Derivation of classical capacity of quantum channel for discrete information source

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Abstract

In this letter, we prove that the classical capacity of quantum channel for $M$ symmetric states is achieved by an uniform distribution on a priori probabilities. We also investigate non-symmetric cases such as a ternary amplitude shift keyed signal set and a 16-ary quadrature amplitude modulated signal set in coherent states.

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It is well known that the conventional information theory has provided many fruitful applications [1, 2]. The quantum theory of communication has lead to some profound and exciting new insights into both physics and communication science. In the pioneering works, the main theme was to formulate a mathematical theory for communication processes which convey classical information by quantum states. As a result, several fundamental results in quantum aspect of communication and information theory have been brought. On the channel coding theorem, Stratonovich [3] and Holevo [4] pointed out that a true capacity of quantum measurement channel is greater than the maximum mutual information with respect to detection operators and a priori probabilities of information symbols, which corresponds to channel capacity in conventional information theory denoted as $C_1$. In addition, Levitin [5] conjectured and Holevo [6] proved that it is bounded by the quantity, called Holevo’s bound, defined by the von Neumann entropy for the ensemble of signal quantum states. Recently, Hausladen et al. [7] proved that the maximum value with respect to a priori probabilities of the Holevo’s bound is the true capacity in the case of pure states by introducing a typical sub-space in addition to the random coding scheme and the square-root measurement as a decoding. Thus we can understand that the capacity of quantum measurement channel for given set of signal quantum states is essentially the maximum value with respect to a priori probabilities of von Neumann entropy. However, a simple question arose such that “how to derive the maximum value with respect to a priori probabilities of the von Neumann entropy for the ensemble when quantum states are in an infinite dimensional space”. The von Neumann entropy of the complete information source which consist of signal quantum states and their a priori probabilities has strong non-linearity for a priori probabilities, because it is given by the eigenvalues of the density operator for the complete information source. Thus this problem is not trivial one. So far, we examined numerical calculation of the von Neumann entropy for several kinds of sets of quantum states [8, 9]. In this letter, we show analytical solution and simplification of the maximization with respect to a priori probability of the von Neumann entropy for some representative finite sets of signal quantum states in an infinite dimensional space.
We now consider a model in which classical information is sent by quantum states, and give a formulation of our problem. Let \{1, 2, \ldots, M\} be an input alphabet and let a letter ‘i’ in the input alphabet correspond to a quantum state \(\hat{\rho}_i\), called a letter state, which satisfy

\[
\hat{\rho}_i \geq 0, \quad \text{Tr}\hat{\rho}_i = 1, \quad \forall i.
\] (1)

A complete information source is defined as an ensemble of letter states with a priori probabilities \(\{\xi_i\} (\xi_i \geq 0, \sum_{i=1}^{M} \xi_i = 1)\), and the statistical property is represented by a density operator

\[
\hat{\rho}(\xi) = \sum_{i=1}^{M} \xi_i \hat{\rho}_i.
\] (2)

The von Neumann entropy of \(\hat{\rho}(\xi)\) is defined as

\[
S(\hat{\rho}(\xi)) = -\text{Tr}\hat{\rho}(\xi) \log \hat{\rho}(\xi)
\]
\[= -\sum_j \lambda_j(\xi) \log \lambda_j(\xi),
\] (3)

where \(\lambda_j(\xi)\) is the eigenvalue of the density operator \(\hat{\rho}(\xi)\). According to the quantum channel coding theorem [7, 10], when the complete information source is connected to the quantum measurement channel, the capacity is given by

\[
C = \max_{\xi} \Delta S(\xi),
\] (4)

where

\[
\Delta S(\xi) = S(\hat{\rho}(\xi)) - \sum_{i=1}^{M} \xi_i S(\hat{\rho}_i).
\] (5)

In general, we can carry out the maximization by the following two steps. (A) Finding the eigenvalues of the density operator as a function of a distribution on a priori probabilities. (B) Finding the maximum value by some optimization techniques. However, it is very difficult to solve analytically because of the following two reasons. One of them is a difficulty of obtaining the eigenvalues of the density operator which is essential to calculate the von Neumann entropy. Other is a nonlinearity of the function ‘\(-p \log p\)’ even if one could get
the eigenvalues. Fortunately, if the prepared states have a certain symmetry in the sense of Ref. [11], the analytical solution can be obtained. In fact, we show that the capacity for the ensemble of $M$ symmetric states is achieved by a uniform distribution on a priori probabilities.

