Existence of Spinorial States in Pure Loop Quantum Gravity

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January 25, 1998

Abstract

We demonstrate the existence of spinorial states in a kinematical theory of canonical quantum gravity without matter. This should be regarded as evidence towards the conjecture that bound states with particle properties appear in association with spatial regions of non-trivial topology. In asymptotically trivial general relativity the momentum constraint generates only a subgroup of the spatial diffeomorphisms. The remaining diffeomorphisms give rise to the mapping class group, which acts as a symmetry group on the phase space. This action induces a unitary representation on the loop state space of the Ashtekar formalism. Certain elements of the diffeomorphism group can be regarded as asymptotic rotations of space relative to its surroundings. We construct states that transform non-trivially under a $2\pi$-rotation: gravitational quantum states with fractional spin.

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1 Introduction

That states with fractional spin may exist in pure quantum gravity was suggested as early as 1959 by Finkelstein and Misner [?]. They believed that certain topological properties of 3-manifolds may give rise to a notion of spinoriality. Decades later, Friedman and Sorkin [?] showed that this possibility is implicit in a canonical formulation of quantum gravity. The purpose of our paper is to provide evidence for this conjecture by finding genuine spinorial states in the loop state space of canonical quantum gravity.

Spinorial states in pure quantum gravity have been discussed in connection with topological geons, bound states describing regions of non-trivial spatial topology, whose origins lie in Wheeler’s work [?]. It is therefore relevant to review the mathematical framework which leads to the idea of a ‘particle of topology’. An underlying issue is the search for a quantum theory given an appropriate classical limit, which is often a highly non-trivial task. In particular, when the classical configuration space of a system is not simply connected it is possible to construct many quantum theories, each with its own physical properties. This is the case for Yang-Mills theories, where the different state spaces are known as $\theta$-sectors. Our approach to quantisation sectors will rest on the canonical viewpoint, where the interplay between geometry and group theory [?, ?, ?] is best appreciated. Let us outline the main points.

A canonical formulation of a Yang-Mills theory is characterised by a time-independent group of gauge transformations $G$, which acts on the phase space of the theory. This action can be viewed as an internal symmetry transformation at each space point. The Gauss law demands that we regard as equivalent all phase space points which are related by the action of the connected component $G_0$ of $G$, while invariance under all of $G$ is not necessarily required. Hence the group $G/G_0$ can act non-trivially on the reduced phase space: the space of orbits under the action of $G_0$. When quantising we are lead to consider a Hilbert space carrying a unitary representation $U$ of this quotient group. Superselection sectors arise when we demand that observables be invariant under the full $G$, which in the quantum theory translates to the requirement that operators should commute with all elements of $U$. In this way we ensure that these operators do not mix states from different unitary irreducible representations of $G/G_0$, so that each of these representations constitutes a separate quantum theory. If the internal symmetry group of the theory is a non-Abelian simple Lie group and the physical 3-space is simply connected, then it can be shown that $G/G_0 = \mathbb{Z}$, where the parameter labelling the irreps $e^{i\theta}$ is the well-known $\theta$-angle from conventional instanton theory.

The same applies to general relativity, where the analogues of the gauge transformations are the diffeomorphisms and the canonical formalism demands only a subgroup of diffeomorphisms to be redundancy transformations. We are left with
the non-trivial action of the mapping-class-group, which we will often call $G$. Interesting phenomena arise when one interprets this group as the symmetry group of bound states. There has been extensive research on the structure of $G$, here we just indicate the basic features.

We recall that in canonical approaches to gravity there is a preferred 3-manifold $\Sigma$ which represents physical space ‘at a given moment in time’. It turns out that at a kinematical level one can attribute particle properties to non-trivial pieces of topology, using $G$ as the asymptotic symmetry group. Let us make this statement a little more concrete. A region of space will give rise to a bound state only when viewed in isolation by a distant observer. We capture this idea by demanding that $\Sigma$ be asymptotically flat. There is a theorem according to which any such 3-manifold $\Sigma$ admits a unique decomposition — up to diffeomorphism — into finitely many basic constituents, that is:

$$\Sigma = \mathbb{R}^3 \# P_1 \# \ldots \# P_n,$$

The symbol $\#$ denotes the connected sum, which is the operation whereby two oriented 3-manifolds are glued together, by removing an open ball from each and identifying the resulting 2-sphere boundaries with an orientation-reversing diffeomorphism. The basic elements $P_n$ in the decomposition are prime manifolds, which means that they cannot be further reduced into the connected sum of other 3-manifolds apart from the 3-sphere (note that $P \# S^3 = P$). For classifications and properties of these manifolds we refer to [?]. We are interested in their interpretation as elementary quantum geons, which finds support in another important theorem: the mapping class group of a manifold without handles, $i.e.$, without primes of the type $S^1 \times S^2$, is a semi-direct product of three subgroups, each with its own geometrical interpretation. Roughly we can describe them as:

- **permutations**, which act by exchanging identical primes;
- **internal diffeomorphisms**, which have support inside a given prime, and hence can be viewed as its internal symmetries;
- and **slides**, which drag primes through other primes along a non-contractible loop in $\Sigma$.

The interested reader will find details in [?]. Among the internal diffeomorphisms, one of particular interest is the $2\pi$ rotation, since it is associated with the notion of spinoriality of a geon. By composing $2\pi$ rotations and exchanges one can investigate correlations between spin and statistics of geons [?]. More generally, properties of topological geons arise via the representation theory of the mapping-class-group of the background 3-space $\Sigma$. Although finite-dimensional representations have been studied and catalogued for a wealth of 3-manifolds, no attempts have been made so far to represent $G$ on a physical Hilbert space. One would nevertheless expect $G(\Sigma)$ to induce an action on the state space in any canonical formulation of quantum gravity based on $\Sigma$. 

3
What type of representation this will be, however, is not known. For example it would be a priori conceivable that any proper gravitational state space carries only a trivial representation of the mapping class group. Luckily, one needs not be so pessimistic, for we will show that the mapping class group induces a nontrivial action on the loop state space of canonical quantum gravity, one of the recent candidates for a gravitational Hilbert space. This is the second main theme of our paper.

A reason why physical representations have not been considered so far is that a geometrodynamical formulation of canonical gravity, where the basic fields are Riemannian metrics on $\Sigma$, lacks a well defined quantum counterpart at the kinematical level. In other words, even neglecting Hamiltonian evolution, it has not been possible to construct a quantum state space rigorously. Over the years this situation has greatly improved. With the introduction of the Ashtekar variables [?], based on connections and triads rather than metrics, it was possible to overcome many of the technical problems that had been hampering the definition of a kinematical state space for quantum general relativity. A genuine Hilbert space was finally introduced, the loop state space. This is the space we will use in our search for physical representations of the mapping class group. In this paper we concentrate on just one aspect of the particle nature of geons: spinoriality. We investigate whether a single prime manifold, when used as the constant slice of canonical quantum gravity, can lead to states with half-integer spin.

We have organised the material as follows. In section 2 we define the quantum Hilbert space where we will construct spinorial states. In section 3 we discuss in detail the definition of spinoriality in terms of diffeomorphisms of 3-manifolds. The results from the first two sections are combined in section 4 to obtain a condition for the existence of spinorial states. Then we demonstrate that these condition are fulfilled for a particular prime, the 3-torus. In this way we provide evidence for the original conjecture. To conclude we indicate how our results can be generalised and we suggest some avenues of future research. Basic knowledge of fibre bundles and also of the Ashtekar formalism is assumed. The references cover general background as well as more specific topics.

2 The kinematical Hilbert space

In this section we describe how the construction of the kinematical Hilbert space used in loop quantum gravity can be adapted to the case where space is no longer compact but asymptotically flat. The restriction to a kinematical space is due to the fact that a satisfactory definition of a quantum Hamiltonian is still lacking and presents a major obstacle in the construction of a complete theory of quantum gravity. Nevertheless it is hoped that essential ‘kinematical’ features will carry over to a
full theory as is often done in discussions of ‘quantum geometry’. Following Baez’s exposition [?], we first introduce an auxiliary Hilbert space, made of ‘cylindrical functions’ of connections. An action of constraint operators can be defined rather naturally on this space, which can be used to reduce it to obtain the final kinematical Hilbert space.

