Mean field theory for collective motion of quantum meson fields

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Abstract

Mean field theory for the time evolution of quantum meson fields is studied in terms of the functional Schrödinger picture with a time-dependent Gaussian variational wave functional. We first show that the equations of motion for the variational wavefunctional can be rewritten in a compact form similar to the Hartree-Bogoliubov equations in quantum many-body theory and this result is used to recover the covariance of the theory. We then apply this method to the O(N) model and present analytic solutions of the mean field evolution equations for an N-component scalar field. These solutions correspond to quantum rotations in isospin space and represent generalizations of the classical solutions obtained earlier by Anselm and Ryskin. As compared to classical solutions new effects arise because of the coupling between the average value of the field and its quantum fluctuations. We show how to generalize these solutions to the case of mean field dynamics at finite temperature. The relevance of these solutions for the observation of a coherent collective state or a disoriented chiral condensate in ultra-relativistic nuclear collisions is discussed.

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§1. Introduction

Several studies have been devoted to the determination of the evolution of a self-interacting scalar field both at the classical and quantum level. Some calculations have considered the possible formation of a disoriented chiral condensate in ultra-relativistic nuclear collisions\(^1\)\(^{-6}\). The evolution of a scalar field is also important to describe some aspects of inflationary models\(^7\)\(^{-12}\).

Most analytical results in this domain concern the classical equations of motion. This is the case in particular for the rotating solutions in isospin space which were investigated some time ago for the non linear sigma model by Enikova, Karloukovski and Velchev\(^{13}\), Blaizot and Krzywicki\(^1\) and Anselm and Ryskin\(^5\). These last authors discussed the observability of a classical coherent pion field created during an ultra relativistic nuclear collision in a finite volume. They examined the effect it would have on the outside volume where it would act as a source. Anselm and Ryskin concluded that a large number of pions, of the order of fifty in typical collisions, would be emitted in an interaction vertex, all of them with almost the same momentum. Furthermore some events would contain mainly neutral pions while some other events would contain in contrast mainly charged pions.

Classical equations are of interest when the pion density is high enough to make a description by a coherent state plausible. An interesting discussion of this limit and several related questions can be found in the section on waves and particles in the book by Peierls on “Surprises in Theoretical Physics”\(^{14}\). In this section Peierls points out that physicists originally encountered only the wave aspects of light and only the corpuscular aspects of particles with mass. He argues that this is no accident at all because the analogy between light and matter has very severe limitations. In the case of neutral pions for instance, the classical field description implies superpositions of states with different numbers i.e. corresponding to substantial energy changes. This implies rapid oscillations in time in the case of free fields. Peierls therefore concludes that the question of the knowledge of the field phase, which is central for a classical description, becomes of academic interest in this particular case.

In order to have some deeper understanding on the domain of validity of the classical field picture and the coherent or collective effects it implies, it is highly desirable to have calculational tools incorporating quantum effects as well as a consistent treatment of interactions. The most natural framework for this purpose is the well known mean field or Hartree-Bogoliubov approach, which assumes that at each time the density matrix of the system is given by the exponential of a quadratic form in the field operators (or that the state of the system is described by a Gaussian functional in the special case of zero temperature).
Several numerical investigations of the mean field equations, which generally involve large amounts of computational work, are now available for a wide variety of geometries and initial conditions. Some analytical results have also been obtained concerning the return of the system to equilibrium in linear response theory\textsuperscript{10} and the behaviour of mean field solutions in a boost invariant geometry at large proper times\textsuperscript{15}. In our recent work\textsuperscript{16} we have used a covariant form of the mean field equations to construct a quantum generalization of the time-dependent running-wave-type solutions of Anselm and Ryskin for the classical equation of motion of non-linear sigma model.

The purpose of the present paper is three-fold: i) First we make a systematic review of the properties of the equations of motion in the Gaussian variational approach. By introducing a generalized (one-body) density matrix, we show that the variational equations of motion can be written in the form of Hartree-Bogoliubov like equations, as encountered in many-body theory. An attractive property of the density matrix is that its eigenvectors are found to be identical to the so-called mode functions. These define at each time a set of creation and annihilation operators having the Gaussian state as a vacuum. The mode functions satisfy a set of Klein-Gordon type equation in the presence of a mean field and they can be exploited to recover the covariance of the theory. ii) The second purpose of this paper is to apply the method to the O(N) sigma model and describe a set of time-dependent solutions which correspond to rotations in the internal symmetry space of the quantum fields. This generalization of the classical Anselm-Ryskin solutions was presented in our recent work\textsuperscript{16} also using the covariant formalism. iii) In this paper we further show (and this is the main result of this present study) that the previous solutions can be extended to finite temperature. These solutions allow one to investigate the phase diagram of the model as a function of thermal excitation energy and/or collective excitation energy.

The present article is organized as follows. We describe the framework of the Gaussian approximation in section 2 in the case of a single component scalar meson field. The equivalence of this approximation with the time dependent Hartree-Bogoliubov equations is briefly reviewed in section 3. In section 4 a covariant form of the mean field equations is presented. It uses the so called mode functions, which are the eigenvectors of the generalized density operator. These functions have remarkable properties needed later on to extend the Anselm Ryskin solutions. In section 5 we consider the case of a scalar field with N components and an O(N) symmetry. We review some of the results obtained earlier in this model for the classical equations of motion and describe the structure of the mean field equations. We show that rotating solutions in isospin space of the mean field equations can be constructed. In section 6 an extension to describe mean field equations at finite temperature in the O(N) model is given and particular analytic solutions of these equations are also presented. They
correspond to quantum rotations at finite temperature in isospin space. They include a self consistent treatment of quantum fluctuations around the classical solutions of Anselm and Ryskin in the special case of zero momentum and zero temperature. The detailed derivation is included in Appendices A ∼ D. Numerical results are presented in section 7. We discuss in section 8 the importance of quantum effects, with special focus on the self consistency conditions occurring in our equations. The relevance of our results regarding the possible formation of a coherent collective state or a disoriented chiral condensate in ultra relativistic collisions is sketched in section 9.