$M$ symmetric states are defined by

$$\hat{\rho}_i = \hat{V}_{i-1} \hat{\rho}_1 \hat{V}_{i-1}^\dagger, \quad \forall i,$$

(6)

where $\hat{V}_{i-1} = \hat{V} \hat{V}_{i-1} = \hat{V}^M = \hat{I}$ and where $\hat{I}$ stands for an identity operator. In this case, $S(\hat{\rho}_1) = S(\hat{\rho}_i)$ for all $i$, so that $\Delta S(\xi) = S(\hat{\rho}(\xi)) - S(\hat{\rho}_1)$. Hence, the maximization problem of $\Delta S(\xi)$ results in that of the von Neumann entropy $S(\hat{\rho}(\xi))$.

Let us represent an arbitrary distribution on a priori probabilities as follows:

$$\xi = (\xi_1, \xi_2, \ldots, \xi_M) \equiv \xi^{(i)},$$

(7)

and its permutations

$$\xi^{(2)} = (\xi_2, \xi_3, \ldots, \xi_1),$$

$$\xi^{(3)} = (\xi_3, \xi_4, \ldots, \xi_2),$$

$$\vdots$$

$$\xi^{(M)} = (\xi_M, \xi_1, \ldots, \xi_{M-1}).$$

(8)

For these distributions $\xi^{(i)}$, the corresponding density operators satisfy the next relation.

$$\hat{\rho}(\xi^{(i)}) = \hat{V}^{-(i-1)} \hat{\rho}(\xi) \hat{V}^{-(i-1)}.$$
This fact means that the capacity for $M$ symmetric states is achieved by a uniform distribution on a priori probabilities, because the right side of Eq.(9) is equal to or greater than the von Neumann entropy for arbitrary distribution $S(\hat{\rho}(\mathbf{\xi}))$. We have thus proved.

This result is useful to derive the channel capacity. When $\hat{\rho}_1$ is a pure state represented as $\hat{\rho}_1 = |\psi\rangle\langle\psi|$, eigenvalues of the information source $\{\lambda_j\}$ can be obtained from the Gram matrix consisting of signal quantum states. Hence the capacity is given explicitly as follows.

$$C = -\sum_{j=1}^{M} \lambda_j \log \lambda_j,$$

(11)

where $\lambda_j$ is

$$\lambda_j = \frac{1}{M} \sum_{k=1}^{M} \langle \psi | \hat{V}^{k-1} \hat{V} | \psi \rangle \exp \left[ -\frac{2j(k-1)\pi i}{M} \right],$$

(12)

and $i = \sqrt{-1}$. The result may be also applicable to the case of mixed states, but it depends on the signal quantum states whether the capacity is analytically obtained like as Eq.(11) and (12).

In the following, let us consider the ‘partially’ symmetric case as a non-symmetric case to see another usage of our result. The word ‘partially’ means that some states in the set of the prepared states have a property like as Eq.(6). In general, we cannot solve analytically in this case. However, the above result may provide a powerful tool for the calculation of the maximization. Here, we consider the case of a ternary coherent-state set, whose states are represented as follows.

$$\hat{\rho}_1 = |0\rangle\langle 0|, \quad \hat{\rho}_2 = |\alpha\rangle\langle \alpha|, \quad \hat{\rho}_3 = |\alpha\rangle\langle -\alpha| = \hat{V} |\alpha\rangle\langle \alpha| \hat{V}^\dagger,$$

