Simple Evaluation of the Chiral Jacobian with the Overlap Dirac Operator

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ABSTRACT

The chiral Jacobian, which is defined with Neuberger’s overlap Dirac operator of the lattice fermion, is explicitly evaluated in the continuum limit without expanding it in the gauge coupling constant. Our calculational scheme is simple and straightforward. We determine a coefficient of the chiral anomaly for general values of the mass parameter and the Wilson parameter of the overlap Dirac operator.

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In a recent studies of the chiral symmetries on a lattice, the lattice chiral Jacobian [1,2]

\[
\ln J = -2ia^4 \sum_x \alpha(x) \text{tr} \gamma_5 \left[ 1 - \frac{1}{2} aD(x) \right] \delta(x,y) \bigg|_{y=x}, \quad \delta(x,y) = \frac{1}{a^4} \delta_{x,y},
\]

(1)

associated with the Dirac operator \(D(x)\) which satisfies the Ginsparg-Wilson relation [3]

\[
D(x)\gamma_5 + \gamma_5 D(x) = aD(x)\gamma_5 D(x),
\]

(2)

plays a central role. The simplest example of such a \(D(x)\) is given by the overlap Dirac operator [4,5],

\[
S = a^4 \sum_x \bar{\psi}(x)iD(x)\psi(x), \quad aD(x) = 1 + X(x) \frac{1}{\sqrt{X(x)X(x)}},
\]

(3)

where \(X(x)\) is the Wilson-Dirac operator

\[
X(x) = i\gamma(x) - \frac{m}{a} + R(x),
\]

(4)

and the lattice covariant derivative \(D_\mu(x)\) and the Wilson term \(R(x)\) are defined by \((\gamma(x) = \sum_\mu \gamma_\mu D_\mu(x)\), and our gamma matrices are anti-Hermitian: \(\gamma_\mu^\dagger = -\gamma_\mu\).)

\[
D_\mu(x) = \frac{1}{2a} \left[ U_\mu(x)e^{a\partial_\nu} - e^{-a\partial_\nu} U_\mu^\dagger(x) \right], \quad R(x) = \frac{r}{2a} \sum_\mu \left[ 2 - U_\mu(x)e^{a\partial_\nu} - e^{-a\partial_\nu} U_\mu^\dagger(x) \right].
\]

(5)

Note that \(\gamma(x) = \gamma(x)\) and \(\gamma(x) = \gamma(x)\), and thus \(X(x) = -i\gamma(x) - m/a + R(x) = \gamma_5 X(x)\gamma_5\).

The chiral Jacobian (1) with \(\alpha(x) = i/2\) represents the index theorem on the lattice with finite lattice spacing [1] and it has been shown to be an integer-valued
topological invariant. It can also be characterized as the Jacobian factor associated with the (local) chiral $U(1)$ transformation [2],

$$
\psi(x) \rightarrow \left\{ 1 + i\alpha(x)\gamma_5 \left[ 1 - \frac{1}{2}aD(x) \right] \right\} \psi(x),
$$

$$
\overline{\psi}(x) \rightarrow \overline{\psi}(x)\left\{ 1 + i\left[ 1 + \frac{1}{2}a\overline{D}(x) \right] \gamma_5 \alpha(x) \right\}.
$$

(6)

Therefore, from the variation of the action (3) under Eq. (6),

$$
\delta S = a^4 \sum_x \alpha(x)\overline{\psi}(x)\left[D(x) + \overline{D}(x)\right] \gamma_5 \left[ 1 - aD(x) \right] \psi(x)
$$

\[= \int d^4x \alpha(x)\partial_\mu j_5^\mu(x),\]

(7)

(where use of the Ginsparg-Wilson relation (2) has been made) one has the Ward-Takahashi identity for the chiral anomaly:

$$
\partial_\mu \langle j_5^\mu(x) \rangle = 2i \mathrm{tr} \gamma_5 \left[ 1 - \frac{1}{2}aD(x) \right] \delta(x,y) \bigg|_{y=x}.
$$

(8)