§2. Mean field approximation in the functional Schrödinger picture

We consider in this section a single component scalar field $\varphi$ described by the Hamiltonian density:

$$
H(x) = \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi(x))^2 + \frac{m_0^2}{2} \varphi^2(x) + \frac{\lambda}{24} \varphi^4(x),
$$

where the operator $\pi(x)$ is the functional derivative $-i \delta / \delta \varphi(x)$. The mean field equations are obtained by assuming that at each time $t$ the wave functional of the system is a Gaussian

$$
\Psi[\varphi(x)] = \mathcal{N} \exp \left( i \langle \bar{\pi} | \varphi - \bar{\varphi} \rangle - \langle \varphi - \bar{\varphi} | (\frac{1}{4G} + i \Sigma) | \varphi - \bar{\varphi} \rangle \right),
$$

where $G$, $\Sigma$, $\bar{\varphi}$, $\bar{\pi}$ define respectively the real and imaginary part of the kernel of the Gaussian and its average position and momentum. We have used the short hand notation

$$
\langle \bar{\pi} | \varphi \rangle = \int \bar{\pi}(x,t) \varphi(x) dx.
$$

The resulting equations are found to be

$$
\dot{G} = 2(G \Sigma + \Sigma G),
$$

$$
\dot{\Sigma} = \frac{1}{8} G^{-2} - 2 \Sigma^2 - \frac{1}{2} \left( -\Delta + m_0^2 + \frac{\lambda}{3} \bar{\varphi}^2 + \frac{\lambda}{2} G(x,x) \right),
$$

$$
\dot{\bar{\varphi}} = -\bar{\pi},
$$

$$
\dot{\bar{\pi}} = \left( -\Delta + m_0^2 + \frac{\lambda}{6} \bar{\varphi}^2 + \frac{\lambda}{2} G(x,x) \right) \bar{\varphi}.
$$

The vacuum state corresponds to the static solution

$$
\Sigma = 0, \quad \bar{\pi} = 0, \quad \bar{\varphi} = \varphi_0, \quad G = G_0,
$$
where
\[ G_0 = \frac{1}{2\sqrt{-\Delta + \mu^2}}. \]

(2.6)

The quantity \( \mu \) has to satisfy the so-called gap equation\(^{25}\)
\[ \mu^2 = m_0^2 + \frac{\lambda}{2} \langle x \mid \frac{1}{2\sqrt{-\Delta + \mu^2}} | x \rangle + \frac{\lambda}{2} \phi_0^2. \]

(2.7)

In the symmetric phase the expectation value of the field \( \phi_0 \) vanishes while in the symmetry broken phase it must be such that
\[ m_0^2 + \frac{\lambda}{2} \langle x \mid \frac{1}{2\sqrt{-\Delta + \mu^2}} | x \rangle + \frac{\lambda}{6} \phi_0^2 = 0. \]

(2.8)

This last equation implies that \( \mu^2 = \lambda \phi_0^2 / 3. \)

The previous equation requires a regularization scheme such as a discretization of the Laplacian operator on a lattice with a mesh size \( \Delta x = 1/\Lambda \) or a cutoff \( \Lambda \) in momentum space. To make the gap equation finite when the scale \( \Lambda \) goes to \( \infty \) a popular prescription is to send the bare coupling constant to zero according to the formula
\[ \frac{1}{2\lambda R} = \frac{1}{\lambda} + \frac{1}{16\pi^2} \log \left( \frac{2\Lambda}{e\mu} \right). \]

(2.9)

An interesting consequence of this prescription is that it also makes the dynamical mean field equations finite\(^{17},^{11},^{12}\). Note however that one difficulty arises because the bare coupling constant becomes infinitesimally small by negative values when the momentum cutoff goes to infinity, so that the model is unstable i.e. not really a viable field theory. This prescription and its limitations have been extensively discussed in the literature\(^{8},^{3},^{25}\). In what follows we will mainly consider, unless otherwise specified, our interacting scalar field model as an effective low energy cutoff theory\(^{3},^{27}\). The model is however a useful one which contains attractive aspects. It has in particular a single bound state whose influence on meson distributions gives rise to interesting effects\(^{17}\). This bound state occurs because the \( \lambda \phi^4 \) coupling is a contact force which can accommodate one bound state only\(^{28}\).

We have in this section considered only pure states i.e. a zero temperature theory. The generalization to statistical mixtures and non zero temperatures is straightforward and the corresponding mean field equations can be found in the literature\(^{8}\).

§3. Dynamics in Hartree-Bogoliubov form

It is sometimes convenient to rewrite the mean field evolution equations in a form which exhibits in a more transparent way the underlying Lorentz invariance of the equations and
which furthermore reduces to a set of linear equations for a non interacting meson gas. Such a form is provided by the familiar time dependent Hartree- Bogoliubov equations \(^{26, 8, 29}\) which are particularly suited to investigate boost invariant configurations. In order to construct these equations we introduce the generalized density matrix \(\mathcal{M}\) which adequately implements the requirements of Lorentz invariance. It is defined by

\[
\mathcal{M}(x, y; t) + \frac{1}{2} = \left( \begin{array}{cc}
    i \langle \hat{\varphi}(x) \hat{\pi}(y) \rangle & \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle \\
    \langle \hat{\pi}(x) \hat{\pi}(y) \rangle & -i \langle \hat{\pi}(x) \hat{\varphi}(y) \rangle 
\end{array} \right),
\]