2.1 The auxiliary Hilbert space

First we review the structure of the classical configuration space of canonical gravity within the real Ashtekar formalism and the boundary conditions imposed on the fields by the requirement of spatial asymptotic flatness. In this formalism the spacetime manifold $\mathcal{M}$ is assumed to have the form $\mathcal{M} = \Sigma \times \mathbb{R}$. The condition of asymptotic flatness marks the topology of $\Sigma$: it is required to have a regular end, i.e., a subset $N$ (the neighbourhood of infinity) homeomorphic to $\mathbb{R} \times S^2$, whose complement (the inner part of $\Sigma$) contains all the distinctive topological features. In our laboratory interpretation the cylinder $N$ represents the transition region between the isolated system under study and the ambient universe.

Consider now the principal bundle of frames $P$ over $\Sigma$, which is a trivial $SO(3)$ bundle\(^1\) (if, as we assume, $\Sigma$ is orientable). Denote the space of smooth connections on $P$ by $\mathcal{A}$. Since our bundle is trivial, it has a global cross section, which can be used to pull back the connections to $so(3)$-valued one forms $A^i_a$ on $\Sigma$, where $i$ denotes an internal $so(3)$ index and $a$ a tensor index. If $\Sigma$ was compact these connections would form the usual configuration space in the Ashtekar formalism; for our non-compact $\Sigma$, the boundary condition of asymptotic flatness implies that the connections ‘go to 0 at infinity’ at an appropriate rate. To make this precise we introduce a radial coordinate system in the trivial region of $\Sigma$, defined with respect to any metric on $\Sigma$ that is flat in this region. It has been shown [?, ?] that in order to ensure the equivalence of the geometrodynamical and connection formulation of asymptotically trivial general relativity we need to impose the following fall-off condition on our connections as $r \to \infty$:

$$A^i_a = \frac{G^i_a}{r^2} + O\left(\frac{1}{r^3}\right)$$\(^{(1)}\)

where $G^i_a$ is the leading order part of $A^i_a$. Let us denote the subspace of $\mathcal{A}$ whose elements satisfy this condition by $\mathcal{A}_{\infty}$.

Our discussion will be greatly simplified if instead of $\Sigma$ we use its one-point compactification $\tilde{\Sigma} = \Sigma \cup \infty$, which is well-defined by the existence of a regular end in $\Sigma$, along with the inclusion $\iota : \Sigma \hookrightarrow \tilde{\Sigma}$. In the remainder of this section all quantities denoted by a tilde will relate to $\tilde{\Sigma}$. The fall off condition in equation (1) allows us to extend all connections in $\mathcal{A}_{\infty}$ to infinity, i.e., for every $A \in \mathcal{A}_{\infty}$ there

\(^1\)Note that we are using the real Ashtekar variables [?].
exists a well-defined connection \( \tilde{A} \) on \( \tilde{\Sigma} \) with pull-back \( \iota^* \tilde{A} = A \). We can define these connections explicitly by doing a coordinate transformation \( r \to v \equiv 1/r \) on the radial coordinates defined above. In these coordinates we have \( \tilde{A}^i_a = G^i_a v^2 + O(v^3) \), which tends smoothly to 0 as \( v \to 0 \). Hence, taking \( v = 0 \) at infinity, we can set \( \tilde{A}^i_a = A^i_a \) on \( \iota(\Sigma) \) and \( \tilde{A}^i_a = 0 \) at \( \infty \), which is a covariant, and hence well-defined, boundary condition.

We are now ready to begin the construction of our Hilbert space. In a finite-dimensional theory this would be a space of square-integrable functions on the classical configuration space. But when considering infinite-dimensional configuration spaces such as \( \mathcal{A}_\infty \) this construction is impeded by the lack of an analogue of the Lebesgue measure. This problem has been addressed in several papers (see [?] and references therein) and eventually solved by defining a Hilbert space over an appropriate completion of \( \mathcal{A}_\infty \), where a measure can be defined. We now make use of the compactification of \( \Sigma \) to adapt the construction given by Baez in [?] to the asymptotically flat case. We will see later that the use of \( \tilde{\Sigma} \) is essential for the construction of spinorial states. The starting point towards a Hilbert space is to select within the space of all functions on \( \mathcal{A}_\infty \) a subset \( \text{Fun}_0(\mathcal{A}_\infty) \), since this will make the definition of an inner-product possible.

So, let \( \gamma : [0, 1] \to \tilde{\Sigma} \) be a piecewise analytic path\(^2\) in our compactified manifold. Any connection \( A \in \mathcal{A}_\infty \) gives rise to a holonomy \( H(\tilde{A}, \gamma) \), that is, to an \( \text{SO}(3) \)-equivariant map between the fibres of \( \tilde{P} \) over the endpoints of the path, where \( \tilde{P} \) is the principle bundle of frames over \( \tilde{\Sigma} \). Since the bundle \( \tilde{P} \) is trivial we can identify its fibres through a global cross-section and view the holonomy along a given path as an \( \text{SO}(3) \) element. One then introduces, in analogy with the cylindrical functions used in field theory, the algebra of functions of \( \mathcal{A}_\infty \) generated by those of the form:

\[
\psi_{f, \Gamma}(A) = f(H(\tilde{A}, \gamma_1), \ldots, H(\tilde{A}, \gamma_n))
\]

where the set of paths \( \Gamma \equiv \{ \gamma_1, \ldots, \gamma_n \} \) is an embedded graph\(^3\) in \( \tilde{\Sigma} \), and \( f \) is a continuous function taking \( \text{SO}(3) \times \cdots \times \text{SO}(3) \) to \( \mathbb{C} \). Notice that the same functional can be defined through infinitely many pairs \( f, \Gamma \). For example, the Wilson loop on some closed path \( \gamma \) is the same as the product of holonomy traces along the two subpaths between any two points in \( \gamma \). Now let \( \text{Fun}(\mathcal{A}_\infty) \) be the completion of \( \text{Fun}_0(\mathcal{A}_\infty) \) in the sup norm (which is well-defined since \( \text{SO}(3)^n \) is compact):

\[
\| \psi \|_\infty = \sup_{A \in \mathcal{A}_\infty} |\psi(A)|,
\]

where we have dropped the subscripts for ease of notation. We continue with the construction of the kinematical Hilbert space by making \( \text{Fun}(\mathcal{A}_\infty) \) into an \( L^2 \) space; this amounts to specifying how to integrate functions. Here one faces the problem of

\(^2\)The analyticity requirement is not strictly necessary but is usually made for simplicity [?]. All our topological considerations will be valid for the more general smooth case.

\(^3\)An embedded graph is a finite set of paths \( \gamma_i : [0, 1] \to \Sigma \) that are embeddings when restricted to \( (0, 1) \) and intersect, if at all, only at their endpoints.
constructing appropriate measures, which as mentioned before is extremely difficult. A way round this problem is to introduce a so-called ‘generalised measure’, which is a bounded linear functional on Fun(\(\mathcal{A}_\infty\)). One such functional, the uniform generalised measure, can be obtained by using the Haar measure on SO(3) and defining:

\[
\int_{\mathcal{A}_\infty} d\mu \psi \equiv \int_{SO(3)^n} f(g_1, \ldots, g_n) dg_1 \cdots dg_n, \tag{2}
\]

where the symbol \(\int d\mu\) signifies a map Fun_0(\(\mathcal{A}_\infty\)) \(\to\) \(\mathbb{C}\). Note that the right hand side does not depend on \(\Gamma\). More importantly it can be shown (c.f. [?, ?, ?, ?]) that if \(\psi_{f,\Gamma} = \psi'_{f,\Gamma'}\), then \(\int d\mu \psi = \int d\mu \psi'\). Since Fun_0(\(\mathcal{A}_\infty\)) is dense in Fun(\(\mathcal{A}_\infty\)), the functional in equation (2) extends uniquely to a continuous linear functional on Fun(\(\mathcal{A}_\infty\)), enabling us to define the norm:

