated \( N = 2 \) model describing type IIA supergravity compactified on \( T_6/Z_3 \). Therefore, in the framework of this latter simpler model, we shall work out the STU model and construct the set of second and first order differential equations defining our solution. In [14] it has been considered the type IIB counterpart of the same model. There, however, the effective \( N = 2 \) supergravity theory was simpler because there were 10 hypermultiplets (which are constant in the solution) and no vector multiplets, the only vector in the game being the graviphoton.

### 2.1 The STU model in the \( SU(3, 3)/SU(3) \times U(3) \) theory and solvable Lie algebras

As it was shown in [4] the hyperscalars do not contribute to the dynamics of our BPS black-hole, therefore, in what follows, all hyperscalars will be set to zero and we shall forget about the quaternionic factor \( \mathcal{M}_{\text{Quat}} \) in \( \mathcal{M}_{T_6/Z_3} \). The latter will then be the scalar manifold of an \( N = 2 \) supergravity describing 9 vector multiplets coupled with the graviton multiplet. The 18 real scalars span the manifold \( \mathcal{M}_{T_6/Z_3} = SU(3, 3)/SU(3) \times U(3) \), while the 10 electric and 10 magnetic charges associated with the 10 vector fields transform under duality in the \( 20 \) (three times antisymmetric) of \( SU(3, 3) \). As anticipated, in order to show how the STU scalar manifold \( \mathcal{M}_{\text{STU}} \) is embedded in \( \mathcal{M}_{T_6/Z_3} \) we shall use the SLA description.

Apparently a great variety of scalar manifolds in extended supergravities in different dimensions are non-compact Riemannian manifolds \( \mathcal{M} \) admitting a solvable Lie algebra description, i.e. they can be expressed as Lie group manifolds generated by a solvable Lie algebra \( \text{Solv} \):

\[
\mathcal{M} = \exp(\text{Solv})
\]

For instance non-compact homogeneous manifolds of the form \( G/H \) (\( H \) maximal compact subgroup of \( G \)) always admit a solvable Lie algebra representation and \( \text{Solv} \) is defined by the
so called Iwasawa decomposition. A solvable algebra $Solv$ is defined as an algebra for which the $k^{th}$ Lie derivative vanishes for a finite $k$:

\[
\begin{align*}
D^{(k)}(Solv) &= 0 \\
D^{(n)}(A) &= [D^{(n-1)}(A), D^{(n-1)}(A)] \\
D^{(1)}(A) &= [A, A]
\end{align*}
\] (2.4)

In the solvable representation of a manifold (2.3) the local coordinates of the manifold are the parameters of the generating Lie algebra, therefore adopting this parametrization of scalar manifolds in supergravity implies the definition of a one to one correspondence between the scalar fields and the generators of $Solv$ [15, 16].

Special Kähler manifolds and Quaternionic manifolds admitting such a description have been classified in the 70’s by Alekseevskii [17]. The simplest example of solvable Lie algebra parametrization is the case of the two dimensional manifold $\mathcal{M} = SL(2, \mathbb{R})/SO(2)$ which may be described as the exponential of the following solvable Lie algebra:

\[
\begin{align*}
SL(2, \mathbb{R})/SO(2) &= \exp(Solv) \\
Solv &= \{\sigma_3, \sigma_+\} \\
[\sigma_3, \sigma_+] &= 2\sigma_+ \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{align*}
\] (2.5)

From (2.5) we can see a general feature of $Solv$, i.e. it may always be expressed as the direct sum of semisimple (the non-compact Cartan generators of the isometry group) and nilpotent generators, which in a suitable basis are represented respectively by diagonal and upper triangular matrices. This property, as we shall see, is one of the advantages of the solvable Lie algebra description since it allows to express the coset representative of an homogeneous manifold as a solvable group element which is the product of a diagonal matrix and the exponential of a nilpotent matrix, which is a polynomial in the parameters. The simple solvable algebra represented in (2.5) is called key algebra and will be denoted by $F$. The scalar manifold of the $STU$ model is a special Kähler manifold generated by a solvable Lie algebra which is the sum of 3 commuting key algebras:

\[
\begin{align*}
\mathcal{M}_{STU} &= \left( \frac{SL(2, \mathbb{R})}{SO(2)} \right)^3 = \exp(Solv_{STU}) \\
Solv_{STU} &= F_1 \oplus F_2 \oplus F_3 \\
F_i &= \{h_i, g_i\}; \quad [h_i, g_i] = 2g_i \\
[F_i, F_j] &= 0
\end{align*}
\] (2.6)
the parameters of the Cartan generators $h_i$ are the dilatons of the theory, while the parameters of the nilpotent generators $g_i$ are the axions. The three $SO(2)$ isotropy groups of the manifold are generated by the three compact generators $\tilde{g}_i = g_i - g_i^\dagger$.

$\mathcal{M}_{T_6/\mathbb{Z}_3}$ is an 18–dimensional Special Kähler manifold generated by a solvable algebra whose structure is slightly more involved:

$$\mathcal{M}_{T_6/\mathbb{Z}_3} = \frac{SU(3,3)}{SU(3) \times U(3) \times U(1)} = \exp(Solv)$$

$$Solv = Solv_{STU} \oplus X \oplus Y \oplus Z$$

(2.7)

The 4 dimensional subspaces $X, Y, Z$ consist of nilpotent generators, while the only semisimple generators are the 3 Cartan generators contained in $Solv_{STU}$ which define the rank of the manifold. The algebraic structure of $Solv$ together with the details of the construction of the $SU(3,3)$ generators in the representation $20$ can be found in Appendix A. Eq. (2.7) defines the embedding of $\mathcal{M}_{STU}$ inside $\mathcal{M}_{T_6/\mathbb{Z}_3}$, i.e. tells which scalar fields have to be put to zero in order to truncate the theory to the $SU(3,3)$ model. As far as the embedding of the isotropy group $SO(2)^3$ of $\mathcal{M}_{STU}$ inside the $\mathcal{M}_{T_6/\mathbb{Z}_3}$ isotropy group $SU(3)_1 \times SU(3)_2 \times U(1)$ is concerned, the 3 generators of the former ($\{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3\}$) are related to the Cartan generators of the latter in the following way:

$$\tilde{g}_1 = \frac{1}{2} \left( \lambda + \frac{1}{2} \left( H_{c_1} - H_{d_1} + H_{c_1+c_2} - H_{d_1+d_2} \right) \right)$$

$$\tilde{g}_2 = \frac{1}{2} \left( \lambda + \frac{1}{2} \left( H_{c_1} - H_{d_1} - 2(H_{c_1+c_2} - H_{d_1+d_2}) \right) \right)$$

$$\tilde{g}_3 = \frac{1}{2} \left( \lambda + \frac{1}{2} \left( -2(H_{c_1} - H_{d_1}) + (H_{c_1+c_2} - H_{d_1+d_2}) \right) \right)$$

(2.8)

where $\{c_i\}, \{d_i\}, i = 1, 2$ are the simple roots of $SU(3)_1$ and $SU(3)_2$ respectively, while $\lambda$ is the generator of $U(1)$. In order to perform the truncation to the $STU$ model, one needs to know also which of the 10 vector fields have to be set to zero in order to be left with the 4 $SU(3)$ vector fields. This information is given by the decomposition of the $20$ of $SU(3,3)$ in which the vector of magnetic and electric charges transform, with respect to $[SL(2,\mathbb{R})]^3$:

$$20 \overset{SL(2,\mathbb{R})^3}{\rightarrow} (2,2,2) \oplus 2 \times [(2,1,1) \oplus (1,2,1) \oplus (1,1,2)]$$

(2.9)

Skewdiagonalizing the 5 Cartan generators of $SU(3)_1 \times SU(3)_2 \times U(1)$ on the $20$ we obtain the 10 positive weights of the representation as 5 components vectors $\tilde{v}^i_{\Lambda}$ ($\Lambda' = 0, \ldots, 9$):

$$\{C(n)\} = \{ \frac{H_{c_1}}{2}, \frac{H_{c_1+c_2}}{2}, \frac{H_{d_1}}{2}, \frac{H_{d_1+d_2}}{2}, \lambda \}$$

$$C(n) \cdot |v_x^{\Lambda_i} \rangle = v_{(x)}^{\Lambda_i} |v_x^{\Lambda_i} \rangle$$

$$C(n) \cdot |v_y^{\Lambda_i} \rangle = -v_{(y)}^{\Lambda_i} |v_y^{\Lambda_i} \rangle$$

(2.10)
Using the relation (2.8) we compute the value of the weights $v^A$ on the three generators $\tilde{g}_i$ and find out which are the 4 positive weights $\tilde{v}^A (\Lambda = 0, \ldots, 3)$ of the $(2, 2, 2)$ in (2.9). The weights $\tilde{v}^A$ and their eigenvectors $|v^A_{x,y}\rangle$ are listed in Appendix A.