(3.1)

where \(\hat{\varphi} = \varphi - \bar{\varphi}\), \(\hat{\pi} = \pi - \bar{\pi}\), \(1\) is the unit (Dirac) matrix and expectation values are calculated with the Gaussian functional \(\Psi(t)\). Explicitly

\[
\mathcal{M} = \left( \begin{array}{cc}
    -2iG \Sigma & G \\
    \frac{1}{4G} + 4G \Sigma & 2iG \Sigma 
\end{array} \right).
\]

(3.2)

From its structure it can be checked that the generalized density matrix satisfies

\[
\mathcal{M}^2 = \frac{1}{4}.
\]

(3.3)

The eigenvalues of the density matrix are thus \(\pm 1/2\). Note that if \((u, v)\) is an eigenvector of \(\mathcal{M}\) with eigenvalue 1/2 i.e.

\[
\begin{pmatrix}
    -2iG \Sigma & G \\
    \frac{1}{4G} + 4G \Sigma & 2iG \Sigma 
\end{pmatrix}
\begin{pmatrix}
    u \\
    v 
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
    u \\
    v 
\end{pmatrix},
\]

(3.4)

then \((u^*, -v^*)\) is also an eigenvector with eigenvalue -1/2. Eigenvectors are conveniently normalized to

\[
\int d\mathbf{x} (u_m^*(\mathbf{x})v_m(\mathbf{x}) + v_m^*(\mathbf{x})u_m(\mathbf{x})) = \pm \delta_{m,n},
\]

(3.5)

with a \(\pm\) sign for eigenvalues \(\pm 1/2\). It can also be checked that for eigenvalues +1/2, \(u\) and \(v\) components of eigenvectors are related by

\[
v(\mathbf{x}) = \left( \frac{1}{2G} + 2i \Sigma \right) u(\mathbf{x}).
\]

(3.6)

The components \(u\) of the eigenvectors are often referred to as mode functions \(^{7, 9}\).

For the particular normalisation condition we have adopted the eigenvectors \(u_n\) and \(v_n\) provide the following spectral decomposition for the generalized density matrix \(\mathcal{M}\)

\[
\mathcal{M} = \frac{1}{2} \sum_{n>0} \begin{pmatrix} u_n & v_n^* \\ v_n & -u_n^* \end{pmatrix} \begin{pmatrix} u_n & v_n^* \\ -v_n & u_n^* \end{pmatrix},
\]

(3.7)

where the sum runs over eigenstates with eigenvalues +1/2 only.
The interpretation of the mode functions can be made transparent from the following observation. For a given Gaussian functional \( \Psi \) specified by its kernel \( G \) and \( \Sigma \) the operators
\[
b_n = \int dx \left( v_n(x) \varphi(x) + u_n(x) \frac{\delta}{\delta \varphi(x)} \right),
\]
(3.8)
can be seen to have the state \( \Psi \) as a vacuum state for eigenvalues +1/2. Indeed because of the relation (3.6) between the components \( u \)'s and \( v \)'s we have
\[
b_n |\Psi\rangle = 0.
\]
(3.9)
Furthermore because of the normalization condition the operators \( b \) and \( b^+ \) can be seen to have canonical commutation relations. The mode functions thus define a Hartree Bogoliubov transformation preserving commutation relations.

The evolution of the density matrix is governed by the following equations
\[
\partial_t G = 2(\Sigma G + G \Sigma),
\]
\[
\partial_t (G^{-1}/4 + 4 \Sigma G \Sigma) = -2(\Gamma G \Sigma + \Sigma G \Gamma),
\]
\[
\partial_t (\Sigma G) = \frac{1}{8} G^{-1} + 2 \Sigma G \Sigma - \frac{1}{2} \Gamma G,
\]
\[
\partial_t (G \Sigma) = \frac{1}{8} G^{-1} + 2 \Sigma G \Sigma - \frac{1}{2} G \Gamma.
\]
(3.10)
In this equation \( \Gamma \) is the mean field
\[
\Gamma = -\Delta + m_0^2 + \frac{\lambda}{2} \varphi^2 + \frac{\lambda}{2} G(x, x).
\]
(3.11)
A remarkable property of the generalized density matrix is that its equation of motion can be written in Liouville-von Neumann form
\[
i \dot{\mathcal{M}} = [\mathcal{H}, \mathcal{M}],
\]
(3.12)
where the generalized Hamiltonian \( \mathcal{H} \) has the particularly simple form
\[
\mathcal{H} = \begin{pmatrix} 0 & 1 \\ \Gamma & 0 \end{pmatrix}.
\]
(3.13)

§4. Mean field equations in covariant form

7
The equation of motion can also be written in terms of the eigenvectors \((u, v)\) of the generalized density matrix:

\[
    i\partial_t \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Gamma & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}.
\]

(4.1)

The normalization condition is preserved by these equations.

Using the explicit form of the mean field operator \(\Gamma\) we see that the mode functions \(u_n\) satisfy a set of coupled Klein-Gordon type equations

\[
    \left(\Box + m_0^2 + \frac{\lambda}{2}\bar{\varphi}^2 + \frac{\lambda}{2}G(x, x)\right)u_n = 0,
\]

(4.2)

while the component \(v\) are given by

\[
    v_n = i\partial_t u_n.
\]

(4.3)

This last equation implies that the normalisation condition can be rewritten in terms of the mode function \(u_n\) only as

\[
    i\int dx \left( u_n^*(x)\partial_t u_m(x) - u_m(x)\partial_t u_n^*(x) \right) = \pm\delta_{m,n},
\]

(4.4)

which can be recognized as the familiar normalization of the solutions of the Klein-Gordon equation \(^{19}\).