The chiral $U(1)$ anomaly (8) with the overlap Dirac operator (3) has explicitly been calculated [6] in the continuum limit by expanding the right-hand side with respect to the gauge coupling constant (see also Ref. [3]). The authors of

\[\uparrow\] We have introduced the notation

$$
a\overline{D}(x) = -1 + \overline{X}(x)\frac{1}{\sqrt{\overline{X}(x)X(x)}},
$$

$$
\overline{X}(x) = i\overline{\nabla} + \frac{m}{a} - \overline{R}(x), \quad X^\dagger(x) = -i\nabla + \frac{m}{a} - R(x),
$$

$$
\overline{D}_\mu(x) = -\frac{1}{2a}\left[ U_\mu(x)e^{-a\partial_\mu} - e^a\partial_\mu U_\mu^\dagger(x) \right],
$$

$$
\overline{R}(x) = \frac{r}{2a} \sum_\mu \left[ 2 - U_\mu(x)e^{-a\partial_\mu} - e^a\partial_\mu U_\mu^\dagger(x) \right].
$$
Ref. [6] also observed that the coefficient of the chiral anomaly can be expressed as a topological object in five-dimensional momentum space, and thus it is stable with respect to a variation of the mass parameter $m$ and the Wilson parameter $r$. When the overlap Dirac operator (3) possesses only one massless pole [4], $0 < m < 2r$, Eq. (8) reproduces the correct magnitude in the continuum field theory, $ig^2 \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}/16\pi^2$ [6]. Also, quite recently, it was shown [7] that the continuum limit of the expression (8) generally displays the correct chiral anomaly (including the coefficient), under general assumptions regarding the Dirac operator, i.e., the Ginsparg-Wilson relation and the absence of doublers.

In this note, we present a short evaluation of the chiral Jacobian (1) or (8) with the overlap Dirac operator (3) in the continuum limit $a \to 0$, but without an expansion in the gauge coupling constant $g$. This is possible because the chiral anomaly is infrared finite and one may simply expand the expression (8) in the lattice spacing $a$. Therefore, the actual calculation does not require a perturbative expansion of the lattice Dirac propagator in $g$, which often becomes cumbersome. Our calculational scheme is basically the same as that of Ref. [8] (similar calculational schemes can be found in Refs. [9] and [10]). We also determine the coefficient of the chiral anomaly for general values of the parameters $m$ and $r$. Although we present the calculation for the overlap Dirac operator, the scheme itself is not sensitive to the specific choice of the Dirac operator and, once an explicit form of the Dirac operator is given, it provides a quick way to evaluate the chiral Jacobian in the continuum limit.

First, we write the integrand of the Jacobian (1), or the opposite of the chiral anomaly (8), as follows (where we use $\text{tr} \gamma_5 = 0$)

\* The gauge field is treated as a non-dynamical background, and the gauge field configuration is assumed to have a smooth continuum limit (the well-defined spatial derivative).
\[ i \text{tr} \gamma_5 a D(x) \delta(x, y) \bigg|_{y=x} \]
\[ = i \text{tr} \gamma_5 \left( i \not{D} - \frac{m}{a} + R \right) \]
\[ \times \left[ - \sum_{\mu} D_\mu^2 + \frac{1}{4} \sum_{\mu, \nu} [\gamma^\mu, \gamma^\nu] [D_\mu, D_\nu] - i[D, R] + \left( \frac{m}{a} - R \right)^2 \right]^{-1/2} \delta(x, y) \bigg|_{y=x}. \]

(9)

We next use \( \delta(x, y) = \int_{-\pi}^{\pi} d^4 k e^{ik(x-y)/a} / (2\pi a)^4 \) and the relation

\[ e^{-ikx/a} D_\mu e^{ikx/a} = \frac{i}{a} s_\mu + \tilde{D}_\mu, \]
\[ e^{-ikx/a} R e^{ikx/a} = \frac{r}{a} \sum_\mu (1 - c_\mu) + \tilde{R}, \]

where \( s_\mu = \sin k_\mu \) and \( c_\mu = \cos k_\mu \) and

\[ \tilde{D}_\mu = \frac{1}{2a} \left[ e^{ik\mu} (U_\mu e^{a\partial_\mu} - 1) - e^{-ik\mu} (e^{-a\partial_\mu} U_\mu^\dagger - 1) \right], \]
\[ \tilde{R} = -\frac{r}{2a} \sum_\mu \left[ e^{ik\mu} (U_\mu e^{a\partial_\mu} - 1) + e^{-ik\mu} (e^{-a\partial_\mu} U_\mu) - 1 \right]. \]

(11)