In this way we achieved an algebraic recipe to perform the truncation to the STU model: setting to zero all the scalars parametrizing the 12 generators $X \oplus Y \oplus Z$ in (2.7) and the 6 vector fields corresponding to the weights $v^\Lambda, \Lambda = 4, \ldots, 9$. Restricting the action of the $[SL(2, \mathbb{R})]^3$ generators $(h_i, g_i, \tilde{g}_i)$ inside $SU(3,3)$ to the 8 eigenvectors $|v^\Lambda_{x,y}\rangle (\Lambda = 0, \ldots, 3)$ the embedding of $[SL(2, \mathbb{R})]^3$ in $Sp(8)$ is automatically obtained 1.

3 First order differential equations: the algebraic approach

Now that the STU model has been constructed out the original $SU(3,3)/SU(3) \times U(3)$ model, we may address the problem of writing down the BPS first order equations. To this end we shall use the geometrical intrinsic approach defined in [4] and eventually compare it with the Special Kähler geometry formalism.

The system of first order differential equations in the background fields is obtained from the Killing spinor conditions (1.2). The expressions of the gravitino and gaugino supersymmetry transformation are:

$$\delta \epsilon \psi_{A|\mu} = \nabla_\mu \epsilon_A - \frac{1}{4} T_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu \epsilon_{AB} \epsilon^B$$
$$\delta \epsilon \lambda^{i|A} = i \nabla_\mu \gamma_\mu \epsilon_A + G^{-1}_{\rho\sigma} \gamma^{\rho\sigma} \epsilon_{AB} \epsilon_B$$ (3.1)

where $i = 1, 2, 3$ labels the three matter vector fields and $A, B = 1, 2$ are the $SU(2)$ R-symmetry indices. Following the procedure defined in [4, 5, 18], in order to obtain a system of first order differential equations out of the killing spinor conditions (1.2) we make the following ansätze for the vector fields:

$$F^{-|\Lambda} = \frac{t^\Lambda}{4\pi} F^-$$
$$t^\Lambda(r) = 2\pi (p^\Lambda + i \ell^\Lambda(r))$$
$$F^\Lambda = 2Re F^{-|\Lambda}; \tilde{F}^\Lambda = -2Im F^{-|\Lambda}$$

$${F^\Lambda} = \frac{p^\Lambda}{2\mu^3} e^{abc} dx^a \wedge dx^b \wedge dx^c - \frac{e^{\Lambda}(r)}{r^3} e^{2\mu} dt \wedge \vec{x} \cdot d\vec{x}$$

1In the $Sp(8)$ representation of the U–duality group $[SL(2, \mathbb{R})]^3$ we shall use the non–compact Cartan generators $h_i$ are diagonal. Such a representation will be denoted by $Sp(8)_D$, where the subscript “D” stands for “Dynkin”. This notation has been introduced in [4] to distinguish the representation $Sp(8)_D$ from $Sp(8)_Y$ (“Y” standing for “Young”) where on the contrary the Cartan generators of the compact isotropy group (in our case $\tilde{g}_i$) are diagonal. The two representations are related by an orthogonal transformation.
\[ F^\Lambda = -\frac{\ell^\Lambda(r)}{2\pi^3} \epsilon_{abc} x^a dx^b \wedge dx^c - \frac{p^\Lambda}{r^3} e^{2\ell t} dt \wedge \vec{x} \cdot d\vec{x} \]  

(3.2)

where

\[ E^- = \frac{1}{2\pi^3} \epsilon_{abc} x^a dx^b \wedge dx^c + \frac{i e^{2\ell t}}{r^3} dt \wedge \vec{x} \cdot d\vec{x} = E^-_{ab} dx^b \wedge dx^c + 2 E^-_{0a} dt \wedge dx^a \]

\[ 4\pi = \int_{S^2_r} E^-_{ab} dx^a \wedge dx^b \]  

(3.3)

Integrating on a two–sphere \( S^2_r \) of radius \( r \) we obtain

\[ 4\pi p^\Lambda = \int_{S^2_r} F^\Lambda = \int_{S^2_r} F^\Lambda = 2\text{Re} t^\Lambda \]  

\[ 4\pi t^\Lambda(r) = -\int_{S^2_r} \tilde{F}^\Lambda = 2\text{Im} t^\Lambda \]  

(3.4)

The difference between the two results is evident. In the first case the integrand is a closed two form and hence the choice of the 2–cycle representative is immaterial. In the second case the integrand is not closed and hence the result depends on the radius of the integration sphere.

As far as the metric \( g_{\mu\nu} \), the scalars \( z^i \) and the Killing spinors \( \epsilon_A(r) \) are concerned, the ansatze we adopt are the following:

\[ ds^2 = e^{2U(r)} dt^2 - e^{-2U(r)} dx^2 \quad \left( r^2 = \vec{x}^2 \right) \]

\[ z^i \equiv z^i(r) \]

\[ \epsilon_A(r) = e^{f(r)} \xi_A \quad \xi_A = \text{constant} \]

\[ \gamma_0 \xi_A = \pm i \epsilon_{AB} \xi^B \]  

(3.5)

As usual we represent the scalars of the \( STU \) model in terms of three complex fields \( \{ z^i \} \equiv \{ S, T, U \} \), parametrizing each of the three factors \( SL(2, \mathbb{R})/SO(2) \) in \( \mathcal{M}_{STU} \). After some algebra, one obtains the following set of first order equations:

\[ \frac{dz^i}{dr} = \mp \left( e^{U(r)} \right) \frac{4\pi r^2}{4\pi r^2} g^{ij} \nabla_j (\mathcal{N} - \mathcal{N})_{\Lambda \Sigma} t^\Sigma \]

\[ = \mp \left( e^{U(r)} \right) \frac{4\pi r^2}{4\pi r^2} g^{ij} \nabla_j (z, \bar{z}, p, q) \]

\[ \frac{dU}{dr} = \mp \left( e^{U(r)} \right) \frac{4\pi r^2}{r^2} (M_{\Sigma} p^\Sigma - L^\Lambda q_\Lambda) = \mp \left( e^{U(r)} \right) \frac{4\pi r^2}{r^2} Z(z, \bar{z}, p, q) \]  

(3.6)

where \( N_{\Lambda \Sigma}(z, \bar{z}) \) is the symmetric usual matrix entering the action for the vector fields in (4.1). The vector \( (L^\Lambda(z, \bar{z}), M_{\Sigma}(z, \bar{z})) \) is the covariantly holomorphic section on the symplectic bundle.
defined on the Special Kähler manifold $\mathcal{M}_{STU}$. Finally $Z(z, \bar{z}, p, q)$ is the local realization on $\mathcal{M}_{STU}$ of the central charge of the $N = 2$ superalgebra, while $Z^i(z, \bar{z}, p, q) = \mathcal{O}_j^i \nabla_j \bar{Z}(z, \bar{z}, p, q)$ are the central charges associated with the matter vectors, the so-called matter central charges. In writing eqs. (3.6) the following two properties have been used:

\begin{equation}
0 = \mathcal{H}_j^i |\Lambda t^* \Sigma - \mathcal{F}_j^i |\Lambda N_{\Sigma} t^* \Sigma
0 = M_{\Sigma} t^* \Sigma - L^i N_{\Lambda} t^* \Sigma
\end{equation}

(3.7)

The electric charges $\ell^\Lambda(r)$ defined in (3.4) are moduli dependent charges which are functions of the radial direction through the moduli $a_i, b_i$. On the other hand, the moduli independent electric charges $q_{\Lambda}$ in eqs. (3.6) are those that together with $p^\Lambda$ fulfill the Dirac quantization condition, and are expressed in terms of $t^\Lambda(r)$ as follows:

\begin{equation}
q_{\Lambda} = \frac{1}{2\pi} \text{Re}(N(z(r), \bar{z}(r)) t(r))_{\Lambda}
\end{equation}

(3.8)

Equation (3.8) may be inverted in order to find the moduli dependence of $\ell_{\Lambda}(r)$. The independence of $q_{\Lambda}$ on $r$ is a consequence of one of the Maxwell’s equations:

\begin{equation}
\partial_a \left( \sqrt{-g} \hat{G}^{00}\right) = 0 \Rightarrow \partial_a \text{Re}(N(z(r), \bar{z}(r)) t(r))_{\Lambda} = 0
\end{equation}