For the self consistent vacuum the mode functions are

\[
    u_k(x) = 1/\sqrt{2\omega_k}\exp(ik\cdot x)/(2\pi)^{3/2},
\]

(4.5)

where \(\omega_k\) is the self consistent energy \(\sqrt{k^2 + \mu^2}\).

The evolution equations for mode functions form a closed set. This is because the spectral decomposition of the generalized density matrix \(\mathcal{M}\) implies

\[
    \langle x|G(t)|x\rangle = \frac{1}{2} \sum_n |u_n(x, t)|^2.
\]

(4.6)

The evolution equations can be written in a more compact and manifestly covariant form:

\[
    m^2(x) = m_0^2 + \frac{\lambda}{2}\bar{\varphi}^2(x) + \frac{\lambda}{2}\langle x|S|x\rangle,
\]

(4.7)

where \(x = (x_0, x_1, x_2, x_3)\) and where \(S\) is the Feynman propagator in the presence of an \(x\) dependent mass

\[
    S = \frac{i}{\Box + m^2(x) + i\varepsilon}.
\]

(4.8)
Indeed the completeness relation of the eigenvectors of the generalized density matrix, valid for each time $t$, reads

$$\sum_{n>0} \begin{bmatrix} u_n & v_n \end{bmatrix} \begin{bmatrix} u_n^* & -v_n^* \end{bmatrix} \begin{bmatrix} -v_n \\ u_n \end{bmatrix} = 1.$$  \hfill (4.9)

This relation implies the following ones

$$\sum_{n>0} u_n^*(x, t) i \partial_t u_n(y, t) - \sum_{n<0} u_n^*(x, t) i \partial_t u_n(y, t) = \delta(x - y),$$  \hfill (4.10)

and

$$\sum_{n>0} u_n^*(x, t) u_n(y, t) - \sum_{n<0} u_n^*(x, t) u_n(y, t) = 0.$$  \hfill (4.11)

From these relations we find that the quantity

$$-i \sum_{n>0} (\theta(t - t') u_n^*(x, t) u_n(y, t') + \theta(t' - t) u_n(x, t) u_n^*(y, t')),$$  \hfill (4.12)

is the inverse of the operator $\partial_{tt} + \Gamma + i\epsilon$, as can be checked by acting with this operator on the previous quantity. As a result

$$\langle x|S|y \rangle = \theta(x_0 - y_0) \sum_{n>0} u_n^*(x, x_0) u_n(y, y_0)$$

$$+ \theta(y_0 - x_0) \sum_{n<0} u_n^*(x, x_0) u_n(y, y_0),$$  \hfill (4.13)

and

$$\langle x, x_0|S|x, x_0 \rangle = \langle x|G(x_0)|x \rangle.$$  \hfill (4.14)

This version of the mean field equations is particularly suited to deal with calculations of the linear response of the meson field to an external source\textsuperscript{15}.

At this point let us indicate briefly how the previous equations can be extended to finite temperatures. The key is that the eigenvalues of the generalized density matrix are constants of the motion. They are equal to $\pm 1/2$ at zero temperature. To have a statistical mixture one has to consider eigenvalues which differ from this value. For instance one can consider eigenvalues of the form $f_n + 1/2$ where the $f$'s are thermal occupation numbers of bosons.

§5. The case of the O(N) model

The $O(N)$ model corresponds to the following Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2} \partial_{\mu} \phi_\alpha \partial^{\mu} \phi_\alpha - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4!} \lambda (\phi^2)^2,$$  \hfill (5.1)
where
\[ \phi^2 = \phi_a \phi_a \quad a = 1, 2 \ldots N \] (5.2)

In the special case of the sigma model \( N=4 \) the index \( a \) corresponds to the isospin quantum number. In what follows we will be refer to it as isospin even when \( N \) differs from 4.

In the classical approximation the evolution equations read
\[ \left( \Box + m_0^2 + \frac{\lambda}{6} \phi^2(x) \right) \phi_a(x) = 0. \] (5.3)

5.1. The classical solutions of Anselm and Ryskin for the sigma model

Plane wave like solutions of these equations were obtained by Anselm and Ryskin\(^5\) who focused on the special case of the sigma model \( N=4 \) and more precisely its non linear version. Their solutions have the following structure
\[ \phi_a(x) = A_a \cos (\omega t - k \cdot r + \theta_a), \] (5.4)

with the conditions
\[ \sum_a A_a^2 \cos(2\theta_a) = \sum_a A_a^2 \sin(2\theta_a) = 0. \] (5.5)

For these solutions the square of the field is a constant \( \phi_0^2 \) in space and time
\[ \phi^2 = \phi_0^2 = \frac{1}{2} \sum_a A_a^2. \] (5.6)

The corresponding energy density is
\[ E = \frac{1}{2} \phi_0^2 (k^2 + \omega^2). \] (5.7)

For \( \omega = k = 0 \) (i.e. in the ground state) the constant \( \phi_0^2 \) is such that
\[ m_0^2 + \frac{\lambda}{6} \phi_0^2 = 0, \] (5.8)

while for non vanishing momenta it must satisfy
\[ \frac{\lambda}{6} \phi_0^2 = -m_0^2 - k^2 + \omega^2. \] (5.9)

We thus see that time like momenta shift the value of the field outside the chiral radius while the opposite is true for space like momenta.

