Molien Function for Duality

Philippe Pouliot

Department of Physics
University of California
Santa Barbara, CA 93106

Abstract

The Molien function counts the number of independent group invariants of a representation. For chiral superfields, it is invariant under duality by construction. We illustrate how it calculates the spectrum of supersymmetric gauge theories.
1. Introduction

It is quite remarkable that certain four dimensional gauge theories can be solved exactly. The examples that have been solved so far ([1] for a review) are quite special: they have lots of symmetries.

A generic theory does not have so many symmetries, so here I introduce a tool which I hope will be useful to the study of more general theories. To be concrete, I will consider only supersymmetric theories.

2. The Molien function

Consider a supersymmetric gauge theory with chiral superfields transforming as a representation $R$ of a group $G$. I make no restrictions on $R$: it can be reducible, and also contain singlets (mesons). Similarly, $G$ can be a product of groups, or it can be the identity, for a confining theory.

The Molien generating function for the representation $R$ is

$$M(z) = \sum_{k=0}^{\infty} c_k z^k$$

where $c_k$ is the number of independent group invariant polynomials of order $k$. It is a holomorphic function.

It turns out that there is a nice way to write down $M(z)$ (see [2] p. 204 for an easy proof):

$$M(z) = \int \frac{d\mu(g)}{\det(1 - zR(g))}.$$ 

The idea of the proof is that one can diagonalize the unitary representation $R$ for any fixed group element $g$; then integration ($\int d\mu(g)$) over the whole group picks out only the singlets in the tensor products $R^\otimes k$.

The function $M$ can be evaluated more explicitly (see the nice paper by Forger [3] for much useful and readable complementary details.)

$$M(z) = \frac{1}{|W|} \int \cdots \int \frac{dw_1}{2\pi i w_1} \cdots \frac{dw_l}{2\pi i w_l} \frac{\Pi_{\alpha}(1 - w^{h(\alpha)})}{\Pi_{\lambda}(1 - z w^{h(\lambda)})}.$$ 

- $|W|$ is the number of elements in the Weyl group
- $l$ is the rank of the group;
• the products are over all the roots $\alpha$ of the group and over all the weights $\lambda$ of the representation $R$;

• and finally, the concise notation $w^{h(\alpha)}$ means $w_1^{h(\alpha_1)} \cdots w_l^{h(\alpha_l)}$, where $h(\alpha_i)$ is the eigenvalue of the root $\alpha$ under the Cartan generator $H_i$.

Another representation of the Molien function coefficients is given in terms of an index:

$$c_k = \frac{1}{|W|} \sum_{\tilde{\lambda}} i(\tilde{\lambda})m_k(\tilde{\lambda})$$

with the $\tilde{\lambda}$ denoting the extended weights and $m_k$ the $k$-extended multiplicities. This terminology is defined in [3]. According to [3], the index might be more efficient for explicit calculations.

3. Duality

For a $\mathcal{N} = 1$ supersymmetric gauge theory with a vanishing superpotential, the Molien function calculated for the gauge group of the theory encodes much information about the low-energy spectrum. It calculates how many gauge invariant independent chiral (holomorphic) operators there are of a given degree in the number of elementary fields. In other words, it contains all the structure of the chiral ring. Obviously, the Molien function is duality invariant: any conjectured dual to that theory must have the same low-energy spectrum, and therefore the same Molien function.

Even if it is not enough to fully characterize a supersymmetric conformally invariant gauge theory, the Molien function could be “the” characteristic function of $\mathcal{N} = 1$ duality. It is definitely interesting because it does not rely on global symmetries. For a generic theory, global symmetries are small and the constraints one can get from them have a limited power: it is well known that satisfying the ’t Hooft anomaly matching conditions is not enough. In string theory, there are no global symmetries anyway.

If the theory has a non-zero superpotential, extra constraints are introduced among the invariants. The definition of the Molien function stays the same, but the integral representation has to be generalized to include the effect of the superpotential (I don’t know how to write it down).