(3.9)

In order to compute the explicit form of eqs. (3.6) in a geometrical intrinsic way [4] we need to decompose the 4 vector fields into the graviphoton $F_{\mu\nu}$ and the matter vector fields $F^i_{\mu\nu}$ in the same representation of the scalars $z^i$ with respect to the isotropy group $H = [SO(2)]^3$. This decomposition is immediately performed by computing the positive weights $\mathcal{V}^\Lambda$ of the $(2, 2, 2)$ on the three generators $\{\mathcal{G}_i\}$ of $H$ combined in such a way as to factorize in $H$ the automorphism group $H_{\text{aut}} = SO(2)$ of the supersymmetry algebra generated by $\lambda = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3$ from the remaining $H_{\text{matter}} = [SO(2)]^2 = \{\mathcal{G}_1 - \mathcal{G}_2, \mathcal{G}_1 - \mathcal{G}_3\}$ generators acting non trivially only on the matter fields. The real and imaginary components of the graviphoton central charge $Z$ will be associated with the weight, say $\mathcal{V}^0$ having vanishing value on the generators of $H_{\text{matter}}$. The remaining weights will define a representation $(2, 1, 1) \oplus (1, 2, 1) \oplus (1, 1, 2)$ of $H$ in which the real and imaginary parts of the central charges $Z_i$ associated with $F^i_{\mu\nu}$ transform and will be denoted by $\mathcal{V}^i$, $i = 1, 2, 3$. This representation is the same as the one in which the 6 real scalar components of $z^i = a_i + ib_i$ transform with respect to $H$. It is useful to define on the tangent space of $\mathcal{M}_{STU}$ curved indices $\alpha$ and rigid indices $\hat{\alpha}$, both running form 1 to 6. Using the solvable parametrization of $\mathcal{M}_{STU}$, which defines real coordinates $\phi^\alpha$, the generators of $\text{Solv}_{STU} = \{T^\alpha\}$ carry curved indices since they are parametrized by the coordinates, but do not transform in a representation of the isotropy group. The compact generators $\mathcal{K} = \text{Solv}_{STU} + \text{Solv}^1_{STU}$ of $[SL(2, \mathbb{R})]^3$ on the other hand transform in the $(2, 1, 1) \oplus (1, 2, 1) \oplus (1, 1, 2)$ of $H$ and we
can choose an orthonormal basis (with respect to the trace) for $\mathbb{K}$ consisting of the generators $\mathbb{K}^\alpha = T^\alpha + T^{\alpha\dagger}$. These generators now carry the rigid index and are in one to one correspondence with the real scalar fields $\phi^\alpha$. There is a one to one correspondence between the non–compact matrices $\mathbb{K}^\alpha$ and the eigenvectors $|v_{x,y}^i\rangle$ ($i = 1, 2, 3$) which are orthonormal basis (in different spaces) of the same representation of $H$:

$$\begin{align*}
\{\mathbb{K}^1, \mathbb{K}^2, \mathbb{K}^3, \mathbb{K}^4, \mathbb{K}^5, \mathbb{K}^6\} & \leftrightarrow \{ |v_{x}^1\rangle, |v_{x}^2\rangle, |v_{x}^3\rangle, |v_{y}^1\rangle, |v_{y}^2\rangle, |v_{y}^3\rangle\} \\
\{\{v^\alpha\}\} & \quad (3.10)
\end{align*}$$

The relation between the real parameters $\phi^\alpha$ of the SLA and the real and imaginary parts of the complex fields $z^i$ is:

$$\{ \phi^\alpha \} \equiv \{-2a_1, -2a_2, -2a_3, \log (-b_1), \log (-b_2), \log (-b_3), \} \quad (3.11)$$

Using the $Sp(8)_D$ representation of $\text{Solv}_{STU}$, we construct the coset representative $\mathbb{L}(\phi^\alpha)$ of $\mathcal{M}_{STU}$ and the vielbein $\mathbb{F}^\alpha$ as follows:

$$\begin{align*}
\mathbb{L}(a_i, b_i) & = \exp (T_\alpha \phi^\alpha) = (1 - 2a_1 g_1) \cdot (1 - 2a_2 g_2) \cdot (1 - 2a_3 g_3) \cdot \exp \left( \sum_i \log (-b_i) h_i \right) \\
\mathbb{F}^\alpha & = \frac{1}{2\sqrt{2}} \text{Tr} \left( \mathbb{K}^\alpha \mathbb{L}^{-1} d\mathbb{L} \right) = \{ -\frac{da_1}{2b_1}, -\frac{da_2}{2b_2}, -\frac{da_3}{2b_3}, \frac{db_1}{2b_1}, \frac{db_2}{2b_2}, \frac{db_3}{2b_3} \} \\
& \quad (3.12)
\end{align*}$$

The scalar kinetic term in the $N = 2$ lagrangian (4.1) is expressed in terms of the vielbein $\mathbb{F}$ in the form $\sum_\alpha (\mathbb{F}_\alpha)^2$. The following relations between quantities computed in the solvable approach and Special Kähler formalism hold:

$$\begin{align*}
\langle \bar{v}^0 \mathbb{L}^\dagger \mathbb{C} \mathbb{M} \rangle & = \sqrt{2} \begin{pmatrix} \text{Re}(g^{ij'} (\bar{h}_{j'}|_A)), -\text{Re}(g^{ij'} (\bar{f}_{j'}^\mathbb{E})) \\ \text{Im}(g^{ij'} (\bar{h}_{j'}|_A)), -\text{Im}(g^{ij'} (\bar{f}_{j'}^\mathbb{E})) \end{pmatrix} \\
\langle \bar{v}^0_1 \mathbb{L}^\dagger \mathbb{C} \mathbb{M} \rangle & = \sqrt{2} \begin{pmatrix} \text{Re}(M_A), -\text{Re}(L^\mathbb{E}) \\ \text{Im}(M_A), -\text{Im}(L^\mathbb{E}) \end{pmatrix} \\
& \quad (3.13)
\end{align*}$$

where in the first equation both sides are $6 \times 8$ matrix in which the rows are labeled by $\alpha$. The first three values of $\alpha$ correspond to the axions $a_i$, the last three to the dilatons $\log(-b_i)$. The columns are to be contracted with the vector consisting of the 8 electric and magnetic charges $|\bar{Q}\rangle_{sc} = 2\pi (p^A, q_E)$ in the *special coordinate* symplectic gauge of $\mathcal{M}_{STU}$. In eqs. (3.13) $\mathbb{C}$ is the symplectic invariant matrix, while $\mathbb{M}$ is the symplectic matrix relating the charge vectors in the $Sp(8)_D$ representation and in the *special coordinate* symplectic gauge:

$$|\bar{Q}\rangle_{Sp(8)_D} = \mathbb{M} \cdot |\bar{Q}\rangle_{sc}$$
Using eqs. (3.13) it is now possible to write in a geometrically intrinsic way the first order equations:

\[ \frac{d\phi_\alpha}{dr} = \left( \mp \epsilon U \right) \sqrt{2\pi} I^\alpha_i \langle \bar{V}^i \mid \mathcal{L} M \mid \bar{Q} \rangle_{sc} - \frac{1}{2\epsilon U} \right) \]

\[ \frac{dU}{dr} = \left( \mp \epsilon U \right) \sqrt{2\pi} \langle \bar{V}^0 \mid \mathcal{L} M \mid \bar{Q} \rangle_{sc} - \frac{1}{2\epsilon U} \right) \]

\[ 0 = \langle \bar{V}^0 \mid \mathcal{L} M \mid \ell \rangle_{sc} \] (3.15)

The full explicit form of eqs. (3.15) can be found in Appendix B where, using eq. (3.8), everything is expressed in terms of the quantized moduli-independent charges \((q_\Lambda, p^\Sigma)\). The fixed values of the scalars at the horizon are obtained by setting the right hand side of the above equations to zero and the result is consistent with the literature ([8]):

\[ (a_1 + ib_1)_{fix} = \frac{p^\Lambda q_\Lambda - 2p^1 q_1 - i\sqrt{f(p,q)}}{2p^2 p^3 - 2p^0 q_1} \]

\[ (a_2 + ib_2)_{fix} = \frac{p^\Lambda q_\Lambda - 2p^2 q_2 - i\sqrt{f(p,q)}}{2p^1 p^3 - 2p^0 q_2} \]

\[ (a_3 + ib_3)_{fix} = \frac{p^\Lambda q_\Lambda - 2p^3 q_3 - i\sqrt{f(p,q)}}{2p^1 p^2 - 2p^0 q_3} \] (3.16)

where \(f(p,q)\) is the \(E_7(7)\) quartic invariant \(I_4(p,q)\) expressed as a function of all the 8 charges (and whose square root is proportional to the entropy of the solution):

\[ f(p,q) = -(p^0 q_0 - p^1 q_1 + p^2 q_2 + p^3 q_3)^2 + 4(p^2 p^3 - p^0 q_1)(p^1 q_0 + q_2 q_3) \] (3.17)