For a vanishing value of the momentum the previous solutions depend, in the case of the sigma model \( N=4 \), on six arbitrary parameters \( (A_a, \theta_a, \omega) \) with three constraints. For \( A_1^2 + A_2^2 \gg 2A_3^2 \) one has a state with mainly charged pions while the opposite is true for \( A_1^2 + A_2^2 \ll 2A_3^2 \). It was argued by Anselm and Ryskin\(^5\) that the observation of such events may be a signature of the formation of a coherent collective state.
5.2. Mean field equations at zero temperature

In the case of the O(N) model the kernels $G$ and $\Sigma$ of the Gaussian wave functional become $N \times N$ matrices while the center $\bar{\phi}$ of the Gaussian (often called condensate) becomes a vector with $N$ components $\bar{\phi}_a$, $a=1,2\ldots N$. In the mean field approximation the evolution of $\bar{\varphi}$ is governed by\textsuperscript{15,18}

$$\left[ \left( \Box + m_0^2 + \frac{\lambda}{6} \bar{\varphi}^2 + \frac{\lambda}{6} \text{tr}S(x, x) \right) \delta_{ab} + \frac{\lambda}{3} S_{ab}(x, x) \right] \bar{\varphi}_b(x) = 0. \quad (5.10)$$

In the previous equation, $S(x, x)$ is a $N \times N$ matrix related to the kernel of the Gaussian wave-functional

$$S_{ab}(x, x) = \langle x|G_{ab}(t)|x \rangle$$

and the trace runs over the flavour indices. $S(x, y)$ is the Feynman propagator

$$S = \frac{i}{\Box + m^2(x) - i\epsilon}, \quad (5.11)$$

where the $N \times N$ mass matrix is

$$m_{ab}^2(x) = \left( m_0^2 + \frac{\lambda}{6} \bar{\varphi}^2(x) + \frac{\lambda}{6} \text{tr}S(x, x) \right) \delta_{ab}$$

$$+ \frac{\lambda}{3} \bar{\varphi}_a(x) \bar{\varphi}_b(x) + \frac{\lambda}{3} S_{ab}(x, x). \quad (5.12)$$

The previous equations (5.10) – (5.12) are non-linear because the motion of the condensate involves the mass matrix $m^2(x)$. Note that the first three terms in equation (5.10) correspond to the classical approximation considered by Anselm and Ryskin. The next two correspond to the contribution of quantum fluctuations whose effect is the object of our study. Generalization of these equations to the case finite temperatures have been discussed in references\textsuperscript{20,21}.

5.3. Quantum generalization of the Anselm Ryskin solutions

Particular solutions of the previous equations which correspond to rotations in isospin space at zero temperature were already presented in reference\textsuperscript{16}. For completeness we give in the present section a short description of the structure of these solutions. They correspond to the following form for the condensate

$$\bar{\varphi}(x) = U(x)\bar{\varphi}^{(0)} = \exp\{i(q \cdot x)\tau_y\} \begin{pmatrix} \varphi_0 \\ 0 \\ \vdots \end{pmatrix},$$
where \( q_\mu = (\omega, q) \) and \( \tau_y \) is a generator of rotation in the subspace of flavour 1 and 2:

\[
\tau_y = \begin{pmatrix}
0 & -i & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The propagator \( S \) is of the form

\[
S(x, y) = U(x)S^{(0)}(x, y)U^\dagger(y),
\]

with

\[
S^{(0)}(x, y) = -\int \frac{d^4 p}{(2\pi)^4} S^{(0)}(p) e^{ip\cdot(x-y)},
\]

and

\[
S^{(0)}(p) = \frac{i}{(p_\mu + q_\mu \tau_y)(p_\mu + q^\mu \tau_y) - M^2 + i\varepsilon}.
\]

It can be seen that the previous expressions solve the mean field equations provided the mass matrix \( M \) in the propagator \( S^{(0)} \) satisfies

\[
M^{2}_{ab} = \left( m^2_0 + \frac{\lambda}{6} \varphi^2_0 + \frac{\lambda}{6} \text{tr}S^{(0)}(x, x) \right) \delta_{ab} + \frac{\lambda}{3} \varphi^{(0)}_a \varphi^{(0)}_b + \frac{\lambda}{3} S^{(0)}_{ab}(x, x).
\] (5.13)

To have a closed set, the previous equations must be supplemented by the relation satisfied by the condensate \( \varphi^{(0)}_0 \)

\[
\left[ -q^2 + m^2_0 + \frac{\lambda}{6} \varphi^2_0 + \frac{\lambda}{6} \text{tr}S^{(0)}(x, x) + \frac{\lambda}{3} S^{(0)}_{11}(x, x) \right] \varphi_0 = 0.
\] (5.14)

The above solutions depend on the 4 parameters \((\omega, q)\). For fixed values of these parameters the above equations provide the values of the matrix elements \( M_{ab} \) and of the condensate \( \varphi_0 \). The matrix elements \( M_{ab} \) are found to be diagonal in the isospin index i.e. \( M_{ab} = M_a \delta_{ab} \).

For rotations in the subspace of flavour 1 and 2 one has \( M_3 = M_4 = \mu \) and one thus needs only to determine the four quantities \( M_1, M_2, \mu \) and \( \varphi_0 \). In reference\(^\text{16}\) it was shown, using a perturbative analysis, that a disappearance of the condensate \( \varphi_0 \), reflecting a resturcation of chiral symmetry, occurs for \( \omega = 0 \) at values of \( q \) corresponding to an energy density of about \((123 \text{ MeV})^4\). We refer the reader to reference\(^\text{16}\) for a discussion of this result. In the next section we focus on an extension of this calculation to the case of rotating solutions in isospin space at finite temperature.
§6. Mean field equations at finite temperature in the O(N) model