\[
\|\psi\|_2 \equiv \left(\int_{\mathcal{A}_\infty} d\mu |\psi|^2\right)^{1/2}, \tag{3}
\]

and the inner product

\[
\langle \psi_1 | \psi_2 \rangle = \int_{SO(3)^n} f_1^*(g_1, \ldots, g_n) f_2(g_1, \ldots, g_n) dg_1 \cdots dg_n, \tag{4}
\]

where, if the original functions \(f_1\) and \(f_2\) have a different number of arguments, say \(f_1 : SO(3)^m \to \mathbb{C}\) with \(m < n\), we trivially extend \(f_1\) to a function on SO(3)^n which does not depend on the last \(n - m\) arguments. The auxiliary Hilbert space \(L^2(\mathcal{A}_\infty)\) is the completion of Fun(\(\mathcal{A}_\infty\)) with respect to this norm. If desired this space can also be regarded as an \(L^2\) space defined with respect to a genuine measure on some completion \(\mathcal{A}_\infty\) of \(\mathcal{A}_\infty\). Since this will not be needed in the rest of our discussion we leave the details to the references.

### 2.2 Constraints and the kinematical Hilbert space

It is well known that canonical general relativity is a theory of constraints. Classically these play the dual role of restricting the phase space and generating physically important flows. In the quantum theory however, following the Dirac procedure, this translates to the sole condition that physical states have to be annihilated by operator versions of the constraints.

A consequence of the extra SO(3) redundancy introduced in the passage from metrics to triads is that there are three types of constraints in the Ashtekar formalism, as opposed to two in geometrodynamics. As mentioned in the introduction we will ignore the Hamiltonian constraint, which is in some sense responsible for dynamics and time evolution. This leaves us with the momentum constraint \(\mathcal{H}_i\), which classically generates diffeomorphisms of the 3-manifold \(\Sigma\), and the Gauss constraint,
which generates gauge transformations of the principal bundle where the connection fields are defined. This geometrical interpretation of the constraints greatly simplifies the passage to the quantum theory. Instead of looking for operator equivalents of the classical constraints, with all the associated operator ordering and regularisation problems, we demand that physical states be invariant under the diffeomorphisms and gauge transformations generated by the constraints.

To recognise the diffeomorphisms generated by the momentum constraint, we need to remember the asymptotic structure of the theory. Let us start by defining two subgroups of the diffeomorphism group of $\Sigma$:

$$D_\infty(\Sigma) \equiv \{ \text{diffeos in } D(\Sigma) \text{ that preserve the asymptotic structure of } A_\infty \}$$

$$D_F(\Sigma) \equiv \{ \text{diffeos in } D(\Sigma) \text{ that are asymptotically trivial} \}$$

We denote their respective identity components by $D_0^\infty$ and $D_0^F$. For example, the diffeomorphism $\phi$ belongs to $D_0^F$ iff there is a continuous curve of diffeos $t \mapsto \Phi(t)$ with $\Phi(t)$ asymptotically trivial for all $t$, such that $\Phi(0) = \text{id}_\Sigma$ and $\Phi(1) = \phi$. We will also call elements of $D_0^F$ small and the remaining elements of $D_F$ large diffeomorphisms. Analogous definitions apply for the gauge transformations.

Since phase-space is made of asymptotically-flat connections, it would be meaningless to consider diffeomorphisms that take us out of this space. Therefore $D_\infty$ is the total group of diffeomorphisms acting on our theory. But the momentum constraint generates only the smaller group $D_0^F$. To see this, recall that, in general, $H_i$ determines flows in phase-space by smearing with vector fields on $\Sigma$. Thus from a vector field $\xi$, one obtains the generating functional $H(\xi) : [A, \pi] \mapsto \int dx^3 \xi^i H_i(A, \pi)$, where $A$ and $\pi$ are the canonical coordinates on the phase space. It is easy to see that the flow generated by a given $H(\xi)$ is precisely that induced by the one parameter subgroup of diffeomorphisms of $\Sigma$ associated with the vector field $\xi$. Without further restrictions, this would lead to all diffeomorphisms in $D_\infty^0$.

What is special about our non-compact theory is that only asymptotically trivial\footnote{The precise fall-off condition will not be relevant to our topological considerations.} vector fields can be used in the smearing, which clearly reduces the flows generated to $D_0^F$. This requirement is made to satisfy the necessary condition that the generating functionals have well-defined functional derivatives, in the sense that no boundary terms appear in their variations with respect to the canonical variables. In a similar way one can derive that the Gauss constraint only generates the gauge transformations in the group $G_0^F$.

Let us now define unitary actions of the total transformation groups on the auxiliary Hilbert space $L^2(A_\infty)$, which are induced naturally from the action of gauge transformations and diffeomorphisms on the space of connections $A_\infty$. First we consider the gauge transformations. In general they are defined to be any principal automorphism of the underlying principal bundle (the base space is kept fixed). Any
such morphism \( g \), pulls back each connection \( A \) on \( P \) to some other connection, thus defining a group action of \( \mathcal{G} \) on \( \mathcal{A} \), which we denote: \( A \to gA \). Since the bundle of frames is trivial, gauge transformations can be identified with the set of \( C^\infty \) maps \( g : \Sigma \to SO(3) \). So in this case the familiar form of the gauge transformations:

\[
A_a(x) \to g(x)A_a(x)g^{-1}(x) + g(x)\partial_a g^{-1}(x),
\]

can be interpreted globally as an automorphism of the space of Lie-algebra valued connection one-forms on \( \Sigma \). The subgroup \( \mathcal{G}_\infty \) is simply the subset of maps from \( \Sigma \) to \( SO(3) \) that respect the boundary conditions of the connection. The action of \( \mathcal{G}_\infty \) on \( \mathcal{A}_\infty \) induces a representation of \( \mathcal{G}_\infty \) on \( \text{Fun}_0(\mathcal{A}_\infty) \) and by extension also on \( \text{Fun}(\mathcal{A}_\infty) \) and \( L^2(\mathcal{A}_\infty) \) given by:

\[
(U_g\varphi)(A) = \varphi(g^{-1}A),
\]

where \( g \in \mathcal{G}_\infty \). The fact that \( U_g \) is a unitary operator follows directly from our definition of the inner product.

A unitary action of \( \mathcal{D}_\infty \) is defined analogously. Diffeomorphisms of \( \Sigma \) lift to the frame bundle \( P \), therefore inducing a transformation of the connections denoted: \( A \to \phi A \) with \( \phi \in \mathcal{D}_\infty \). In terms of the connection one-forms \( A^i_a \) on \( \Sigma \) this action is given by the ordinary pull-back by \( \phi \). We define the representation of \( \mathcal{D}_\infty \) on \( L^2(\mathcal{A}_\infty) \) by:

\[
(U_\phi\varphi)(A) = \varphi(\phi^{-1}A).
\]

Unitarity also follows from the diffeomorphism invariance of the generalised measure.

To implement the constraints we now have to demand that physical states are left invariant by the actions of \( \mathcal{G}_0^F \) and \( \mathcal{D}_0^F \). Gauge invariance is simply achieved by restricting to the subspace \( L^2(\mathcal{A}_\infty/\mathcal{G}_0^F) \subset L^2(\mathcal{A}_\infty) \) of gauge invariant functions. A particular example of such a state, which will be of importance later on is given by the Wilson loop:

\[
\varphi(A) = tr[\rho(H(\tilde{A}, \ell))], \quad (5)
\]

where \( \ell \) is a loop in \( \tilde{\Sigma} \) based at infinity and \( \rho \) is any representation of \( SO(3) \). A natural treatment of gauge-invariant quantities was in fact one of the main motivations for using loop variables.