(13)

where $\hat{V} = \exp[-\pi i \hat{a}^\dagger \hat{a}]$ and where $\hat{a}$ and $\hat{a}^\dagger$ are the photon annihilation and creation operators, respectively. In communication theory, this is called a ternary amplitude shift keyed (3ASK) signal set [13]. The problem is the maximization of the von Neumann entropy $S(\hat{\rho}(\mathbf{\xi}))$ with respect to $\mathbf{\xi}$. We can assume that a priori probability of signal 1 is a certain
\( \xi_1 = \xi_1' \) at first. Under this condition, we take a distribution of fixed \( \xi_1' \) and arbitrary \( \xi_2 \) and \( \xi_3 \), so we have

\[
\xi = (\xi_1', \xi_2, \xi_3) \equiv \xi^{(1)},
\]

and its permutation

\[
\xi^{(2)} \equiv (\xi_1', \xi_3, \xi_2).
\]

In this case, the next relation is obtained.

\[
\hat{\rho}(\xi^{(2)}) = \hat{V} \hat{\rho}(\xi^{(1)}) \hat{V}^\dagger.
\]

From the invariance for unitary transformation and the concavity of von Neumann entropy, we have

\[
S(\hat{\rho}(\xi)) \leq S(\hat{\rho}(\xi')) \equiv S',
\]

where

\[
\xi' = \frac{1}{2}\xi^{(1)} + \frac{1}{2}\xi^{(2)} = (\xi_1', \frac{1 - \xi_1'}{2}, \frac{1 - \xi_1'}{2}).
\]

That is, the maximum \( S' \) under the condition is achieved by \( \xi_2 = \xi_3 = (1 - \xi_1')/2 \). So, two a priori probabilities of signal 2 and 3 are equal at the optimum distribution at least. Then, we can transform the original problem to the maximization of the von Neumann entropy of the following density operator.

\[
\hat{\rho}'(\xi_1) = \xi_1 \hat{\rho}_1 + \frac{1 - \xi_1}{2} \hat{\rho}_2 + \frac{1 - \xi_1}{2} \hat{\rho}_3.
\]

As a result, the variable becomes only \( \xi_1 \). The eigenvalues of this operator are

\[
\lambda_1(\xi_1) = \frac{1}{2}(1 - \kappa^4)(1 - \xi_1),
\]

\[
\lambda_2(\xi_1) = \frac{1}{4}\left\{ (1 + \xi_1) + \kappa^4(1 - \xi_1) \right. \\
-2\sqrt{\frac{1}{4} \left\{ (1 + \xi_1) + \kappa^4(1 - \xi_1) \right\}^2 - 2(1 - \kappa^2)^2(1 - \xi_1)\xi_1} \left. \right\},
\]

\[
\lambda_3(\xi_1) = \frac{1}{4}\left\{ (1 + \xi_1) + \kappa^4(1 - \xi_1) \right. \\
+2\sqrt{\frac{1}{4} \left\{ (1 + \xi_1) + \kappa^4(1 - \xi_1) \right\}^2 - 2(1 - \kappa^2)^2(1 - \xi_1)\xi_1} \left. \right\}.
\]
where $\kappa = \langle 0 | \alpha \rangle = \exp[-|\alpha|^2/2]$. Using these eigenvalues, the maximization problem is written as follows:

$$C = \max_{0 \leq \xi_1 \leq 1} \left\{ - \sum_{i=1}^{3} \lambda_i(\xi_1) \log \lambda_i(\xi_1) \right\}. \tag{23}$$

Unfortunately, it is still difficult to solve analytically. So we examined numerical calculation, and numerical solution is shown in FIG.1. From FIG.1 it turns out that $\xi_1 = 0$ when $|\alpha|^2$ is very small, and $\xi_1$ arises when $|\alpha|^2$ exceeds approximately 0.21. When $|\alpha|^2$ is quite large, all a priori probabilities converge into 1/3 respectively.