The mean field equations at finite temperature in the O(N) model are presented and discussed in Appendix A. The construction of rotating solutions in isospin space at finite temperature is presented in Appendix B. These solutions are constructed in two successive steps. The first step is to write the mean field evolution equations in a rotating frame in isospin space. The second one is to look for static solutions in the rotating frame. In the present section we will just give a short description of the structure of the solutions. As compared to the previous section, devoted to the zero temperature case, the basic structure of the coupled gap equations (5.13) and the condensate equation (5.14) is unchanged. The only difference is that the propagator in the rotating frame now involves boson occupation numbers:

\[ S(p) = (2n(p_0) + 1) \frac{i}{(p_0 - \omega \tau_y)^2 - (p - q \tau_y)^2 - M^2 + i\epsilon} , \]  

where

\[ n(p_0) = \frac{1}{e^{\beta p_0} - 1} , \]  

as shown in Appendix C, see Eqs.(C.22) and (C.18). Here, \( \omega \) is the frequency of isospin rotation and \( q \) is its wave number. The appearance of the occupation number (6.2) breaks the Lorentz invariance of our previous results for the vacuum case. The non-zero value \( q \) means the space-oscillation is realized in isospin rotating frame where the condensate is static and uniform. The above expression contains a specific information of the boundary conditions by the \( \epsilon \)-prescription for the solution of a generalized Klein-Gordon equation as shown in detail in Appendix D using the mode-expansion of the propagator, (D.3). It is shown that the gap equations can be written in the form as in the vacuum (5.13) with fluctuation terms involving the matrix:

\[ S_{ab}(x, x) = \int \frac{d^4p}{(2\pi)^3} S_{ab}(p; x_0 = 0) , \]

where

\[ S(p; x_0 = 0) = -\int \frac{dp_0}{2\pi i} (2n(p_0) + 1) \frac{1}{(p_0 - \omega \tau_y)^2 - (p - q \tau_y)^2 - M^2 + i\epsilon} . \]

The integral with respect to \( p_0 \) can be carried out analytically: it picks up the poles of the integrand in the lower complex \( p_0 \)-plane as in the vacuum case. We present below the results for the pure time-like rotation (\( q^2 = \omega^2 > 0 \), \( q = 0 \)) and for the pure space-like rotation (\( q^2 = -q^2 < 0 \), \( \omega = 0 \)). For both cases, off-diagonal matrix elements of \( S_{ab}(p, x_0 = 0) \) vanish after integration over \( p_0 \) or \( p \). This implies that the matrix \( M^2_{ab} \) is diagonal as in the case for the vacuum.
The gap equations can therefore be written explicitly as

\[ M_1^2 = m_0^2 + \frac{\lambda}{2} \varphi_0^2 + \frac{\lambda}{6} \sum_{c=1}^{N} S_{cc}(x, x) + \frac{\lambda}{3} S_{11}(x, x) \]

\[ M_2^2 = m_0^2 + \frac{\lambda}{6} \varphi_0^2 + \frac{\lambda}{6} \sum_{c=1}^{N} S_{cc}(x, x) + \frac{\lambda}{3} S_{22}(x, x) \]

\[ \mu^2 \equiv M_3^2 = M_4^2 = \cdots = M_N^2 \]

\[ = m_0^2 + \frac{\lambda}{6} \varphi_0^2 + \frac{\lambda}{6} \sum_{c=1}^{N} S_{cc}(x, x) + \frac{\lambda}{3} S_{33}(x, x) , \]

(6.5)

while the condensate equation becomes

\[ \frac{\lambda}{6} \varphi_0^2 = \omega^2 - q^2 - m_0^2 - \frac{\lambda}{6} \sum_{c=1}^{N} S_{cc}(x, x) - \frac{\lambda}{3} S_{11}(x, x) . \]

(6.6)

By using the mass \( M_1^2 \), the above condensate equation can be recast into a simpler form:

\[ \frac{\lambda}{3} \varphi_0^2 = -\omega^2 + q^2 + M_1^2 . \]

(6.7)

The explicit forms of the integrands of the diagonal matrix elements of \( S_{ab}(p, x_0 = 0) \) are given below for two specific cases.

**Case 1** Pure time-like isospin rotations \( (q = 0) \):

\[ S_{11}(p, x_0 = 0) = (2n(E_+)) \left[ \frac{1}{4E_+} + \frac{4\omega^2 + M_1^2 - M_2^2}{4E_+(E^2_+ - E^2_)} \right] \]

\[ + (2n(E_-)) \left[ \frac{1}{4E_-} - \frac{4\omega^2 + M_1^2 - M_2^2}{4E_- (E^2_+ - E^2_-)} \right] , \]

\[ S_{22}(p, x_0 = 0) = (2n(E_+)) \left[ \frac{1}{4E_+} + \frac{4\omega^2 + M_2^2 - M_1^2}{4E_+(E^2_+ - E^2_)} \right] \]

\[ + (2n(E_-)) \left[ \frac{1}{4E_-} - \frac{4\omega^2 + M_2^2 - M_1^2}{4E_- (E^2_+ - E^2_-)} \right] , \]

(6.8)

where

\[ E^2_\pm(p) = p^2 + \frac{M_1^2 + M_2^2}{2} + \omega^2 \pm \left( \frac{M_1^2 - M_2^2}{2} \right)^2 + 4\omega^2 \left( M_1^2 + M_2^2 \right) . \]

(6.9)
We note that $E_{\pm}$ are also eigenvalues for $\mathcal{H}(\omega)$ in (B.3). Also, the matrix elements $S_{ab}(x, x)$ for $a, b \geq 3$ are given as

$$S_{ab}(x, x) = \delta_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\mathbf{p}^2 + \mu^2}} \left( n(\sqrt{\mathbf{p}^2 + \mu^2} + \frac{1}{2}) \right) \quad \text{(for } a, b = 3, 4, \cdots, N) \right. \quad (6.10)$$