In the case of the momentum constraint things get more complicated, for it can be shown that there are no non-trivial diffeomorphism invariant states in \( L^2(\mathcal{A}_\infty/\mathcal{G}_F^0) \). A similar situation arises even in elementary quantum mechanics when considering constraint operators with a continuous spectrum. These have non-normalisable eigenvectors, which means in particular that the space of states satisfying the constraint — eigenvectors with eigenvalue 0 — is empty. This problem is overcome in two steps\(^5\) [?, ?]. First one considers distributional solutions to the momentum

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\(^5\)In the series of steps given towards the definition of \( H_{kin} \), we have omitted motivations and details. These come mainly from functional analysis and the interested reader should consult the literature.
constraint. So instead of imposing the constraint directly on $L^2(\mathcal{A}_\infty/G^0_F)$, one imposes it on the topological dual of a dense subset thereof, which can be taken to be Fun($\mathcal{A}_\infty/G^0_F$). Secondly one selects a subspace $\mathcal{F}$ of Fun($\mathcal{A}_\infty/G^0_F$)′ on which a natural product of an inner product is possible. The essential point for us is that there is an induced action of $D_\infty$ on Fun($\mathcal{A}_\infty/G^0_F$)′ and hence $\mathcal{F}$, defined by:

$$ (U^t_\phi \psi)(\varphi) = \psi(U^t_\phi \varphi), \quad (6) $$

where $U_\phi$ represents the action of the diffeomorphism $\phi$ on Fun($\mathcal{A}_\infty/G^0_F$), $\varphi \in$ Fun($\mathcal{A}_\infty/G^0_F$) and $\psi \in$ Fun($\mathcal{A}_\infty/G^0_F$)′. Our final reduced Hilbert space $H_{\text{kin}}$ is then given by the subspace of functionals in $\mathcal{F}$ which satisfy:

$$ \psi'(U_\phi \varphi) = \psi'(\varphi) \quad (7) $$

for all $\varphi$ and $\phi \in D^0_F$. Equivalently we can view $H_{\text{kin}}$ as containing functionals on the quotient Fun($\mathcal{A}_\infty/G^0_F$)/$D^0_F$. A way to construct such elements is via diffeomorphism group averages on vectors in Fun($\mathcal{A}_\infty/G^0_F$), see [?]. An alternative procedure through generalised measures can be found in [?]. It is shown that many non-trivial elements actually exist.

3 Spinorial manifolds

As we have discussed in the introduction there is a natural symmetry group that appears in gauge theories, which is related to configuration space topology and results in the emergence of the well-known $\theta$-sectors. The analogous situation in general relativity will be explored in more detail in this section. We will introduce the mapping class group and show how it can be used to define a notion of spin for the gravitational states.

We recall that the origin of $\theta$-sectors in gauge theories is a discrepancy between all allowed gauge transformations (following Giulini we denote these invariances) and those generated by the constraints (redundancies). When considering general relativity the analogue of the gauge transformations are the diffeomorphisms. Indeed we have seen that only the group $D^0_F$ was required to be a redundancy group of our physical system, while we were free to define an action of the larger group $D_\infty$, which induces non-trivial transformations of the rigid structure of $\Sigma$ at infinity. After construction of our final kinematical Hilbert space $H_{\text{kin}}$ we are thus left with a residual group action of $S = D_\infty(\Sigma)/D^0_F(\Sigma)$. Associated to $S$ is its discrete normal subgroup $G = D_F/D^0_F$: the mapping class group of $\Sigma$.

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6In the Ashtekar variables we could have also considered $\theta$-sectors arising as a result of $SO(3)$ gauge invariance, but it can be shown that in our construction of the Hilbert space there is no residual gauge action. This provides an example that even if classically we have the necessary structure to give rise to $\theta$-sectors (c.f. [?]), this does not necessarily imply that these are realised in a quantum theory.
In order to interpret the role of this group remember that usually observables in general relativity are assumed to be invariant under all diffeomorphisms. A way to proceed would be to regard $S$ as a residual gauge group and use it to reduce the state space further, either classically or quantum mechanically. We won’t follow this route, as it evidently rules out any interesting effects of the type we are looking for. Another possibility is to consider $S$ as a proper symmetry group rather than a gauge group. This less restrictive approach is the one to take when interested in detecting superselection sectors. It is physically motivated by the observation that our point at infinity does not represent an actual infinity, instead it is a convenient model for the environment of our system. This means that diffeomorphisms lying in $S$, which are non-trivial at infinity correspond to genuine physical changes relative to this environment. Therefore we will let $S$ act properly on our state space. The outcome of this proper action is what we will be investigating. For more discussion on this point we refer to [?, ?, ?].

We will now take a closer look at these symmetries, especially their relation to concepts of rotation and spinoriality. It turns out that in order to investigate the dependence of $S$ on the topology of $\Sigma$ it is convenient to make use of the one-point compactification $\tilde{\Sigma}$ of $\Sigma$ introduced earlier, since the natural subgroups of diffeomorphisms in $\Sigma$ have analogues in the closed 3-manifold $\tilde{\Sigma}$, which can be characterised in a very concise manner. So let us define a few more subgroups of the respective diffeomorphism groups and establish some homotopy equivalences. If $\phi$ is a diffeomorphism of some manifold $M$ and $p$ a point in the manifold, then $\phi_*|p$ denotes the push-forward isomorphism from $T_p M$ onto $T_{\phi(p)} M$. This general linear transformation will be said to be in $SO(3)$ if its matrix representative with respect to some fixed frame is in $SO(3)$.

$$\begin{align*}
D_{\infty}(\Sigma) &\equiv \{ \phi \in D(\Sigma) \text{ such that } \phi(\infty) = \infty \text{ and } \phi_*|\infty \in SO(3) \} \\
D_F(\Sigma) &\equiv \{ \phi \in D(\Sigma) \text{ such that } \phi(\infty) = \infty \text{ and } \phi_*|\infty = 1 \}
\end{align*}$$

(8)

Each of these groups contains its identity component as a normal subgroup. These definitions are well motivated since the extra motions in $D_{\infty}(\Sigma)$, over those in $D_F(\Sigma)$, are rigid rotations at infinity, which are needed to define spinoriality. Indeed, it can be shown that the spaces $D_{\infty}(\Sigma)$ and $D_{\infty}(\tilde{\Sigma})$ are homotopically equivalent, as are $D_F(\Sigma)$ and $D_F(\tilde{\Sigma})$ [?]. In particular this implies that the mapping class group $G(\Sigma)$ is isomorphic to $D_F(\Sigma)/D_F^0(\Sigma) = \pi_0(D_F^0)$, where the isomorphism is simply given by:

$$\sigma : G(\tilde{\Sigma}) \rightarrow G(\Sigma) : [\phi] \mapsto [\phi|_{\Sigma}],$$

(9)

where $[\cdot]$ denotes an appropriate equivalence class of diffeomorphisms. Our final physical results will refer to the space $\Sigma$, but the above isomorphism enable us to use $\tilde{\Sigma}$ as an equivalent ground where to discuss the role of $G$. From now on all diffeomorphism groups will refer to $\tilde{\Sigma}$ unless stated otherwise.

To see that the mapping class group naturally leads to a concept of spinoriality of gravitational states, we need a precise statement of what we understand both by
a rotation and by spinoriality. If we adopted the general definition of a rotation as an element of a transformation group isomorphic to $SO(3)$, we would be precluding the possibility of spinoriality. Spinoriality means that a system together with its environment is not returned to its initial state after the system is rotated by $2\pi$. This can also be a macroscopic property as is suggested by Dirac’s string problem (c.f. [?]): under certain boundary conditions, the $2\pi$-rotation of three strings cannot be undone while their $4\pi$-rotation still can. The key to allow for such effects is to look at rotations as continuous sequences of transformations, i.e., as curves in $SO(3)$. Of course, what this implies is that we may be actually talking about $SU(2)$, the universal cover of $SO(3)$.