As the next example, let us investigate a 16-ary coherent-state set called a quadrature amplitude modulated (QAM) signal set [14]. The complete information source is represented as follows:

$$\begin{align*}
\{ & \hat{\rho}_{1,1}, \hat{\rho}_{2,1}, \hat{\rho}_{3,1}, \hat{\rho}_{4,1}, \hat{\rho}_{1,2a}, \hat{\rho}_{2,2a}, \hat{\rho}_{3,2a}, \hat{\rho}_{4,2a}, \\
& \hat{\xi}_{1,1}, \hat{\xi}_{2,1}, \hat{\xi}_{3,1}, \hat{\xi}_{4,1}, \hat{\xi}_{1,2a}, \hat{\xi}_{2,2a}, \hat{\xi}_{3,2a}, \hat{\xi}_{4,2a}, \\
& \hat{\rho}_{1,2b}, \hat{\rho}_{2,2b}, \hat{\rho}_{3,2b}, \hat{\rho}_{4,2b}, \hat{\rho}_{1,3}, \hat{\rho}_{2,3}, \hat{\rho}_{3,3}, \hat{\rho}_{4,3}, \\
& \hat{\xi}_{1,2b}, \hat{\xi}_{2,2b}, \hat{\xi}_{3,2b}, \hat{\xi}_{4,2b}, \hat{\xi}_{1,3}, \hat{\xi}_{2,3}, \hat{\xi}_{3,3}, \hat{\xi}_{4,3} \} \tag{24}
\end{align*}$$

where

$$\begin{align*}
\hat{\rho}_{1,1} &= |\alpha + i\alpha\rangle\langle\alpha + i\alpha|, \\
\hat{\rho}_{1,2a} &= |3\alpha + i\alpha\rangle\langle3\alpha + i\alpha|, \\
\hat{\rho}_{1,2b} &= |\alpha + 3i\alpha\rangle\langle\alpha + 3i\alpha|, \\
\hat{\rho}_{1,3} &= |3\alpha + 3i\alpha\rangle\langle3\alpha + 3i\alpha|, \\
\hat{\rho}_{2,1} &= \hat{V}_{A}^{1} \hat{\rho}_{1,1} \hat{V}_{A}^{1}\dagger, \quad \hat{\rho}_{3,1} = \hat{V}_{A}^{2} \hat{\rho}_{1,1} \hat{V}_{A}^{2}\dagger, \\
\hat{\rho}_{4,1} &= \hat{V}_{A}^{3} \hat{\rho}_{1,1} \hat{V}_{A}^{3}\dagger, \quad \hat{\rho}_{2,2a} = \hat{V}_{A} \hat{\rho}_{1,2a} \hat{V}_{A}\dagger, \\
\hat{\rho}_{3,2a} &= \hat{V}_{A}^{2} \hat{\rho}_{1,2a} \hat{V}_{A}^{2}\dagger, \quad \hat{\rho}_{4,2a} = \hat{V}_{A}^{3} \hat{\rho}_{1,2a} \hat{V}_{A}^{3}\dagger, \\
\hat{\rho}_{2,2b} &= \hat{V}_{A} \hat{\rho}_{1,2b} \hat{V}_{A}\dagger, \quad \hat{\rho}_{3,2b} = \hat{V}_{A}^{2} \hat{\rho}_{1,2b} \hat{V}_{A}^{2}\dagger, \\
\hat{\rho}_{4,2b} &= \hat{V}_{A}^{3} \hat{\rho}_{1,2b} \hat{V}_{A}^{3}\dagger, \quad \hat{\rho}_{2,3} = \hat{V}_{A} \hat{\rho}_{1,3} \hat{V}_{A}\dagger, \\
\hat{\rho}_{3,3} &= \hat{V}_{A}^{2} \hat{\rho}_{1,3} \hat{V}_{A}^{2}\dagger, \quad \hat{\rho}_{4,3} = \hat{V}_{A}^{3} \hat{\rho}_{1,3} \hat{V}_{A}^{3}\dagger,
\end{align*} \tag{25}$$

and where $\alpha$ is real and
\[ \hat{V}_A = \exp \left[ -\frac{\pi i}{2} a^\dagger a \right]. \]  

(26)