Equation (9) can then be written as

\[ i \text{tr} \gamma_5 a D(x) \delta(x, y) \bigg|_{y=x} \]
\[ = -\frac{i}{a^4} \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \text{tr} \gamma_5 \left[ \hat{s} + m + r \sum_\mu (c_\mu - 1) - ia\not{D} - a\tilde{R} \right] \]
\[ \times \left\{ \sum_\nu (s_\nu - ia\tilde{D}_\nu)^2 + \left[ m + r \sum_\nu (c_\nu - 1) - a\tilde{R} \right]^2 \right\}^{-1/2} \]
\[ + \frac{a^2}{2} \sum_{\nu, \rho} \gamma^\nu \gamma^\rho [\tilde{D}_\nu, \tilde{D}_\rho] - ia^2 [\not{D}, \tilde{R}] \bigg]. \]

(12)

It is easy to find the \( a \to 0 \) limit of this expression, because \( \gamma_5 \) requires at least four gamma matrices (\( \text{tr} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4\epsilon^{\mu\nu\rho\sigma} \) and \( \epsilon^{1234} = 1 \)). Finally, by
parameterizing the link variable as $U_\mu(x) = \exp[i a g A_\mu(x)]$ and noting the relations

$$\tilde{D}_\mu = c_\mu D^C_\mu + O(a), \quad \tilde{R} = -i r \sum_\mu s_\mu D^C_\mu + O(a),$$

where $D^C_\mu = \partial_\mu + i g A_\mu$ is the covariant derivative of the continuum theory, and

$$[\tilde{D}_\mu, \tilde{D}_\nu] = i g c_\mu c_\nu F_{\mu\nu} + O(a), \quad [\tilde{D}_\mu, \tilde{R}] = g r c_\mu \sum_\nu s_\nu F_{\mu\nu} + O(a),$$

we find

$$\left. i \lim_{a \to 0} \text{tr} \gamma_5 a D(x) \delta(x, y) \right|_{y = x} = -\frac{ig^2}{16\pi^2} I(m, r) \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}(x).$$

The lattice integral $I(m, r)$ is given by

$$I(m, r) = \frac{3}{8\pi^2} \int_{-\pi}^\pi d^4k \prod_\mu c_\mu \left[ m + r \sum_\nu (c_\nu - 1) + r \sum_\nu \frac{s_\nu^2}{c_\nu} \right] \times \left\{ \sum_\rho s_\rho^2 + \left[ m + r \sum_\rho (c_\rho - 1) \right]^2 \right\}^{-5/2}$$

$$= \frac{3}{8\pi^2} \sum_{\epsilon_\mu = \pm 1} \left( \prod_\mu \epsilon_\mu \right) \int_{-1}^1 d^4s \times \left\{ m + r \sum_\nu [\epsilon_\nu (1 - s_\nu^2)^{1/2} - 1] + r \sum_\nu s_\nu^2 \epsilon_\nu (1 - s_\nu^2)^{-1/2} \right\} B(s)^{-5/2},$$

where

$$B(s) \equiv \sum_\mu s_\mu^2 + \left\{ m + r \sum_\mu [\epsilon_\mu (1 - s_\mu^2)^{1/2} - 1] \right\}^2.$$  

In writing the last expression of Eq. (16), we have changed the integration variable from $k_\mu$ to $\sin k_\mu$ by splitting the integration region into $-\pi/2 \leq k_\mu < \pi/2$ and $\pi/2 \leq k_\mu < 3\pi/2$ in each direction. Thus the original integration region has been split into $2^4 = 16$ blocks; the four vector $\epsilon_\mu = (\pm 1, \pm 1, \pm 1, \pm 1)$ specifies the individual block. Note that $\cos k_\mu$ is expressed as $c_\mu = \epsilon_\mu (1 - s_\mu^2)^{1/2}$. 