The last of eqs. (3.15) expresses the reality condition for \(Z(\phi, p, q)\) and it amounts to fix one of the three \(SO(2)\) gauge symmetries of \(H\) giving therefore a condition on the 8 charges. Without spoiling the generality (up to \(U\)-duality) of the black–hole solution it is still possible to fix the remaining \([SO(2)]^2\) gauges in \(H\) by imposing two conditions on the phases of the \(Z^i(\phi, p, q)\).
For instance we could require two of the $Z_i(\phi,p,q)$ to be imaginary. This would imply two more conditions on the charges, leading to a generating solution depending only on 5 parameters as we expect it to be [19]. Hence we can conclude with the following:

**Statement 3.1** Since the radial evolution of the axion fields $a_i$ is related to the real part of the corresponding central charge $Z_i(\phi,p,q)$ (see (3.6)), up to $U$ duality transformations, the 5 parameter generating solution will have 3 dilatons and 1 axion evolving from their fixed value at the horizon to the boundary value at infinity, and 2 constant axions whose value is the corresponding fixed one at the horizon (double fixed).

4 The solution: preliminaries and comments on the most general one

In order to find the solution of the $STU$ model we need also the equations of motion that must be satisfied together with the first order ones. We go on using the Special Kähler formalism in order to let the comparison to previous papers being more immediate. Let us first compute the field equations for the scalar fields $z_i$, which can be obtained from an $N = 2$ pure supergravity action coupled to 3 vector multiplets. From the action [20]:

\[
S = \int d^4x \sqrt{-g} \mathcal{L}
\]

\[
\mathcal{L} = R[g] + h_{ij}^\star \partial_\mu z^i \partial_\mu z^j + \left( \text{Im} N_{\Lambda\Sigma} F^\Lambda F^\Sigma \right) + \left( \text{Re} N_{\Lambda\Sigma} F^\Lambda \tilde{F}^\Sigma \right)
\]

\[
g_{\mu\nu} = \text{diag}(e^{2U}, e^{-2U}, e^{-2U}, e^{-2U})
\]

where $h_{ij}^\star(z, \bar{z})$ denotes the realization of the metric on the scalar manifold in a local coordinate chart. Maxwell’s equations :

The field equations for the vector fields and the Bianchi identities read:

\[
\partial_\mu \left( \sqrt{-g} \tilde{G}^{\mu\nu} \right) = 0
\]

\[
\partial_\mu \left( \sqrt{-g} \tilde{F}^{\mu\nu} \right) = 0
\]

Using the ansatze (3.2) the second equation is automatically fulfilled while the first equation, as it was anticipated in section 3, requires the quantized electric charges $q_\Lambda$ defined by eq. (3.8) to be $r$-independent (eq. (3.9)).

Scalar equations :

varying with respect to $z^i$ one gets:

\[
-\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} h_{ij} \partial_\nu z^j \right) + \partial_i (h_{kj}) \partial_\mu z^k \partial_\nu z^j g^{\mu\nu} + (\partial_i \text{Im} N_{\Lambda\Sigma} F^\Lambda F^{\Sigma}) + (\partial_i \text{Re} N_{\Lambda\Sigma} F^\Lambda \tilde{F}^{\Sigma}) = 0
\]

(4.3)
which, once projected onto the real and imaginary parts of both sides, read:

\[
\frac{e^{2U}}{4b_i^2} \left( a''_i + 2 \frac{a'_i}{r} - 2 \frac{a''_i}{b_i} \right) = -\frac{1}{2} \left( (\partial_a \text{Im} \mathcal{N}_{\Sigma}) F^\Lambda_{\Sigma} F^{\Sigma | \cdot} + (\partial_a \text{Re} \mathcal{N}_{\Sigma}) F^\Lambda_{\Sigma} \tilde{F}^{\Sigma | \cdot} \right)
\]

\[
\frac{e^{2U}}{4b_i^2} \left( b''_i + \frac{2b'_i}{r} + \frac{(a_i'^2 - b_i'^2)}{b_i} \right) = -\frac{1}{2} \left( (\partial_b \text{Im} \mathcal{N}_{\Lambda}) F^\Lambda_{\Sigma} F^{\Sigma | \cdot} + (\partial_b \text{Re} \mathcal{N}_{\Lambda}) F^\Lambda_{\Sigma} \tilde{F}^{\Sigma | \cdot} \right)
\] (4.4)

Einstein equations:

Varying the action (4.1) with respect to the metric we obtain the following equations:

\[
R_{MN} = -h_{ij} \partial_M z_i \partial_N z_j + S_{MN}
\]

\[
S_{MN} = -2 \text{Im} \mathcal{N}_{\Lambda \Sigma} \left( F^\Lambda_M F^{\Sigma |} - \frac{1}{4} g_{MN} F^\Lambda \tilde{F}^{\Sigma |} \right) +
\]

\[
-2 \text{Re} \mathcal{N}_{\Lambda \Sigma} \left( F^\Lambda_M \tilde{F}^{\Sigma |} - \frac{1}{4} g_{MN} F^\Lambda \tilde{F}^{\Sigma |} \right)
\] (4.5)

Projecting on the components \((M, N) = (0, 0)\) and \((M, N) = (a, b)\), respectively, these equations can be written in the following way:

\[
U'' + \frac{2}{r} U' = -2 e^{-2U} S_{00}
\]

\[
(U')^2 + \sum_i \frac{1}{4b_i^2} ((b_i')^2 + (a_i')^2) = -2 e^{-2U} S_{00}
\] (4.6)

where:

\[
S_{00} = -\frac{2 e^{4U}}{(8\pi)^2 r^4} \text{Im} \mathcal{N}_{\Lambda \Sigma} (p^\Lambda p^\Sigma + \ell(r)^\Lambda \ell(r)^\Sigma)
\] (4.7)

In order to solve these equations one would need to explicitate the right hand side expression in terms of scalar fields \(a_i, b_i\) and quantized charges \((p^\Lambda, q_\Sigma)\). In order to do that, one has to consider the ansatz for the field strengthes (3.2) substituting to the moduli-dependent charges \(q_\Lambda(r)\) appearing in the previous equations their expression in terms of the quantized charges obtained by inverting eq.(3.8):

\[
\ell^\Lambda(r) = \text{Im} N^{-1 | \Lambda \Sigma} \left( q_\Sigma - \text{Re} N_{\Sigma \Omega} p^\Omega \right)
\] (4.8)

Using now the expression for the matrix \(N\) in eq. (A.9) of Appendix A, one can find the explicit expression of the scalar fields equations of motion written in terms of the quantized \(r\)-independent charges. In Appendix B we report the full explicit expression of the equations of motion for both the scalars and the metric. Let us stress that in order to find the 5 parameter generating solution of the STU model it is not sufficient to substitute to each charge, in the scalar fixed values of eq.(3.16), a corresponding harmonic function \((q_i \to H_i = 1 + q_i/r)\). As already explained, the generating solution should depend on 5 parameters and 4 harmonic
functions, as in \cite{10}. In particular, as explained above, 2 of the 6 scalar fields parametrizing the \textit{STU} model, namely 2 axion fields, should be taken to be constant. Therefore, in order to find the generating solution one as to solve the two systems of eq.s (B.1) (first order) and (B.2) (second order) explicitely putting as an external input the information on the constant nature of 2 of the 3 axion fields. As it is evident from the above quoted system of eq.s, it is quite difficult to give a not double extreme solution of the combined system that is both explicit and manageable. It is our aim, however, to work it out in a forthcoming paper \cite{21}.