In our case we have to investigate how subgroups of the transformation groups $S$ and $G$ are in correspondence to path classes in $SO(3)$. To this end we need to make use of the following principal fibre bundle:

$$D_F \rightarrow D_\infty \xrightarrow{p} SO(3)$$

and the associated fibration:

$$D_F/D^0_F = G \rightarrow S \xrightarrow{p} SO(3)$$

The homotopy lifting theorem ensures that any loop in $SO(3)$ lifts to a unique path through a given point in $S$. In particular, the end point of a lift depends only on the homotopy class of the original loop. Therefore the lift $\bar{\gamma}$ of any non-trivial loop $\gamma$ in $SO(3)$ determines a homomorphism of $\pi_1(SO(3)) = \mathbb{Z}$ into the fibre $G$, given by $\mu([\gamma]) = [\bar{\gamma}(1)]_G$. Spinoriality is decided according to the two possible outcomes:

(i) $\mathbb{Z}$ is faithfully represented, that is $\bar{\gamma}(1)$ is not in the connected component of $D_F$;

(ii) $\mathbb{Z}$ is trivially represented, that is $\bar{\gamma}(1) \in D^0_F$.

Which of the two occurs depends on the topology of the manifold $\tilde{\Sigma}$, which will be respectively deemed as spinorial and non-spinorial. In fact it can be shown (see e.g. [?, ?]) that for manifolds where (i) holds\footnote{The $\mathbb{Z}$ action is generated by $(-1, -1) \in G \times SU(2)$, where the -1 on the right generates the centre of $SU(2)$, while the -1 on the left belongs to some $\mathbb{Z} \subset G$.}, $S \cong \{G \times SU(2)\}/\mathbb{Z}$; while for manifolds where (ii) holds we have simply $S \cong G \times SO(3)$. Hence a spinorial
A manifold is one whose associated symmetry group contains $SU(2)$ but not $SO(3)$ as a subgroup. The remaining manifolds are non-spinorial. A complete classification according to spinoriality has been established for all known 3-manifolds.

Now that we have a criterion for spinoriality, let us emphasise its geometrical content, so that we can visualise the elements of $S$ and $G$ involved. We do so by explicitly lifting a non-trivial loop in $SO(3)$ to a $2\pi$-rotation of a neighbourhood of infinity in $\tilde{\Sigma}$, which is a rotation parallel to spheres ?? henceforth referred to as a ‘prime rotation’. Consider a region $B$ around infinity small enough to be homeomorphic to $\mathbb{R}^3$ with associated radial coordinates. Consider also the 2-spheres $S_1$ and $S_2$ at values $r_1 < r_2$ of the radial distance $r$ from infinity in $B$. They bound two concentric balls $B_1 \subset B_2$. Let $f : B \rightarrow [0, 1]$ be a continuous decreasing function of the radial coordinate with

$$f = \begin{cases} 1 & \text{if } r < r_1 \\ 0 & \text{if } r > r_2 \end{cases}$$

Then choose an axis in $B$ through the point at infinity and let $R(\theta)$ be the rotation of $B$ by $\theta$ around that axis. The $2\pi$-rotation of $B_1$ gives a diffeomorphism $R$ of $\tilde{\Sigma}$ defined by:

$$R \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in \tilde{\Sigma} - B \\ R(2\pi f(r))\mathbf{x} & \text{if } \mathbf{x} \in B \end{cases}$$

(13)

In particular every point in $B_1$ is left invariant by $R$ so that $R \in \mathcal{D}_F$. The $2\pi$-rotation is moreover in $\mathcal{D}_0^\infty$, since the curve of diffeos $R_t$ with

$$R_t \mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in \tilde{\Sigma} - B \\ R(t2\pi f(r))\mathbf{x} & \text{if } \mathbf{x} \in B \end{cases}$$

has $R_t(\infty) = \infty$, $R_0 = id_{\tilde{\Sigma}}$ and $R_1 = R$.

The above defines a curve $R_t$ of diffeomorphisms from $id_{\tilde{\Sigma}}$ to the $2\pi$ rotation $R$, which projects\(^8\) via $p$ to a non-trivial loop in $SO(3)$. All the diffeomorphisms in the curve are representative elements of classes in $S$ and the endpoint $R$ is moreover an element of a class in $G$. Thus $\mathbb{Z}_2$ is trivially represented if and only if $R \in \mathcal{D}_F^0$. So in summary we can say that a 3-manifold is spinorial if a $2\pi$ rotation of a neighbourhood of infinity cannot be deformed to the identity along a curve of diffeomorphisms that leave the neighbourhood pointwise invariant\(^9\).

---

\(^8\)Note that any other choice of spheres $S_1$ and $S_2$, as well as any choice of rotation axis, leads upon projection to a homotopically equivalent loop and therefore defines the same element of $G$.

\(^9\)Note that in topological considerations of diffeomorphism groups we may replace frame-fixing diffeomorphisms with those that leave an arbitrarily small neighbourhood of infinity fixed [??].
4 Spinorial states

We come now to the main part of the paper, where we put together all the information gathered so far in order to derive sufficient conditions for the existence of spinorial states. In quantum theory spinoriality is used to denote states with half integer angular momentum, according to the representation theory of $SU(2)$. Following this idea, one could try to define self-adjoint angular momentum operators on the Hilbert space. This would involve finding operator equivalents of classical ADM expressions and using these to define spinoriality, as was done originally in [?]. At the end of the day, one is really making use of the rotations generated by the angular momentum operators. Indeed we know that in quantum theory any weakly continuous unitary representation of a one parameter group of symmetries will uniquely define a self-adjoint operator. This motivates the definition of spinoriality we will be using: spinorial states will be those that change sign under the $2\pi$ rotation in some unitary representation of the rotation group.

As we have argued in the previous section a $2\pi$ rotation is an element of the symmetry group $G$ that is obtained by lifting a non-trivial loop in $SO(3)$. The endpoint $R$ of the lift is a representative of the equivalence class $[R]$ of diffeomorphisms. Whether these diffeomorphisms act on $\Sigma$ or $\tilde{\Sigma}$ should be clear from the context. Because of the natural isomorphism between $G(\tilde{\Sigma})$ and $G(\Sigma)$ given by equation (9), the same notion will be used in either case. We are interested in the action of $[R]$ on $H_{kin}$. Since $H_{kin}$ is invariant under $D^0_F$, equation (6) induces a well-defined representation of $S$, and hence of its subgroup $G$. In particular we can write:

$$[R] \psi'([\varphi]) = \psi'([U^*_R\varphi]), \quad (15)$$

where $[\varphi]$ is a class in $\text{Fun}(A_\infty/G_0^0)/D^0_F$ and $\psi' \in H_{kin}$. Then, a sufficient condition for the existence of spinorial states is:

$$[R] \psi'_1 = \psi'_2 \neq \psi'_1$$

since then the state $\psi'_1 - \psi'_2$ will be spinorial. Indeed, the $4\pi$ rotation is always trivial, so that $[R]^2 \psi'_1 = \psi'_1$, and therefore $[R](\psi'_1 - \psi'_2) = -(\psi'_1 - \psi'_2)$. We see immediately that this is only possible if $[R]$ is not the identity, i.e., if $R$ is not in $D^0_F$, which means that $\tilde{\Sigma}$ must be spinorial. That spinoriality of $\tilde{\Sigma}$ is a necessary condition for the existence of spinorial states was already established in [?]. Whether it is also sufficient is an open question, which can only be addressed in the presence of a concrete physical Hilbert space. In the following we use the properties of $H_{kin}$ to derive a topological criterion which ensures the existence of spinorial states.