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Our next task is to evaluate the lattice integral $I(m, r)$, Eq. (16). To reproduce the correct coefficient of the chiral anomaly for a single fermion from Eq. (8) and Eq. (15), one would expect $I(m, r) = 1$ at least in some parameter region. In fact, the integral $I(m, r)$ does not vary under an infinitesimal variation of the parameters $m$ and $r$. To see this, we note the identity

\[
\left\{ m + r \sum_\mu \left[ \epsilon_\mu (1 - s_{\mu}^2)^{1/2} - 1 \right] + r \sum_\mu s_{\mu}^2 \epsilon_\mu (1 - s_{\mu}^2)^{-1/2} \right\} \\
\times \left\{ m + r \sum_\nu \left[ \epsilon_\nu (1 - s_{\nu}^2)^{1/2} - 1 \right] \right\}
\]

\[= B(s) + \frac{1}{5} B(s)^{7/2} \sum_\mu s_{\mu} \frac{\partial}{\partial s_{\mu}} B(s)^{-5/2},\]

which is analogous to the identity utilized for the chiral anomaly of the Wilson fermion [11,9]. Having obtained Eq. (18), it is straightforward to see

\[
\frac{\partial}{\partial m} I(m, r) = -\frac{3}{8\pi^2} \sum_{\epsilon_\mu = \pm 1} \left( \prod_\mu \epsilon_\mu \right) \int_{-1}^{1} d^4 s \left( 4 + \sum_\nu s_\nu \frac{\partial}{\partial s_\nu} \right) B(s)^{-5/2}, \tag{19}
\]

and, similarly,

\[
\frac{\partial}{\partial r} I(m, r)
\]

\[= -\frac{3}{8\pi^2} \sum_{\epsilon_\mu = \pm 1} \left( \prod_\mu \epsilon_\mu \right) \int_{-1}^{1} d^4 s \left( 4 + \sum_\nu s_\nu \frac{\partial}{\partial s_\nu} \right) \left[ \epsilon_\mu (1 - s_\mu^2)^{1/2} - 1 \right] B(s)^{-5/2}. \tag{20}
\]

When $m \neq 0, 2r, 4r, 6r$ or $8r$, $B(s)^{-5/2}$ in Eqs. (19) and (20) is regular in the integration region. Therefore we can safely perform the partial integration, and $\partial I(m, r) / \partial m = \partial I(m, r) / \partial r = 0$ immediately follows. Note that the surface terms are canceled out, because they are independent of $\epsilon_\mu$. This stability of the coefficient of the chiral anomaly is consistent with the result of Ref. [6] that the coefficient of the triangle diagram for the chiral anomaly is expressed as a topological object in the momentum space.
The above proof of the stability of $I(m, r)$ indicates how we should proceed: $I(m, r)$ can be regarded as a function of the ratio of two parameters $\alpha = m/r$ and the Wilson parameter $r$. For fixed $\alpha \neq 0, 2, 4, 6$ or $8$, $I(m, r)$ is independent of the value of $r$. Therefore, we may evaluate it with a certain limiting value of $r$. We consider the limit $r \to 0$. By rescaling the integration variable in Eq. (16) as $s_\mu \to rs_\mu$, we have

$$I(\alpha r, r) = \frac{3}{8\pi^2} \sum_{\epsilon_\mu = \pm 1} \left( \prod_{\mu} \epsilon_\mu \right) \int_{-1/r}^{1/r} d^4 s \left\{ \alpha + \sum_\nu \left[ \epsilon_\nu (1 - r^2 s_\nu^2)^{-1/2} - 1 \right] \right\}^{1/2} \left( \sum_\rho s_\rho^2 + \left\{ \alpha + \sum_\rho \left[ \epsilon_\rho (1 - r^2 s_\rho^2)^{1/2} - 1 \right] \right\} \right\}^{5/2}. \tag{21}$$

To consider the $r \to 0$ limit, we divide the integration region $[-1/r, 1/r]^4$ of Eq. (21) into a four-dimensional cylinder $C(L) \equiv S^3 \times [-L, L]$ and the remaining $R(L) \equiv [-1/r, 1/r]^4 - C(L)$. The radius of $S^3$ is $L (L < 1/r)$, and the direction of the cylinder is taken along the $\nu$-direction in the numerator. Then it is possible to show that the $r \to 0$ limit of the latter integral, $I(\alpha r, r)_{R(L)}$, vanishes as $L \to \infty$.*