**Case 2)** Pure space-like isospin rotations ($\omega = 0$):

The expressions of the diagonal matrix elements, $S_{11}(x, x)$ and $S_{22}(x, x)$ are slightly different from those derived in the time-like case, while those of $S_{cc}(x, x)$ for $c \geq 3$ are unmodified:

$$S_{11}(\mathbf{p}, x_0 = 0) = (2n(E_+) + 1) \left[ \frac{1}{4E_+} + \frac{M_1^2 - M_2^2}{4E_+(E_+ - E_-^2)} \right]$$

$$+ (2n(E_-) + 1) \left[ \frac{1}{4E_} - \frac{M_1^2 - M_2^2}{4E_-(E_+ - E_-^2)} \right],$$

$$S_{22}(\mathbf{p}, x_0 = 0) = (2n(E_+) + 1) \left[ \frac{1}{4E_+} - \frac{M_1^2 - M_2^2}{4E_+(E_+ - E_-^2)} \right]$$

$$+ (2n(E_-) + 1) \left[ \frac{1}{4E_-} + \frac{M_1^2 - M_2^2}{4E_-(E_+ - E_-^2)} \right],$$

where

$$E_{\pm}^2(\mathbf{p}) = \mathbf{p}^2 + \mathbf{q}^2 + \frac{M_1^2 + M_2^2}{2} \pm \sqrt{\left( \frac{M_1^2 - M_2^2}{2} \right)^2 + 4(\mathbf{p} \cdot \mathbf{q})^2}. \quad (6.11)$$

$\S 7$. Numerical results

We can numerically solve the gap equation (6.6) with (6.3), (6.8) and (6.10) for $q^2 = \omega^2 \geq 0$, or (6.3), (6.11) and (6.10) for $q^2 = -|\mathbf{q}|^2 < 0$ in a self-consistent manner. We set the number of isospin $N = 4$. A three-dimensional momentum cutoff $\Lambda = 1$ GeV is introduced. For this cutoff, the model parameters used here, $\lambda$ and $m_0^2$, are taken so as to reproduce the chiral condensate $\varphi_0$ to be the pion decay constant $f_\pi = 93$ MeV and $M_1$ to be the sigma meson mass $M_\sigma = 500$ MeV at $q_\mu = 0$ and $T = 0$. Then, condensate equation at $T = q_\mu = 0$ gives $\lambda = 86.7$ and $m_0^2 = -(1019.5 \text{MeV})^2$. One-dimensional numerical integration for $|\mathbf{p}|$ in Eq.(6.8) is performed for $q^2 = \omega^2 \geq 0$ case. For $q^2 = -|\mathbf{q}|^2 < 0$, two-dimensional integration is required because the square of the inner product $(\mathbf{p} \cdot \mathbf{q})^2$ appears in (6.11) with (6.12). However, we replace $\cos^2 \theta$ with its angle-averaged value $\langle \cos^2 \theta \rangle = 1/3$. We safely perform one-dimensional integration for $|\mathbf{p}|$ with the above-mentioned replacement because the resultant curves for condensate $\varphi_0$ versus $|\mathbf{q}|$ are indistinguishable at $T = 0$. We checked numerically that there is no visible difference at $T = 0$. 

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Fig. 1. The chiral condensate \( \varphi_0 \) with \( q^2 \) is depicted as a function of temperature. The vertical axis represents the chiral condensate \( \varphi_0 \) (MeV) and the horizontal axis is temperature (MeV). The solid curve corresponds to the case \( q_\mu = 0 \). Upside five curves from the solid curve correspond to the cases \( \sqrt{q^2} = \omega = 20, 40, 60, 80 \) and 100 MeV, and downside two curves from the solid curve correspond to the cases \( \sqrt{-q^2} = |q| = 20 \) and 40 MeV.

The chiral order parameter with a certain \( q^2 \) is shown as a function of temperature in Fig. 1. The increase of temperature leads to the decrease of the amplitude of the chiral order parameter \( \varphi_0 \) at each \( q^2 \). Namely, chiral symmetry is partially restored at finite temperature. At a certain critical temperature \( T_c \), a solution of the gap equation does not exist suddenly. At \( T > T_c \), only trivial solution \( \varphi_0 = 0 \) exists. The order of phase transition is like the first order. In our numerical calculation, \( T_c \)'s are 97.5, 101.3, 110.9, 123.1, 136.3 and 149.7 MeV for \( \sqrt{q^2} = \omega = 0, 20, 40, 60, 80 \) and 100 MeV, respectively, and 93.4 and 76.9 MeV for \( \sqrt{-q^2} = |q| = 20 \) and 40 MeV, respectively.

Let us consider the case of time-like \( q^2 \), that is, \( q^2 = \omega^2 \geq 0 \) with \( |q| = 0 \). As \( \omega \) increases at each \( T < T_c \), the condensate \( \varphi_0 \) also increases. In this case the condensate rotates uniformly with angular frequency \( \omega \) in isospin space. As a result, the chiral symmetry breaking is enhanced.
Fig. 2. The chiral condensate $\varphi_0$ with temperature $T$ is depicted as a function of three-momentum $|q|$. The vertical axis represents the chiral condensate $\varphi_0$ (MeV) and the horizontal axis represents the magnitude of three momentum $|q|$ (MeV). The solid curve corresponds to the zero temperature case. Three curves from the solid curve correspond to the cases $T=40, 60$ and 80 MeV.

In the case of space-like $q^2$, that is, $q^2 = -|q|^2 < 0$ with $\omega = 0$, the increasing $|q|$ results in the decreasing $\varphi_0$ at each $T < T_c$. Further, the critical momentum $|q_c|$ exists as is shown in Fig. 2. Namely, the gap equation becomes to have no non-trivial solution at a certain value $|q_c|$. The transition seems the first order phase transition. For example, $|q_c|$’s are 50.0, 49.9, 49.7, 47.2 and 37.7 MeV for $T = 0, 20, 40, 60$ and 80 MeV, respectively.