Looking at equation (15), we see that if we find a vector $[\varphi] \in \text{Fun}(A_\infty/G^0_F)/D^0_F$ such that $[U_R\varphi] \neq [\varphi]$, then there will be a spinorial state in $H_{kin}$. In particular, any functional in $F$ that takes on different values on these two equivalence classes will have the desired property.
In order to find a suitable vector in $\text{Fun}(\mathcal{A}_\infty / \mathcal{G}_F^0)/\mathcal{D}_F^0$, we look for loops in $\tilde{\Sigma}$ which are not $\mathcal{D}_F$-ambient-isotopic$^{10}$ to their images under the $2\pi$-rotation, since then cylindrical functions based on these loops will transform non-trivially. We make this precise by looking at states based on loops through infinity. Consider the cylindrical function $\varphi \in \text{Fun}(\mathcal{A}_\infty / \mathcal{G}_F^0)$ defined in equation (5). $R$ acts on $\varphi$ according to:

$$(U_R\varphi)(A) = \text{tr}[\rho(H(R^{-1}\tilde{A}, \ell))]$$

where we’ve used the symbol $R$ for the $2\pi$-rotation as a diffeomorphism in $\Sigma$ and in $\tilde{\Sigma}$. From the definition of the holonomy it follows that:

$$(U_R\varphi)(A) = \text{tr}[\rho(H(\tilde{A}, R^{-1} \circ \ell))].$$

But we know that given two distinct loops $\ell_1$ and $\ell_2$, there is always a connection whose holonomy along $\ell_1$ is different from that along $\ell_2$. Now the elements in the equivalence class $[\varphi]$ are related by the action of a small diffeomorphism. Hence it follows that if $\ell$ and $R^{-1} \circ \ell$ are not related by an element of $\mathcal{D}_F^0$, then the states corresponding to these loops will lie in different classes and $[U_R\varphi] \neq [\varphi]$. We conclude that the $2\pi$-rotation will be represented non-trivially on $H_{\text{kin}}$ provided there is a loop $\ell$ in $\tilde{\Sigma}$ such that no diffeomorphism in $\mathcal{D}_F^0$ takes $\ell$ into its $2\pi$-rotation $R \circ \ell$; or in words, if two $R$-related loops $\ell$ and $R \circ \ell$ are not ambient isotopic relative to infinity.

Now it becomes clear why loops based at infinity are needed. As explained earlier we can choose $R$ to be any prime rotation around $\infty$: they all define the same element of $G$. Thus if a loop $\ell$ doesn’t go through infinity we can define $R$ in a neighbourhood of infinity disjoint from $\ell$. Therefore $R \circ \ell$ and $\ell$ are trivially ambient isotopic. This has an obvious physical interpretation: in order to obtain spinorial effects the rotation of our isolated system needs to be somehow communicated to the ambient environment. This can only be done if the states themselves are highly non local in the sense that they ‘extend to infinity’.

### 4.1 Spinorial states and the fundamental group

In this section we show that the prime manifold $\tilde{\Sigma} = T^3$ contains loops not $\mathcal{D}_F$-isotopic to their $2\pi$ rotation. As we have seen this guarantees the existence of spinorial states in the Hilbert space $H_{\text{kin}}$ associated with $T^3$. When discussing how these results can be generalised we are lead to a condition that may be crucial for spinoriality of states: the loops — when considered as elements of $\pi_1(\tilde{\Sigma})$ — are not simply generated. As we will see this condition ties in naturally with the fact that all known spinorial primes have a non-cyclic fundamental group.

---

$^{10}$ An $(\mathcal{D}_F -)$ ambient isotopy of a manifold $M$ is a collection $\{I_t\}, t \in [0, 1]$ of homeomorphisms (diffeomorphisms in $\mathcal{D}_F$) of $M$ onto itself such that the map $I : M \times [0, 1] \to M : I(x, t) = I_t(x)$ is continuous. Two embeddings $e, e'$ in $M$ are ambient isotopic if there exists an ambient isotopy $I$ of $M$ such that $I_0 = \text{id}_M$ and $I_1 e = e'$. 

15
4.1.1 Existence of spinorial states in $T^3$

The three-torus $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$ is known to be a spinorial manifold. In the following we will consider a loop $\ell$ in $T^3$ and assume that it is $D_{T}$-isotopic to $R \circ \ell$, where $R$ is a prime rotation in $T^3$. This will lead to a contradiction, demonstrating that $\ell$ is spinorial. The idea is to lift the loops and isotopies to $\mathbb{R}^3$, where with a little work one can identify the structure with the Dirac string, which we briefly describe.

Consider a solid object with three elastic strings attached, these strings are tied to three separate posts. If we twist the object by $2\pi$, around the axis of one of the strings say, then we obtain a twisted set of strings referred to as the Dirac string. The twisting cannot be undone by passing the strings over and around the object, which provides an intuitive picture of spinoriality. We will use this fact to conclude that $R \circ \ell$ cannot be untwisted.

The Dirac string problem has a rigorous formulation in the language of braid theory [?, ?], which we will be using below to show how our problem reduces to it. First some definitions. A braid in $S^2$ is a system of $n$ embedded arcs $x = \{x_1, \ldots, x_n\}$ in $S^2 \times [0, 1]$ such that: (i) each arc $x_i$ intersects each intermediate sphere $S_t$ exactly once, (ii) the arcs $x_1, \ldots, x_n$ intersect each intermediate sphere $S_t$ in exactly $n$ different points. Two braids $B_1, B_2$ in $S^2$ are said to be $s$-isotopic if there is an ambient isotopy $I$ of $S^2 \times [0, 1]$ which takes $B_1$ to $B_2$, such that the intermediate images of the braid, $I_t B_1$, during the isotopy define $S^2$-braids and the endpoints of the braid are kept fixed. The braid group $B_k(S^2)$ is defined through the natural composition of $s$-isotopy classes of braids with $k$ strings. A representative of the trivial braid consists of $k$ radial strings from the inner to the outer sphere. A simple presentation of the Dirac braid is obtained from the trivial braid of three strings by a $2\pi$-rotation of the inner sphere with respect to the outer sphere [?]. The result is a non-trivial braid, which is often denoted $\Delta$.

$T^3$ can be presented as the unit cube centred around 0 in $\mathbb{R}^3$ with opposite sides identified. Let 0 be the preferred point called $\infty$ and consider two embedded loops $\alpha, \beta : [0, 1] \to T^3$ at infinity, given by:

$$
\alpha(s) = [h(s), 0, 0] \\
\beta(s) = [0, 0, h(s)]
$$

where $h : [0, 1] \to [-\frac{1}{2}, \frac{1}{2}]$ is the function:

$$
h(s) = \begin{cases} 
  s & 0 \leq s \leq \frac{1}{2} \\
  s - 1 & \frac{1}{2} < s \leq 1
\end{cases}
$$

Our loop $\ell$ here is the composition $\alpha \beta$. We choose $R$ to be a prime rotation around the $z$-axis, centred at the origin and leaving a ball of radius $\epsilon$ invariant. $R$
maps $\beta$ into itself and maps $\alpha$ to a new loop which we denote by $\alpha'$. We have the following:

**Theorem 1** There is no diffeomorphism in $D^0_F(T^3)$ taking $\alpha\beta$ to $\alpha'\beta$.

**Proof** We use the same notation for the elements involved in the prime rotation as in section 3, choosing the radii of the balls $B_1$ and $B_2$ to be $\epsilon$ and $\epsilon'$ respectively. As anticipated, we proceed by contradiction. So suppose there is a diffeomorphism in $D^0_F$ taking $\alpha\beta$ into $\alpha'\beta$. This implies the existence of an isotopy $I : T^3 \times [0, 1] \to T^3$ that leaves the ball $B_1$ pointwise invariant and satisfies $I(\alpha\beta, 1) = \alpha'\beta$ and $I(x, 0) = x$ for all $x \in T^3$.