5 The solution: a simplified case, namely $S = T = U$

In order to find a fully explicit solution we can deal with, let us consider the particular case where $S = T = U$. Although simpler, this solution encodes all non-trivial aspects of the most general one: it is regular, i.e. has non-zero entropy, and the scalars do evolve, i.e. it is an extreme but \textit{not} double extreme solution. First of all let us notice that eq.s (B.1) remain invariant if the same set of permutations are performed on the triplet of subscripts $(1, 2, 3)$ in both the fields and the charges. Therefore the solution $S = T = U$ implies the positions $q_1 = q_2 = q_3 \equiv q$ and $p^1 = p^2 = p^3 \equiv p$ on the charges and therefore it will correspond to a solution depending on (apparently only) 4 charges $(p^0, p, q_0, q)$ instead of 8. Moreover, according to this identification, what we do expect now, is to find a solution which depends on (apparently) only 3 independent charges and 2 harmonic functions. Notice that this is not simply an axion–dilaton black–hole: such a solution would have a vanishing entropy differently from our case. The fact that we have just one complex field in our solution is because the three complex fields are taken to be equal in value. The equations (B.1) simplify in the following way:

$$
\begin{align*}
\frac{da}{dr} &= \pm \left( \frac{\epsilon^{(r)}}{r^2} \right) \frac{1}{\sqrt{-2b}} \left( bq - 2abp + \left( a^2 b + b^3 \right) p^0 \right) \\
\frac{db}{dr} &= \pm \left( \frac{\epsilon^{(r)}}{r^2} \right) \frac{1}{\sqrt{-2b}} \left( 3aq - \left( 3a^2 + b^2 \right) p + \left( a^3 + ab^2 \right) p^0 + q_0 \right) \\
\frac{dU}{dr} &= \pm \left( \frac{\epsilon^{(r)}}{r^2} \right) \left( \frac{1}{2\sqrt{2(-b)^3/2}} \right) \left( 3aq - \left( 3a^2 - 3b^2 \right) p + \left( a^3 - 3ab^2 \right) p^0 + q_0 \right) \\
0 &= 3bq - 6abp + \left( 3a^2 b - b^3 \right) p^0
\end{align*}
$$

(5.1)

where $a \equiv a_i$, $b \equiv b_i$ ($i = 1, 2, 3$). In this case the fixed values for the scalars $a, b$ are:

$$
a_{\text{fix}} = \frac{pq + p^0 q_0}{2p^2 - 2p^0 q}
$$

16
\[ b_{\text{fix}} = -\frac{\sqrt{f(p, q, p^0, q_0)}}{2(p^2 - p^0 q)} \]

where \( f(p, q, p^0, q_0) = 3 p^2 q^2 + 4 p^3 q_0 - 6 p p^0 q q_0 - p^0 \left(4 q^3 + p^0 q_0^2 \right) \) \hspace{1cm} (5.2)

Computing the central charge at the fixed point \( Z_{\text{fix}}(p, q, p^0, q_0) = Z(a_{\text{fix}}, b_{\text{fix}}, p, q, p^0, q_0) \) one finds:

\[
\begin{align*}
Z_{\text{fix}}(p, q, p^0, q_0) &= |Z_{\text{fix}}|^e^\theta \\
|Z_{\text{fix}}(p, q, p^0, q_0)| &= f(p, q, p^0, q_0)^{1/4} \\
\sin \theta &= \frac{p^0 f(p, q, p^0, q_0)^{1/2}}{2(p^2 - qp^0)^{3/2}} \\
\cos \theta &= \frac{-2 p^3 + 3 p p^0 q + p^0 q_0}{2 (p^2 - q p^0)^{3/2}} \hspace{1cm} (5.3)
\end{align*}
\]

The value of the \( U \)–duality group quartic invariant (whose square root is proportional to the entropy) is:

\[
I_4(p, q, p^0, q_0) = |Z_{\text{fix}}(p, q, p^0, q_0)|^4 = f(p, q, p^0, q_0) \hspace{1cm} (5.4)
\]

We see form eqs.(5.3) that in order for \( Z_{\text{fix}} \) to be real and the entropy to be non vanishing the only possibility is \( p^0 = 0 \) corresponding to \( \theta = \pi \). It is in fact necessary that \( \sin \theta = 0 \) while keeping \( f \neq 0 \). We are therefore left with 3 independent charges \((q, p, q_0)\), as anticipated.

### 5.1 Solution of the 1st order equations

Setting \( p^0 = 0 \) the fixed values of the scalars and the quartic invariant become:

\[
\begin{align*}
a_{\text{fix}} &= \frac{q}{2p} \\
b_{\text{fix}} &= -\frac{\sqrt{3q^2 + 4q_0 p}}{2p} \\
I_4 &= (3q^2 p^2 + 4q_0 p^3) \hspace{1cm} (5.5)
\end{align*}
\]

From the last of eq.s (5.1) we see that in this case the axion is double fixed, namely does not evolve, \( a \equiv a_{\text{fix}} \) and the reality condition for the central charge is fulfilled for any \( r \). Of course, also the axion equation is fulfilled and therefore we are left with two axion–invariant equations for \( b \) and \( U \):

\[
\begin{align*}
\frac{db}{dr} &= \pm \frac{e^U}{r^2 \sqrt{-2b}} (q_0 + \frac{3q^2}{4p} - b^2 p) \\
\frac{dU}{dr} &= \pm \frac{e^U}{r^2 (-2b)^{3/2}} (q_0 + \frac{3q^2}{4p} + 3b^2 p) \hspace{1cm} (5.6)
\end{align*}
\]

17
which admit the following solution:

\[
\begin{align*}
    b(r) &= -\sqrt{\frac{(A_1 + k_1/r)}{(A_2 + k_2/r)}} \\
    e^{\mu} &= \left(\frac{A_2 + k_2}{r}\right)^3(A_1 + k_1/r)^{-1/4} \\
    k_1 &= \pm \frac{\sqrt{2}(3q^2 + 4q_0p)}{4p} \\
    k_2 &= \pm \sqrt{2p}
\end{align*}
\]

In the limit \( r \to 0 \):

\[
\begin{align*}
    b(r) &\to -\left(\frac{k_1}{k_2}\right)^{1/2} = b_{\text{fix}} \\
    e^{\mu(r)} &\to r(\frac{k_1^3}{k_2})^{-1/4} = r f^{-1/4}
\end{align*}
\]

as expected, and the only undetermined constants are \( A_1, A_2 \). In order for the solution to be asymptotically minkowskian it is necessary that \((A_1 A_2^3)^{-1/4} = 1\). There is then just one undetermined parameter which is fixed by the asymptotic value of the dilaton \( b \). We choose for simplicity it to be \(-1\), therefore \( A_1 = 1, A_2 = 1 \). This choice is arbitrary in the sense that the different value of \( b \) at infinity the different universe (\( \equiv \)black–hole solution), but with the same entropy. Summarizing, before considering the eq.s of motion, the solution is:

\[
\begin{align*}
    a &= a_{\text{fix}} = \frac{q}{2p} \\
    b &= -\sqrt{\frac{(1 + k_1/r)}{(1 + k_2/r)}} \\
    e^{\mu} &= \left[(1 + k_1/r)(1 + k_2/r)^3\right]^{-1/4}
\end{align*}
\]

with \( k_1 \) and \( k_2 \) given in (5.7).

\section*{5.2 Solution of the 2nd order equations}

In the case \( S = T = U \) the structure of the \( \mathcal{N} \) matrix (A.9) and of the field strenghts reduces considerably. For the period matrices one simply obtains:

\[
\begin{align*}
    \text{Re} \mathcal{N} &= \begin{pmatrix}
        2a^3 & -a^2 & -a^2 & -a^2 \\
        -a^2 & 0 & a & a \\
        -a^2 & a & 0 & a \\
        -a^2 & a & a & 0
    \end{pmatrix}, \\
    \text{Im} \mathcal{N} &= \begin{pmatrix}
        3a^2 b + b^3 & -(ab) & -(ab) & -(ab) \\
        -(ab) & b & 0 & 0 \\
        -(ab) & 0 & b & 0 \\
        -(ab) & 0 & 0 & b
    \end{pmatrix}
\end{align*}
\]
while the dependence of $\ell^A(r)$ from the quantized charges simplifies to:

$$
\ell^A(r) = \begin{pmatrix}
-3a^2p+3aq+q_0 \\
-3a^3p+b^2q+3a^2q+a(-2b^2p+q_0) \\
-3a^3p+b^3q+3a^2q+a(-2b^2p+q_0) \\
-3a^3p+a^4p+b^2q+3a^2q+a(-2b^2p+q_0)
\end{pmatrix}
$$  \(5.10\)

Inserting (5.10) in the expressions (3.2) and substituting the result in the eq.s of motion (4.4) one finds:

$$
\begin{align*}
(a'' - 2a'b' + 2a')/b & = 0 \\
(b'' + 2b' + (a'^2 - b'^2)/b) & = -b^2 e^{2\mathcal{U}} (p^2 - (-3a^2p+3aq+q_0)^2)/r^4
\end{align*}
$$  \(5.11\)

The equation for $a$ is automatically fulfilled by our solution (5.8). The equation for $b$ is fulfilled as well and both sides are equal to:

$$
(k_2 - k_1) e^{4\mathcal{U}} \left( k_1 + k_2 + \frac{2k_1k_2}{r} \right)
$$  \(5.12\)

If $(k_2 - k_1) = 0$ both sides are separately equal to 0 which corresponds to the double fixed solution already found in [8].