Signal constellation of 16QAM is shown in FIG.2. Based on \( \hat{V}_A \) and the concavity of von Neumann entropy, we have

\[ S(\hat{\rho}(\xi)) \leq S(\hat{\rho}(\xi')), \quad \forall \xi, \]  

(27)

where

\[ \xi' = (\xi_1, \xi_1, \xi_1, \xi_1, \xi_2a, \xi_2a, \xi_2a, \xi_2a, \xi_2b, \xi_2b, \xi_2b, \xi_2b, \xi_3, \xi_3, \xi_3), \]  

(28)

and where

\[
\begin{align*}
\xi_1 & \equiv (\xi_{1,1} + \xi_{2,1} + \xi_{3,1} + \xi_{4,1})/4, \\
\xi_{2a} & \equiv (\xi_{1,2a} + \xi_{2,2a} + \xi_{3,2a} + \xi_{4,2a})/4, \\
\xi_{2b} & \equiv (\xi_{1,2b} + \xi_{2,2b} + \xi_{3,2b} + \xi_{4,2b})/4, \\
\xi_3 & \equiv (\xi_{1,3} + \xi_{2,3} + \xi_{3,3} + \xi_{4,3})/4.
\end{align*}
\]  

(29)

Here we define the anti-unitary operator \( \hat{V}_B \) by

\[ \hat{V}_B |\beta\rangle = |\beta^*\rangle, \]  

(30)

where \( \beta^* \) means the complex conjugate of \( \beta \). Because of an invariance of von Neumann entropy for the anti-unitary transformation \( \hat{V}_B \) and the concavity, we have the following inequality:

\[ S(\hat{\rho}(\xi')) \leq S(\hat{\rho}(\xi'')), \]  

(31)

where

\[ \xi'' = (\xi_1, \xi_1, \xi_1, \xi_1, \xi_2, \xi_2, \xi_2, \xi_2, \xi_2, \xi_2, \xi_2, \xi_3, \xi_3, \xi_3), \]  

(32)
and where $\xi_2 \equiv (\xi_{2a} + \xi_{2b})/2$. That is, the optimization parameters becomes only $\xi_1$ and $\xi_2$, since $\xi_3 = (1 - 4\xi_1 - 8\xi_2)/4$. Then, the maximization problem of the von Neumann entropy for 16QAM coherent-state signal is written as
\[
C = \max_{0 \leq \xi_1 + 2\xi_2 \leq \frac{1}{4}} S(\hat{\rho}(\xi_2)).
\] (33)

Although we cannot obtain the analytical solution, it is easy to find the solution numerically because the number of optimization parameters is only two. Numerical solution is shown in FIG.3. It can be seen from FIG.3 that the optimum a priori distribution is classified into three cases. When $|\alpha|^2$ is very small, $\xi_1$ and $\xi_2$ are 0, respectively. When $|\alpha|^2 \gtrsim 0.04$, $\xi_2$ arises and when $|\alpha|^2 \gtrsim 0.12$, $\xi_1$ arises. When $|\alpha|^2$ is quite large, $\xi_i$ converges into $1/16$.

As concluding remark, we demonstrated that the classical capacity for $M$ symmetric states is achieved by the uniform distribution of a priori probabilities. Although the analytical result is given if and only if the set of quantum states consist of symmetric states, our result could apply to the other types of the set of quantum states, in which we can easily find the numerical solution. Unfortunately we have no method for more general cases, but recently it is pointed out that our result is applicable to the group covariant case [15].

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REFERENCES


FIG. 1. Channel capacity for ternary symmetric coherent-state signal. (a) capacity. (b) optimum distribution.
FIG. 2. Signal constellation of 16QAM signal. $\hat{x}_C \equiv (\hat{a} + \hat{a}^\dagger)/2, \hat{x}_S \equiv (\hat{a} - \hat{a}^\dagger)/2i$. Dots stand for amplitudes of coherent states.
FIG. 3. Channel capacity for 16QAM coherent-state signal. (a) capacity. (b) optimum distribution

\[ \xi_3 = 1 - \frac{4\xi_1 + 8\xi_2}{4} \]