We now lift the isotopy to $\mathbb{R}^3$, the universal covering of $T^3$, as illustrated in figures 1a and 1b. Since $\mathbb{Z}^3$ acts freely on $\mathbb{R}^3$ we have the following principal bundle:

$$
\mathbb{Z}^3 \longrightarrow \mathbb{R}^3 \\
\downarrow p \\
\mathbb{R}^3/\mathbb{Z}^3
$$

where $d(x)$ denotes the decimal part\(^\text{11}\) of $x$. First, for each of the loops $\alpha$, $\beta$ and $\alpha'$ there is a unique path: $\mathbb{R} \to \mathbb{R}^3$ through the origin, which we denote $\bar{\alpha}$, $\bar{\beta}$ or $\bar{\alpha}'$, defined by:

$$
p(\bar{\alpha}(s + z)) = \alpha(s) \quad \text{and} \quad \bar{\alpha}(0) = 0
$$

with analogous expressions for $\bar{\beta}$ and $\bar{\alpha}'$. Here $s \in [0, 1]$ and $z \in \mathbb{Z}$. These covering paths are just compositions of the unique lifts of $\alpha$, $\beta$ and $\alpha'$ through different points of the fibre $p^{-1}(\infty)$. It is immediate that $\bar{\alpha}$ is just the $x$-axis and $\bar{\beta}$ the $z$-axis. Secondly we lift the isotopy itself, using the homotopy lifting property for fibre bundles (theorem 11.3 in Steenrod \(?)\). It is easy to see that if the original homotopy is an isotopy, then so is the lifted homotopy. Thus the isotopy $I$ lifts to an isotopy $\bar{I}$ of $\mathbb{R}^3$ which covers $I$, that is:

$$
p(\bar{I}(x, s)) = I(p(x), s)
$$

for $x \in \mathbb{R}^3$ and all $s \in [0, 1]$. This isotopy has the property that it leaves all points in $p^{-1}(\infty)$ fixed, as well as a solid ball of radius $\epsilon$ around each of these points. Indeed, $I$ fixes all points in $B_1$ so that for each $x \in p^{-1}(B_1)$, $\bar{I}(x, s)$ lies in the fibre over $p(x)$ for all $s$. But since $\bar{I}$ is continuous and the fibre is discrete $\bar{I}$ must leave all points in the fibre fixed. It is also easy to verify that:

$$
\bar{I}(\bar{\alpha}, 1) = \bar{\alpha}', \quad \bar{I}(\bar{\beta}, 1) = \bar{\beta}
$$

and hence the lifted isotopy takes the cover of $\alpha$ into the cover of $\alpha'$ and the cover of $\beta$ into itself. Since all points in $\mathbb{R}^3$ with integer coordinates are fixed during the

\(^{11}\)\(d(x) \equiv x - \lfloor x \rfloor\), where $\lfloor x \rfloor$ is the largest integer that is smaller than $x$. 

17
isotopy and since \( \bar{I} \) is periodic, there is an integer radius \( N \) such that for every \( n \) the images of the segments \( \bar{\alpha}_n = \bar{\alpha}(\lfloor n, n + 1 \rfloor) \) and \( \bar{\beta}_n = \bar{\beta}(\lfloor n, n + 1 \rfloor) \) during the isotopy, stay within a ball of radius \( N \) around \( \bar{\alpha}(n + 1/2) \) and \( \bar{\beta}(n + 1/2) \) respectively.

We may simplify \( \bar{\alpha}' \), by composing \( \bar{I} \) with another isotopy \( \bar{I}' \) of \( \mathbb{R}^3 \) that reverses the prime rotation around every point on \( \bar{\alpha}' \) with integer coordinates except 0. This straightens up all the segments \( \bar{\alpha}_m \) except for \( \bar{\alpha}_{-1} \) and \( \bar{\alpha}_1 \). We conclude that there is an isotopy \( K = \bar{I}' \circ \bar{I} \) of \( \mathbb{R}^3 \) that takes \( \bar{\alpha} \bar{\beta} \) onto a new path \( \bar{\alpha}'' \bar{\beta} \), where \( \bar{\alpha}''(t) = \bar{\alpha}'(t) \) for \( |t| \leq \frac{1}{2} \) and \( \bar{\alpha}''(t) = \bar{\alpha}(t) \) for \( |t| > \frac{1}{2} \) (c.f. figure 1c). In addition we are free to impose: (i) \( K \) leaves all points in \( \mathbb{R}^3 \) with integer coordinates as well as the ball \( B_1 \) fixed; (ii) during the whole of \( K \) the curve \( \bar{\alpha} \) remains within \( C_{\bar{\alpha}} \) and \( \bar{\beta} \) remains within \( C_{\bar{\beta}} \), where \( C_{\bar{\alpha}} \) and \( C_{\bar{\beta}} \) are solid cylinders of radius \( N \) around the \( x \) and \( z \)-axis respectively.

The contradiction is now apparent since triviality of \( \bar{\alpha}'' \bar{\beta} \), together with the properties of \( K \), would imply triviality of the Dirac string. To see this more explicitly, consider the following six curves: \( [\epsilon, \infty) \rightarrow \mathbb{R}^3 \):

\[
\gamma_1(t) = \bar{\alpha}(t), \quad \gamma_2(t) = \bar{\beta}(t), \quad \gamma_3(t) = \bar{\beta}(-t) \\
\gamma_1'(t) = \bar{\alpha}''(t), \quad \gamma_2'(t) = \bar{\beta}(t), \quad \gamma_3'(t) = \bar{\beta}(-t)
\]

Choose some \( m \in \mathbb{Z}, m > N \). If we identify the shell in \( \mathbb{R}^3 \) between the two spheres of radius \( \epsilon \) and \( m \) with the product \( S^2 \times [0, 1] \) then the restrictions of the above curves to the interval \( [\epsilon, m] \) define two braids \( \mathcal{B} = \{x_i\} \) and \( \mathcal{B}' = \{x'_i\} \) in \( S^2 \). In fact \( \mathcal{B}' \) is precisely the Dirac braid. It follows that there is no s-isotopy taking the set of three curves \( \{x_i\} \) to the set \( \{x'_i\} \). Notice however that the intermediate strings in our isotopy \( K \) need not define a genuine \( S^2 \) braid, since they can leave the outer sphere \( S_m \) of radius \( m \). Thus at a first glance, the action of the isotopy \( K \) on the braid \( \mathcal{B} \) seems to be more general than s-isotopy. However, as in the case of braids in \( \mathbb{R}^2 \) where s-isotopy and isotopy are equivalent [?], we can convince ourselves that the properties of \( K \) prevent it from using this extra freedom.

Indeed, the only way to relate two braids which, like \( \mathcal{B} \) and \( \mathcal{B}' \), are not equivalent under s-isotopy is by pulling one of the strings \( x_i \) outside the sphere \( S_m \) and over one of the other two\footnote{This move would undo the Dirac twist if one of the curves \( \gamma_i \) had finite length and \( K \) was not restricted by condition (ii) above as shown in figure 2.}. But this is not possible because of property (ii) above and because \( m > N \) implies:

\[
C_{\bar{\alpha}} \cap C_{\bar{\beta}} \cap (\mathbb{R}^3 - B_m) = \emptyset,
\]

where \( B_m \) is the solid ball of radius \( m \) centred at the origin.

We conclude that \( K \) makes use of no additional generators to those in \( \mathcal{B}(S^2) \). It follows from non-triviality of the Dirac braid, \( \mathcal{B}' \), that \( K, \bar{I} \) and \( I \) do not exist \( \square \).
Figure 1: Figure a) depicts a schematic prime rotation about the vertical axis in $T^3$. Figure b) represents the lift $\bar{I}$ to $R^3$ of the isotopy $I$. Figure c) shows the action of the isotopy $K$ on the three semi-infinite strings $\{\gamma_i\}$. The restriction of the curves to the ball $B_m$ of radius $m$ is identified with the ordinary Dirac string $\Delta$. 
Figure 2: If considering curves of finite length, the Dirac twist can be undone if no restrictions are imposed on the isotopy taking the initial to the final configurations.

4.1.2 Generalisations and conjectures

In showing that the above isotopy $I$ does not exist an essential point is that the loop classes $[\alpha]$ and $[\beta]$ are generated by different elements of $\pi_1(T^3)$. We can also define spinorial states using loops generated by the same element, provided their isotopy class encodes the non-cyclicity of $\pi_1(T^3)$, as shown in figure 3. On the other hand, given any generator $[\gamma]$ of $\pi_1(\tilde{\Sigma}, \infty)$, if a loop $\ell$ is in the class $[\gamma]^k$ for some integer $k$ and moreover $\ell$ is contained in a single embedded solid 2-torus, then there is a diffeomorphism in $D^k_\ell(\Sigma)$ taking $\ell$ to $\ell' = R \circ \ell$.