Let us now consider the Einstein’s equations. From equations (4.6) we obtain in our simpler case the following ones:

$$
\begin{align*}
\mathcal{U}'' + \frac{2}{r} \mathcal{U}' & = (\mathcal{U}')^2 + \frac{3}{4b^2} \left( (b')^2 + (a')^2 \right) \\
\mathcal{U}'' + \frac{2}{r} \mathcal{U}' & = -2e^{-2\mathcal{U}} S_{\text{BH}}
\end{align*}
$$  \(5.12\)

The first of eqs.(5.12) is indeed fulfilled by our ansatze. Both sides are equal to:

$$
\frac{3(k_2 - k_1)^2}{16 r^4 (H_1)^2 (H_2)^2}
$$  \(5.13\)

Again, both sides are separately zero in the double-extreme case $(k_2 - k_1) = 0$. The second equation is fulfilled, too, by our ansatz and again both sides are zero in the double-extreme case. Therefore we can conclude with the following:

**Statement 5.1** Eq.5.8 yields a $\frac{1}{8}$ supersymmetry preserving solution of N=8 supergravity that is **not double extreme** and has a **finite entropy**:

$$
S_{\text{BH}} = 2\pi \left( q_0 p^3 + \frac{3}{4} b^2 q^2 \right)^{1/2}
$$  \(5.14\)

depending on three of the 5 truly independent charges.
6 Conclusions

This paper aimed at the completion of a programme started almost two years ago, namely the classification and the construction of all BPS saturated black-hole solutions of $N = 8$ supergravity (that is either M–theory compactified on $T^7$, or what amounts to the same thing type IIA string theory compactified on $T^6$). Such solutions are of three kinds:

1. 1/2 supersymmetry preserving solutions
2. 1/4 supersymmetry preserving solutions
3. 1/8 supersymmetry preserving solutions

The first two cases were completely worked out in [5]. For the third case there existed an in depth study in [4] which had established the minimal number of charges and fields having a dynamical role in the solution and also the identification of the generating solution with an $N = 2$ STU model. The actual structure of this STU black–hole solution however was still missing and so was its explicit embedding into the $N = 8$ theory. The present paper, relying on the techniques of Solvable Lie algebras has filled such a gap.

In this paper we have written the explicit form of the rather involved differential equations one needs to solve in order to obtain the desired result. We also provided a solution of these equations which is **not double extreme** and has a **finite entropy** depending on 3 charges. Finally we have indicated how the fully general solution depending on 5 non trivial charges can be worked out, leaving its actual evalution to a future publication. This 5 parameter solution is presumibely related via U–duality transformations to those found in [10]. In that case the generating solutions were obtained within the supergravity theory describing the low energy limit of toroidally compactified heterotic string theory, therefore they were carrying only NS–NS charges. Our group–theoretical embedding in the $N = 8$ theory, on the other hand, allows one to obtain quite directly the macroscopic description of pure Ramond–Ramond black–holes which can be interpreted microscopically in terms of D–branes only[22].

It should be stressed that the 1/8 SUSY preserving case is the only one where the entropy can be finite and where the horizon geometry is

$$AdS_2 \times S^2$$

Correspondingly our results have a bearing on two interesting and related problems:

1. Assuming the validity of the $AdS/CFT$ correspondence [23] we are lead to describe the 0–brane degrees of freedom in terms of superconformal quantum mechanics [24]. Can the entropy we obtain as an invariant of the U–duality group be described microscopically in this way?
2. Can we trace back the solvable Lie algebra gauge fixing we need to single out the relevant degrees of freedom to suitable wrappings of higher dimensional $p$–branes?

These questions are open and we propose to focus on them.

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**Appendix A: Geometry of $M_{T_6/Z_3}$ and $M_{STU}$: solvable description and Special Kähler formalism.**

The solvable Lie algebra description of a non–compact Riemannian manifold $M$ is based on the following theorem [17]:

**Theorem:** If a non–compact Riemannian manifold $M$ has a solvable subgroup $\exp(Solv)$ of the isometry group acting transitively on it, then $M$ admits a solvable description, i.e. it can be identified with the solvable group of isometries:

$$M = \exp(Solv)$$  \hspace{1cm} (A.1)

For instance all homogeneous manifolds of the form $G/H$ ( $G$ non–compact semisimple Lie group and $H$ its maximal compact subgroup) fulfill the hypothesis of the above theorem and their generating $Solv$ is defined by the Iwasawa decomposition:

$$G = H \oplus Solv$$

$$Solv = C_K \oplus Nil$$  \hspace{1cm} (A.2)

where $G$ and $H$ are the Lie algebras generating $G$ and $H$ respectively, $C_K$ is the subalgebra generated by the non compact Cartan generators of $G$ and $Nil$ is the subspace of $G$ consisting of the nilpotent generators related to roots which are strictly positive on $C_K$.

Applying the decomposition (A.2) to the manifold $M_{T_6/Z_3}$ one obtains:

$$SU(3,3) = [SU(3)_1 \oplus SU(3)_2 \oplus U(1)] \oplus Solv$$

$$Solv = F_1 \oplus F_2 \oplus F_3 \oplus X \oplus Y \oplus Z$$

$$F_i = \{h_i, g_i\} \hspace{0.5cm} i = 1, 2, 3$$
\[ X = X^+ \oplus X^-, \; Y = Y^+ \oplus Y^-, \; Z = Z^+ \oplus Z^- \]
\[ [h_i, g_j] = 2g_i, \; i = 1, 2, 3 \]
\[ [F_1, F_2] = 0, \; i \neq j \]
\[ [h_3, Y^\pm] = \pm Y^\pm, \; [h_3, X^\pm] = \pm X^\pm \]
\[ [h_2, Z^\pm] = \pm Z^\pm, \; [h_2, X^\pm] = X^\pm \]
\[ [h_1, Z^\pm] = Z^\pm, \; [h_1, Y^\pm] = Y^\pm \]
\[ [g_1, X] = [g_1, Y] = [g_1, Z] = 0 \]
\[ [g_2, X] = [g_2, Y] = [g_2, Z^+] = 0, \; [g_2, Z^-] = Z^+ \]
\[ [g_3, Y^+] = [g_3, X^+] = [g_3, Z] = 0 \]
\[ [g_3, Y^-] = Y^+; \; [g_3, X^-] = X^+ \]
\[ [F_1, X] = [F_2, Y] = [F_3, Z] = 0 \]
\[ [X^-, Z^-] = Y^- \quad (A.3) \]

as explained in section (2.1) the solvable subalgebra \( \text{Solv}_{STU} = F_1 \oplus F_2 \oplus F_3 \) is the solvable algebra generating \( \mathcal{M}_{STU} \). Denoting by \( \alpha_i, i = 1, \ldots, 5 \) the simple roots of \( SU(3, 3) \), using the canonical basis for the \( SU(3, 3) \) algebra, the generators in (A.3) have the following form:

\[
\begin{align*}
\alpha_1 = H_{\alpha_1} & \quad g_1 = iE_{\alpha_1} \\
\alpha_3 = H_{\alpha_3} & \quad g_2 = iE_{\alpha_3} \\
\alpha_5 = H_{\alpha_5} & \quad g_3 = iE_{\alpha_5} \\
X^+ &= \begin{pmatrix} X_1^+ = i(E_{-\alpha_4} + E_{\alpha_3 + \alpha_4 + \alpha_5}) \\ X_2^+ = E_{\alpha_3 + \alpha_4 + \alpha_5} - E_{-\alpha_4} \end{pmatrix} \\
X^- &= \begin{pmatrix} X_1^- = i(E_{\alpha_3 + \alpha_4} + E_{-(\alpha_4 + \alpha_5)}) \\ X_2^- = E_{\alpha_3 + \alpha_4} - E_{-(\alpha_4 + \alpha_5)} \end{pmatrix} \\
Y^+ &= \begin{pmatrix} Y_1^+ = i(E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} + E_{-(\alpha_2 + \alpha_3 + \alpha_4)}) \\ Y_2^+ = E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5} - E_{-(\alpha_2 + \alpha_3 + \alpha_4)} \end{pmatrix} \\
Y^- &= \begin{pmatrix} Y_1^- = i(E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} + E_{-(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)}) \\ Y_2^- = E_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} - E_{-(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)} \end{pmatrix} \\
Z^+ &= \begin{pmatrix} Z_1^+ = i(E_{\alpha_1 + \alpha_2 + \alpha_3} + E_{-\alpha_2}) \\ Z_2^+ = E_{\alpha_1 + \alpha_2 + \alpha_3} - E_{-\alpha_2} \end{pmatrix} \\
Z^- &= \begin{pmatrix} Z_1^- = i(E_{\alpha_1 + \alpha_2} + E_{-(\alpha_2 + \alpha_3)}) \\ Z_2^- = E_{\alpha_1 + \alpha_2} - E_{-(\alpha_2 + \alpha_3)} \end{pmatrix} \quad (A.4) \end{align*}
\]