* The proof goes as follows: The absolute value of the integral over $R(L)$, $\left| I(\alpha r, r)_{R(L)} \right|$, can be bounded by a linear combination of

$$4\pi \int_0^{\sqrt{3}/r} d\rho \rho^2 \int_0^{1/r} dz \left\{ (1 - r^2 z^2)^{-1/2} \right\} (\rho^2 + z^2)^{-5/2}$$

and $\int_L^{\sqrt{3}/r} d\rho \int_{-L}^L dz$ with the same integrand. After integrating over $\rho$, this integral is bounded by

$$\frac{4\pi}{3} \int_L^{1/r} dz \left\{ z^{-2} (1 - r^2 z^2)^{-1/2} \right\} (1 + r^2 z^2/3)^{-3/2} \leq \frac{4\pi}{3} \int_L^{1/r} dz \left\{ z^{-2} (1 - r^2 z^2)^{-1/2} \right\} < \frac{4\pi}{3} \int_L^{1/r} dz \left\{ z^{-2} (1 - r^2 z^2)^{-1/2} \right\} \rightarrow 4\pi L$$

Similarly, the integral over $\int_L^{\sqrt{3}/r} d\rho \int_{-L}^L dz$ has a bound whose limit as $r \to 0$ is $4\pi/L$. Therefore, we have

$$\lim_{r \to 0} \left| I(\alpha r, r)_{R(L)} \right| < \frac{\text{const}}{L}. \tag{8}$$
Therefore, the $r \to 0$ limit of the integral (21) can be evaluated by restricting the integration region to $C(L)$, by taking the $L \to \infty$ limit. Namely,

\[
\lim_{r \to 0} I(\alpha r, r) = \frac{3}{8\pi^2} \sum_{\mu=\pm 1} \left( \prod_{\mu} \epsilon_{\mu} \right) \lim_{L \to \infty} \lim_{r \to 0} \int_{-L}^{L} d^4 s \\
\times \frac{\alpha + \sum_{\nu} [\epsilon_{\nu}(1 - r^2 s_{\nu}^2)^{-1/2} - 1]}{\left( \sum_{\rho} s_{\rho}^2 + \left[ \alpha + \sum_{\rho} [\epsilon_{\rho}(1 - r^2 s_{\rho}^2)^{1/2} - 1] \right]^2 \right)^{5/2}}
\]

\[
= \frac{3}{8\pi^2} \sum_{\mu=\pm 1} \left( \prod_{\mu} \epsilon_{\mu} \right) \frac{\alpha + \sum_{\nu} (\epsilon_{\nu} - 1)}{\left( \sum_{\rho} s_{\rho}^2 + [\alpha + \sum_{\rho} (\epsilon_{\rho} - 1)]^2 \right)^{5/2}}
\]

\[
= \frac{1}{2} \sum_{\mu=\pm 1} \left( \prod_{\mu} \epsilon_{\mu} \right) \frac{\alpha + \sum_{\nu} (\epsilon_{\nu} - 1)}{\alpha + \sum_{\rho} (\epsilon_{\rho} - 1)}
\]

In this way, the lattice integral (16) is given by

\[
I(m, r) = \lim_{r \to 0} I(\alpha r, r)
\]

\[
= \theta(m/r) - 4\theta(m/r - 2) + 6\theta(m/r - 4) - 4\theta(m/r - 6) + \theta(m/r - 8),
\]

(23)

where $\theta(x)$ is the step function. As anticipated, the integral $I(m, r)$ varies only stepwise depending on the ratio $m/r$. Finally, by recalling Eqs. (15) and (1), we have the chiral Jacobian in the continuum limit,

\[
\lim_{a \to 0} \ln J
\]

\[
= -\frac{ig^2}{16\pi^2} \left[ \theta(m/r) - 4\theta(m/r - 2) + 6\theta(m/r - 4) - 4\theta(m/r - 6) + \theta(m/r - 8) \right]
\times \int d^4 x \alpha(x) \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}(x).
\]

(24)

It is interesting to note that the quantity inside the square brackets has a simple physical meaning: It is sum of the chiral charge [11] of the massless degrees of
freedom. In particular, when the Dirac operator has only one massless pole \[4\], \(0 < m < 2r\), Eq. (24) reproduces the correct coefficient as a single fermion, as was first shown in Ref. [6]. For other ranges of \(m\), our expression (24) completely agrees with the general result of the analysis of Ref. [7] of the overlap Dirac operator.

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Note added:

The detailed analysis of chiral (gauge) anomalies in the overlap formulation in general is performed in Ref. [12]. The dynamical implication of the special values of the mass parameter \(m\), \(m = 0, 2r, 4r, 6r\) and \(8r\) is fully investigated in Ref. [13]. I would like to thank the authors of these papers for information. Almost the same result as ours is obtained independently by Dr. D. H. Adams in Ref. [14].

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