Figure 3: This is a picture of a singly-generated loop $\ell$ that is not ambient isotopic to $R \circ \ell$ through a curve of diffeos that leave $B_1$ invariant.

This suggests that a sufficient condition for the existence of spinorial states may be established through the relationship between spinoriality of a prime and its fundamental group. We start by restating the rather intuitive result that any prime with cyclic fundamental group is non-spinorial. It depends on the Poincare conjecture [?] according to which each closed, connected, simply connected 3-manifold is homeomorphic to $S^3$. As usual $\Sigma$ is any prime, $R$ any prime rotation around the preferred point $\infty$ of $\Sigma$ and $D_F$ the group of diffeomorphisms of $\Sigma$ that leave a ball around $\infty$ invariant.
Lemma 1 Let ℓ be any loop through infinity in a prime \( \tilde{\Sigma} \) with \( \pi_1(\tilde{\Sigma}) \) cyclic, then there is a diffeomorphism in \( D^0_\ell(\tilde{\Sigma}) \) taking \( \ell \) to \( R \circ \ell \).

Proof It suffices to establish that any prime with cyclic fundamental group is non-spinorial. The proof is divided in two cases:

a) If \( \pi_1(\tilde{\Sigma}) = \mathbb{Z}_p \), \( \tilde{\Sigma} \) is non-spinorial since it has a finite fundamental group with cyclic 2-Sylow subgroup [?].

b) If \( \pi_1(\tilde{\Sigma}) = \mathbb{Z} \), then the Poincare conjecture implies that \( \tilde{\Sigma} \) is either \( S^1 \times S^2 \) or the non-orientable handle, usually denoted \( S^1 \times S^2 \), both of which are non-spinorial.

Note that in any case, we can always untwist the rotation of a null-homotopic loop through infinity, by simply communicating the \( 2\pi \)-rotation outside a ball that contains the loop. This is as expected, for such a loop can be regarded as lying in the regular end of \( \Sigma \) which means that it does not encode any of its non-trivial topology. Hence there are many asymptotic non-spinorial states in any given spinorial manifold.

A converse of the lemma above would be of great interest since it would provide a sufficient condition for the existence of spinorial states in \( H_{\text{kin}} \). We know that all known orientable primes are spinorial if and only if they have a non-cyclic fundamental group. For non-orientable primes, such as \( P^2 \times S^1 \), which is non-spinorial and has non-cyclic \( \pi_1 \), this is not necessarily the case. This motivates:

Conjecture 1 Let \( \tilde{\Sigma} \) be an orientable prime. Consider two embedded loops \( \alpha \) and \( \beta \) in \( \tilde{\Sigma} \) that intersect only at \( \infty \) and such that \([\alpha]\) and \([\beta]\) are distinct generators of \( \pi_1(\tilde{\Sigma}) \). Then there is no diffeomorphism in \( D^0_\ell(\tilde{\Sigma}) \) taking \( \ell = \alpha \beta \) to \( R \circ \ell = \alpha' \beta' \).

The above result would clearly generalise to the case where \([\alpha]\) and \([\beta]\) are not generators but are generated by distinct elements of \( \pi_1(\tilde{\Sigma}) \). A more important corollary would be the following:

Conjecture 2 Every spinorial prime \( \tilde{\Sigma} \) contains spinorial loops and therefore \( H_{\text{kin}}(\tilde{\Sigma}) \) contains spinorial states.

That this conjecture would follow from the previous one is clear, since by Lemma 1 we know that any spinorial manifold has non-cyclic fundamental group. Conjecture 1 can be easily seen to hold when \( \tilde{\Sigma} \) is of the type \( \mathbb{R}^3/H \), where \( H \) is a discrete subgroup of the affine group in 3 dimensions that acts freely and properly discontinuously on \( \mathbb{R}^3 \). The proof follows from a slight generalisation of the arguments.
Figure 4: This is a picture of $P^2 \times S^1$. The disk represents $P^2$ so opposite points in its boundary are identified. Thus the loop $\alpha$ generates $\mathbb{Z}_2$ and the loop $\beta$ generates $\mathbb{Z}$. A $2\pi$-rotation turns $\alpha\beta$ into $\alpha'\beta$, which can be isotoped back to the original loop by sliding the point along the disk boundary.

leading to theorem 1. More general results, concerning all prime manifolds, will be given elsewhere. They require further techniques of algebraic topology, in particular of obstruction theory.

We conclude this section by illustrating explicitly how conjecture 1 can be refuted if orientability of $\tilde{\Sigma}$ is not required as a premise. The fundamental group of the non-orientable prime $P^2 \times S^1$ is $\mathbb{Z}_2 \times \mathbb{Z}$. In figure 4 we start with the product $\ell$ of its two distinct generators $\alpha$ and $\beta$, apply a prime rotation that deforms $\ell$ into $\ell'$ and finally isotope back to $\ell$ without moving the ball at infinity.

5 Conclusion

In the present report we have used the $2\pi$-rotation of an $ADM$-slice relative to its surroundings to define genuine states with half-integer spin in a kinematical Hilbert space for quantum gravity. This reinforces the interpretation of the mapping class group as an asymptotic symmetry group for bound states in a quantum theory of gravity.

Asymptotic symmetry transformations arise only in connection with asymptoti-
cally flat spaces. However, the topological features of these transformations can be
analysed in the one-point compactifications of the corresponding manifolds. This
construction also allows a natural definition of a Hilbert space for asymptotically
trivial gravity, which will be studied in more detail along with possible alternatives
in future. A natural action of the mapping class group can be defined on this Hilbert
space and we have shown that it is precisely the states through the added point at
infinity which carry this representation: localised states are left invariant.

Some elements of the diffeomorphism group can be regarded as rotations of space
with respect to its surroundings, leading to a notion of spinoriality of a manifold and
of the quantum states associated with it. We have shown that spinorial states exist in
a given manifold provided some loop based at infinity transforms non-trivially under
the $2\pi$-rotation, in the sense that there is no ambient isotopy that leaves invariant a
neighbourhood of infinity and takes the original loop to the $2\pi$-rotated loop. Since
in a non-spinorial manifold the $2\pi$-rotations are trivial in the mapping-class-group,
a necessary condition for the existence of spinorial loop states is spinoriality of the
$\text{ADM}$-slice. We believe that this condition should also be sufficient.

In quantum field theory one attempts to classify particles according to irreducible
representations of appropriate symmetry groups. In analogy, when decomposing a
given representation of the mapping class group into irreducibles, one may hope that
some of the quantum charges labelling the resulting sectors admit an interpretation
as particle properties. In the future we intend to address this question by reducing
the actions of the mapping-class-group on the loop state space. Although for our
purposes in this paper it sufficed to look at single primes, certain particle properties
can only be described in more general connected sums. For example, the distinction
between fermionic and bosonic representations of $G$ depends on the appearance of
identical primes within a given connected sum. Hence we expect manifolds with
several primes to play an important role in our search for multiplet structures.

Another interesting project would be to study the extent to which the asymptotic
spatial symmetries that we have seen arise in a canonical formulation of quantum
gravity, are present in the sum-over-histories formalism. As mentioned in section 2,
the Hamiltonian constraint remains one of the main unsolved problems in canonical
quantum gravity. Perhaps by relaxing some of the canonical structure, it would
be possible to implement dynamics in a way that allows for topology change. The
canonical Hilbert space structure would break down at the critical time levels where
topological transitions occur. These hypersurfaces would account for interactions
and the particle properties would vary according to the topology of earlier and later
slices.
Acknowledgements

We are greatly indebted to Chris Isham for supplying the original motivation to study topological geons and for countless discussions thereafter. We are also very grateful to John Baez and Nico Giulini for providing valuable comments on an earlier draft of our report. Finally we would like to thank Fay Dowker for introducing us to the topic of geons, Rafael Sorkin for his incisive remarks and Bill Thurston and Ric Ancel for improving our intuition on three-manifold topology.