We compute the \( SU(3, 3) \) generators in the 20 representation of the group, which is symplectic.
The weights \( \vec{v}^{A'} \) of this representation, computed on the Catan subalgebra \( C \) of \( SU(3)_1 \oplus SU(3)_2 \oplus U(1) \) are:

\[
\vec{v}^{A'} = v^{A'}(\frac{H_{c1}}{2}, \frac{H_{c1+c2}}{2}, \frac{H_{d1}}{2}, \frac{H_{d1+d2}}{2}, \lambda)
\]

\[
v^0 = \{0, 0, 0, 0, \frac{3}{2}\}
\]

\[
v^1 = \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}
\]

\[
v^2 = \{0, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}\}
\]

\[
v^3 = \{\frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}\}
\]

\[
v^4 = \{\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}\}
\]

\[
v^5 = \{0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}\}
\]

\[
v^6 = \{\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}
\]

\[
v^7 = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}\}
\]

\[
v^8 = \{0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}
\]

\[
v^9 = \{\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}\}
\]

(\text{A.5})

these weights have been ordered in such a way that the first four define the \((2,2,2)\) of \(SL(2, \mathbb{R})^3 \subset SU(3,3)\) and \(v^0\) is related to the graviphoton, for its restriction on the Cartan generators \( H_{c1}, H_{c1+c2}, H_{d1}, H_{d1+d2} \) of \( H_{\text{matter}} = SU(3)_1 \oplus SU(3)_2 \) is trivial.

After performing the restriction to the \(Sp(8)_D\) representation of \([SL(2, \mathbb{R})]^3\) described earlier in the paper, the orthonormal basis \(|v^A_{x,y}\rangle\) (\(A = 0, 1, 2, 3\)) is:

\[
|v^1_{x}\rangle = \{0, 0, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}
\]

\[
|v^2_{x}\rangle = \{0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}
\]

\[
|v^3_{x}\rangle = \{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0\}
\]

\[
|v^4_{x}\rangle = \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0\}
\]

\[
|v^1_{y}\rangle = \{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0\}
\]

\[
|v^2_{y}\rangle = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\}
\]

23
\begin{align*}
\langle v_3^y \rangle &= \{0, 0, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\} \\
\langle v_4^y \rangle &= \{0, 0, 0, 0, 1, -\frac{1}{2}, 0, \frac{1}{2}\}
\end{align*}

(A.6)

The $Sp(8)_D$ representation of the generators of $Solv_{STU}$ are:

\[
\begin{align*}
\h_1 &= \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}; & \quad g_1 &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
\h_2 &= \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}; & \quad g_2 &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
\h_3 &= \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}; & \quad g_3 &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}

The first order BPS equations may be equivalently formulated within a Special Kähler description of the manifold $M_{STU}$. In the special coordinate symplectic gauge, all the geometrical quantities defined on $M_{STU}$ may be deduced form a cubic prepotential $F(X)$:

\[
\{z^i\} = \{S, T, U\}; \quad \Omega(z) = \begin{pmatrix} X^A(z) \\ F_\Sigma(z) \end{pmatrix}
\]

\[
X^A(z) = \begin{pmatrix} 1 \\ S \\ T \\ U \end{pmatrix}
\]
The covariantly holomorphic symplectic section $V(z, \bar{z})$ and its covariant derivative $U_i(z, \bar{z})$ are:

\[
V(z, \bar{z}) = \begin{pmatrix} L^A(z, \bar{z}) \\ M^\Sigma(z, \bar{z}) \end{pmatrix} = e^{\kappa(z, \bar{z})/2}\Omega(z, \bar{z})
\]

\[
U_i(z, \bar{z}) = \begin{pmatrix} f_i^A(z, \bar{z}) \\ h_{i\Sigma}(z, \bar{z}) \end{pmatrix} = \nabla_i V(z, \bar{z}) = (\partial_i + \frac{\partial\kappa}{2})V(z, \bar{z})
\]

\[
\bar{U}_i(z, \bar{z}) = \begin{pmatrix} \bar{T}_i^A(z, \bar{z}) \\ \bar{h}_{i\Sigma}(z, \bar{z}) \end{pmatrix} = \nabla_i \bar{V}(z, \bar{z}) = (\partial_i + \frac{\partial\kappa}{2})\bar{V}(z, \bar{z})
\]

\[
M^\Sigma(z, \bar{z}) = \mathcal{N}_\Sigma(z, \bar{z})L^A(z, \bar{z})
\]

\[
h_{i\Sigma}(z, \bar{z}) = \mathcal{N}_\Sigma(z, \bar{z})f_i^A(z, \bar{z})
\]  

(A.7)

The real and imaginary part of $\mathcal{N}$ in terms of the real part $a_i$ and imaginary part $b_i$ of the complex scalars $z^i$ are:

\[
\text{Re}\mathcal{N} = \begin{pmatrix} 2a_1a_2a_3 & -(a_2a_3) & -(a_1a_3) & -(a_1a_2) \\ -a_2a_3 & 0 & a_3 & a_2 \\ -a_1a_3 & a_3 & 0 & a_1 \\ -(a_1a_2) & a_2 & a_1 & 0 \end{pmatrix}
\]

\[
\text{Im}\mathcal{N} = \begin{pmatrix} a_1^2 & b_{12}b_{13} & b_{12}b_{23} & b_{13}b_{23} \\ b_{12}b_{13} & a_2 & a_1 & 0 \\ b_{12}b_{23} & a_1 & a_3 & 0 \\ b_{13}b_{23} & a_2 & a_3 & a_1 \end{pmatrix}
\]  

(A.8)

Using the above defined quantities, the first order BPS equations may be written in a complex notation equivalent to (3.6):

\[
\frac{dS}{dr} = \pm \left(\frac{e^{\mu(r)}}{r^2}\right) i\sqrt{\text{Im}(S)} 2\text{Im}(T)\text{Im}(U) \left( q_0 + \nabla q_3 - \nabla p^2 S + q_1 S + T \left( -\left( T p_1 \right) + q_2 + \nabla p^0 S - p^3 S \right) \right)
\]

\[
\frac{dT}{dr} = \pm \left(\frac{e^{\mu(r)}}{r^2}\right) i\sqrt{\text{Im}(T)} 2\text{Im}(S)\text{Im}(U) \left( q_0 + \nabla q_3 - \nabla p^1 T + q_2 T + S \left( -\left( T p_1^2 \right) + q_1 + \nabla p^0 T - p^3 T \right) \right)
\]

\[
\frac{dU}{dr} = \pm \left(\frac{e^{\mu(r)}}{r^2}\right) i\sqrt{\text{Im}(U)} 2\text{Im}(S)\text{Im}(T) \left( q_0 + \nabla q_2 - \nabla p^1 U + q_3 U + S \left( -\left( T p_1^3 \right) + q_1 + \nabla p^0 U - p^2 U \right) \right)
\]
Setting \( t \) and \( s \) Appendix B: the full set of first and second order differential equations

The central charge \( Z(z, \overline{z}, p, q) \) being given by:

\[
Z(z, \overline{z}, p, q) = -\left( \frac{1}{2\sqrt{2}(|\text{Im}(S)|\text{Im}(T)|\text{Im}(U)|)^{1/2}} \right) [q_0 + S \left( T U p^0 - U p^2 - T p^3 + q_1 \right) + T \left( -\left( U p^1 \right) + q_2 \right) + U q_3] 
\]

(A.11)