4. Illustration

Aside from the duality application, the Molien function provides a technique to grind out the spectrum of a theory, along with plethysms, branching rules and other counting
arguments. I will illustrate some of the uses with the simplest examples. Start with \( \mathcal{N} = 1 \) supersymmetric \( SU(2) \) gauge theory with one flavor of fundamentals \( Q_i \) (two doublets) [1].

Evaluating \( M(z) \) with the integral representation readily gives

\[
M = \frac{1}{1-z^2} = 1+z^2+z^4+\cdots.
\]

This generating function is characteristic of a freely generated ring with one invariant: there’s one polynomial of order 2, namely \( Q_1Q_2 \), and one of order 4, \( (Q_1Q_2)^2 \), and so on.

With 4 doublets,

\[
M(z) = \frac{1-z^4}{(1-z^2)^6} = 1+6z^2+20z^4+50z^6+\cdots.
\]

The coefficient 6 indicates that the ring is generated by the invariants \( V_{ij} = Q_iQ_j \). At order \( z^4 \), we learn that the \( V_{ij} \) are not independent, but there is one constraint among them, the famed pf \( V = \Lambda^4 \). Studying the following coefficients shows that there are no more constraints.

With 6 doublets,

\[
M(z) = \frac{1+6z^2+6z^4+z^6}{(1-z^2)^9} = 1+15z^2+(120-15)z^4+(680-189-1)z^6+\cdots.
\]

This is already more complicated. There are 15 invariants \( V_{ij} \), and 15 constraints (syzygies) \( \epsilon^{ijklmn} V_{kl} V_{mn} \), but there are constraints amongst the constraints and so on.

More generally, with \( d \) doublets,

\[
M_d = \sum_{k=0}^{\infty} \dim \left( \underbrace{\Box \cdots \Box}_{k} \right) z^k
\]

where the tensor under the \( SU(2d) \) symmetry has \( k \) horizontal boxes.

5. New Example

As the rank of the group increases, the formulas for the Molien function become rapidly cumbersome to evaluate. For the integral representation, one is faced with high order poles to be evaluated by the residue theorem. A trick is to settle for less than the full generating function, and get only the first few \( c_k \): one takes derivatives with respect to \( z \), and then set \( z = 0 \), before evaluating the residues at the \( w_i \), for which the poles are
now automatically all at $w_i = 0$. To go beyond that, perhaps one could reexpress these integrals using Littlewood’s Schur functions, or use the index formula of [3].

Here for simplicity, I will only calculate the spectrum of the supersymmetric $SU(2)$ gauge theories with one matter field in the 4-dimensional representation $S$ and $2k$ doublet fields $Q_i$.

- $k = 0$. This theory was studied in [4]. There is just one invariant, quartic, so $M = 1/(1 - z^4)$.

When $k > 0$, the theories are not asymptotically free. That does not make them uninteresting, because they can still be the free duals of strongly coupled theories.

- $k = 1$

$$M = \frac{1 - z^2 + 5z^4 - z^6 + z^8}{(1 - z^4)^3(1 - z^2)^2}$$

At this stage, we see the invariants $Q^2$, $SQ^3$, $S^2Q^2$, $S^3Q^3$ and $S^4$. The invariants $SQ^3$, $S^2Q^2$, $S^3Q^3$ are fully symmetric in their flavor indices. They generate the full ring, but they are not independent. Checking this result for $k > 1$, we see that these invariants still form a full set, but there are more constraints.

- $k = 2$

$$M = \frac{1 + 2z^2 + 28z^4 + 23z^6 + 78z^8 + 23z^{10} + 28z^{12} + 2z^{14} + z^{16}}{(1 - z^4)^6(1 - z^2)^4}.$$ $M = \frac{1 + 9z^2 + 101z^4 + 319z^6 + 1020z^8 + 1475z^{10} + 2091z^{12} + 1475z^{14} + 1020z^{16} + 319z^{18} + 101z^{20} + 9z^{22} + z^{24}}{(1 - z^4)^9(1 - z^2)^6}$.

Acknowledgments

This work is supported by NSF Grant PHY97-22022.

References