Appendix B: the full set of first and second order differential equations

Setting \( z^i = a_i + ib_i \) eqs.(A.10) can be rewritten in the form:

\[
\begin{align*}
\frac{da_1}{dr} &= \pm \frac{\epsilon^{(i)}(r)}{r^2} \sqrt{-\frac{b_1}{2b_2b_3}} [-b_1q_1 + b_2q_2 + b_3q_3 + (-(a_2a_3b_1) + a_1a_3b_2 + a_1a_2b_3 + b_1b_2b_3) p^0 + \\
&\quad + (-a_3b_2 - a_2b_3) p^1 + (a_3b_1 - a_1b_3) p^2 + (a_2b_1 - a_1b_2) p^3] \\
\frac{db_1}{dr} &= \pm \frac{\epsilon^{(i)}(r)}{r^2} \sqrt{-\frac{b_1}{2b_2b_3}} [a_1q_1 + a_2q_2 + a_3q_3 + (a_1a_2a_3 + a_1a_3b_1 + a_2b_1 + a_2b_3 - a_1b_2b_3) p^0 + \\
&\quad + (-a_2a_3 + b_2b_3) p^1 - (a_1a_3 + b_1b_3) p^2 - (a_1a_2 + b_1b_2) p^3 + q_0] \\
\frac{da_2}{dr} &= (1, 2, 3) \rightarrow (2, 1, 3) \\
\frac{da_3}{dr} &= (1, 2, 3) \rightarrow (3, 2, 1) \\
\frac{db_3}{dr} &= (1, 2, 3) \rightarrow (3, 2, 1) \\
\frac{dT}{dr} &= \pm \frac{\epsilon^{(i)}(r)}{r^2} \sqrt{-\frac{b_1}{2b_2b_3}} \frac{1}{(b_1b_2b_3)^{1/2}} [a_1q_1 + a_2q_2 + a_3q_3 + (a_1a_2a_3 - a_3b_1b_2 - a_2b_1b_3 - a_1b_2b_3) p^0 + \\
&\quad - (a_2a_3 - b_2b_3) p^1 - (a_1a_3 - b_1b_3) p^2 - (a_1a_2 - b_1b_2) p^3 + q_0] \\
0 &= b_1q_1 + b_2q_2 + b_3q_3 + (a_2a_3b_1 + a_1a_3b_2 + a_1a_2b_3 - b_1b_2b_3) p^0 - (a_3b_2 + a_2b_3) p^1 \\
&\quad - (a_3b_1 + a_1b_3) p^2 - (a_2b_1 + a_1b_2) p^3 
\end{align*}
\]

(B.1)

The explicit form of the equations of motion for the most general case is:

Scalar equations:

\[
\left( a_i'' - 2 \frac{a_i'b_i'}{b_i} + 2 \frac{a_i'}{r} \right) = -\frac{2b_1 e^{2U}}{r^4} \left[ a_1b_2b_3 \left( p_0^2 - \ell(r) \right)^2 \right] + b_2 \left( -(b_3 p_0 p_1) + b_3 \ell(r) \ell(r) \right) + \\
\frac{\epsilon^{(i)}(r)\sqrt{-\frac{b_1}{2b_2b_3}} \frac{1}{(b_1b_2b_3)^{1/2}} [a_1q_1 + a_2q_2 + a_3q_3 + (a_1a_2a_3 - a_3b_1b_2 - a_2b_1b_3 - a_1b_2b_3) p^0 + \\
&\quad - (a_2a_3 - b_2b_3) p^1 - (a_1a_3 - b_1b_3) p^2 - (a_1a_2 - b_1b_2) p^3 + q_0] \\
\]
\[
\left( b_\ell'' + \frac{2 b'_1}{r} + \frac{(a_1^2 - b_1^2)}{b_1} \right) = -\frac{e^2 U}{b_2 b_3 r^4} \left[ -(a_1^2 b_2^2 b_3^2 p_0^2) + b_1^2 b_2^2 b_3^2 p_0^2 + 2 a_1 b_2^2 b_3^2 p_0^1 + b_1^2 b_2^2 b_3^2 p_0^2 \right. \\
\left. - b_2^2 b_3^2 p_0^2 + b_1^2 b_2^2 p_0^2 + b_2^2 b_3^2 p_0^2 + a_1 b_2^2 b_3^2 \ell(r)_0^2 + b_1^2 b_2^2 b_3^2 \ell(r)_0^2 + b_2^2 b_3^2 \ell(r)_0^2 + a_1^2 b_1^2 b_2^2 (p_0^2 - \ell(r)_0^2) + a_2 b_2^2 b_3^2 \\
\left. (p_0^2 - \ell(r)_0^2) - 2 a_1 b_2^2 b_3^2 \ell(r)_0 \ell(r)_1 + b_2^2 b_3^2 \ell(r)_1 + b_1^2 b_2^2 \ell(r)_2 + b_2^2 b_3^2 \ell(r)_2 + a_2 b_1^2 b_2^2(- (p_0^2) + \ell(r)_0 \ell(r)_2) + b_1^2 b_2^2 \ell(r)_3 + 2 a_3 b_1^2 b_2^2(- (p_0^2) + \ell(r)_0 \ell(r)_3) \right] \\
\left( a_\ell'' - 2 \frac{a_1' b_2'}{b_2} + \frac{a_\ell'}{r} \right) = (1, 2, 3) \to (2, 1, 3) \\
\left( b_\ell'' + \frac{2 b_2'}{r} + \frac{(a_2^2 - b_2^2)}{b_2} \right) = (1, 2, 3) \to (2, 1, 3) \\
\left( a_\ell'' - 2 \frac{a_2' b_3'}{b_3} + \frac{a_\ell'}{r} \right) = (1, 2, 3) \to (3, 2, 1) \\
\left( b_\ell'' + \frac{2 b_3'}{r} + \frac{(a_3^2 - b_3^2)}{b_3} \right) = (1, 2, 3) \to (3, 2, 1)
\]

Einstein equations:
\[
U'' + \frac{2}{r} U' = -2 e^{-2U} S_{00} \\
(U')^2 + \sum_i \frac{1}{4b_i^2} ((b_i')^2 + (a_i')^2) = -2 e^{-2U} S_{00}
\]

where the quantity $S_{00}$ on the right hand side of the Einstein eqs. has the following form:
\[
S_{00} = \frac{e^4 U}{4 b_1 b_2 b_3 r^4} \left( a_1^2 b_2^2 b_3^2 p_0^2 + 2 b_1^2 b_2^2 b_3^2 p_0^1 + b_2^2 b_3^2 p_0^2 + b_1^2 b_2^2 b_3^2 p_1^2 + b_1^2 b_2^2 b_3^2 p_2^2 + b_1^2 b_2^2 b_3^2 \ell(r)_0^2 + b_1^2 b_2^2 b_3^2 \ell(r)_1^2 + b_1^2 b_2^2 b_3^2 \ell(r)_2^2 + b_1^2 b_2^2 b_3^2 \ell(r)_3^2 - 2 a_2 b_2 b_3 \ell(r)_0 \ell(r)_2 - 2 a_3 b_1 b_3 \ell(r)_1 \ell(r)_3 \right) \\
\]

The explicit expression of the $\ell_\Lambda(r)$ charges in terms of the quantized ones is computed from eq. (4.8):
\[
\ell_\Lambda(r) = \left( \begin{array}{c}
q_0 + a_1 \left( a_2 a_3 p_0 + a_3 p_0 - a_2 p_0 + q_1 \right) + a_2 \left( -(a_3 p_0^2) + q_2 \right) + a_3 q_3 \\
a_1^2 \left( a_2 a_3 p_0 - a_3 p_0 - a_2 p_0 + q_1 \right) + a_1^2 \left( a_2 a_3 p_0 - a_3 p_0 - a_2 p_0 + q_1 \right) + a_1 \left( q_0 + 2 a_2 \left( -(a_3 p_0^2) + q_2 \right) + a_3 q_3 \right) \\
a_1 \left( a_2 a_3 p_0 - a_3 p_0 - a_2 p_0 + q_1 \right) + a_1 \left( q_0 + 2 a_2 \left( -(a_3 p_0^2) + q_2 \right) + a_3 q_3 \right) \\
a_3 q_0 + a_1 \left( -(a_3 p_0^2) - a_3 p_0 - a_2 a_3 p_0^3 + q_1 \right) - a_2 \left( a_3 p_0^2 + a_3 p_0^2 + q_2 \right) + a_3 q_3 + b_3 q_3 \\
\end{array} \right)
\]

(B.3)
References