It is interesting to compare these critical momenta for space-like condensate to the corresponding values in the classical limit. In classical case, $|q_c|^2 = M^2/2$ is expected at $T = 0$ from the classical gap equation. However, the quantum effect leads to more rapid change of chiral condensate as was pointed out by perturbation theory in the previous paper.\textsuperscript{16) This is expected because quantum fluctuation smears out the effective potential and makes symmetry breaking more difficult to reach. The coupling to the quantum fluctuation works to suppress the appearance of static condensate with longer wavelength. This would have important implications for the dynamics of chiral condensate in high energy nuclear collisions.
§8. Importance of quantum effects

In the work of Anselm and Ryskin, which uses the framework of the classical equations of motion, it was pointed out that the existence of rotating solutions in isospin space may have observable consequences in ultra-relativistic collisions. For instance it may be possible to produce a classical state of the pion field in which all pions nearly carry the same momentum, which would manifest itself by the observation of pion “jets”. Since Anselm and Ryskin’s analysis was restricted to the classical picture, an important question is to find out how the previous scenario is affected by quantum fluctuations. A possible way to evaluate the importance of these effects is to compare the rotational contributions to the energy density arising from classical and quantum terms. The energy density at a given momentum is indeed proportional to the pion number density times the pion energy. In the present framework the rotational contribution $E_R$ to the energy density (due to non-vanishing $\Sigma$ and $\bar{\pi}$) at zero temperature is

$$E_R(x) = \frac{1}{2}\pi^2 + 2 \text{Trace}(x|\Sigma G \Sigma|x). \quad (8.1)$$

(See, (A·3) with $\xi = \sigma = 0$ at zero temperature in Appendix A.) The first term is of classical origin while the second term is purely quantal. Using the fact that $G_{ab} = S_{ab}$ with $n(E) = 0$ at zero temperature and by solving the equation of the second line in (A·11) to get $\Sigma$ with $i\hat{G}^{-1}$ being $i\omega[G^{-1}, \tau_y]$ in isospin rotating frame, this contribution can be rewritten as

$$E_R(x) = \frac{1}{2}\omega^2 \left(\varphi_0^2 + \mu^4/24\pi^2 M^2\right), \quad (8.2)$$

with

$$I = \int \frac{dp}{(2\pi)^3} \frac{(S_{22}(p, x_0 = 0) - S_{11}(p, x_0 = 0))^2}{S_{22}(p, x_0 = 0) + S_{11}(p, x_0 = 0)}. \quad (8.3)$$

Let us evaluate this expression in the vicinity of $\omega \simeq 0$ at $T = 0$. In this case we replace the kernels $S$ by $1/2\sqrt{p^2 + M_1^2}$ with the result

$$I \simeq \mu^4 \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{(p^2 + M_1^2)(p^2 + M_2^2)}} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2}\right)^{-3}, \quad (8.4)$$

where $M_1^2 = M^2 + \mu^2$ and $M_2^2 = M^2 - \mu^2$. As a first approximation we replace the square masses in the denominator by their averages $M^2$ so that

$$I \simeq \frac{1}{4}\mu^4 \int \frac{dp}{(2\pi)^3} \frac{1}{(p^2 + M^2)^5/2} = \frac{\mu^4}{4\pi^2 M^2}. \quad (8.5)$$

This gives the approximate contribution to the rotational energy density

$$E_R(x) \simeq \frac{1}{2}\omega^2 \left(\varphi_0^2 + \frac{\mu^4}{24\pi^2 M^2}\right). \quad (8.6)$$
Taking the value of the condensate $\varphi_0$ to be the pion decay constant $f_\pi = 93$ MeV, $M_1$ to be the sigma meson mass $M_\sigma = 500$ MeV and $M_2$ to be of the order of the pion mass $M_\pi = 140$ MeV we find that quantum effects give only a small contribution (4.8\%) to the energy density. We can also evaluate the quantum contribution (8.3\%) directly in our numerical calculation with three-momentum cutoff. We conclude that the quantum effects yield only a very small contribution about 2.8\% in the range of $\omega = 0 \sim 100$ MeV.

We thus expect the number of pions outside of the condensate to be small. This implies that, in this case, the discussion of Anselm and Ryskin is unaffected by quantum fluctuations. Let us recall however that quantum fluctuations still play an important role since they drive the chiral radius away from its vacuum value as excitation energy increases.

§9. Discussion

In the present paper we have shown that it is possible to construct analytically quantum coherent states which are solutions of the mean field evolution equations in the case of an N-component scalar field. We have proceeded in two steps. We have first constructed rotating solutions in isospin space in the zero temperature case. In a second step we have generalized our results to the case of finite temperature. These solutions represent the quantum generalization of the classical solutions obtained earlier for the sigma model by Anselm and Ryskin\(^5\).

As compared to the work of Anselm and Ryskin we have found that new effects arise as a result of the coupling between the motion of the mean (classical) value of the field and its quantum fluctuations. In particular restaurauration of chiral symmetry is affected by quantum fluctuations. Another difference is that, while classical solutions correspond to a distribution of mesons with almost the same momentum, a new component containing all possible momenta appears in the quantum solution. However we have found that the number of mesons outside of the condensate is small. As a result the conclusion of Anselm and Ryskin about the observability of the coherent classical state remains unaffected.

Further questions in the line of the study by Anselm and Ryskin are worth additional investigations. For instance it would be interesting to consider an initial configuration in which the coherent solution we have built is present in a finite volume. By considering this field as a source for the external volume it would be possible to find out the distribution of the emitted pions as discussed by Anselm and Ryskin\(^5\).

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Appendix A

Mean field equations of motion at finite temperature

Let us consider in this Appendix the quantum and thermal generalization of the classical equations of the O(N) model. Following the Eboli, Jackiw and Pi prescription, let us introduce a Gaussian density matrix in the functional Schrödinger picture:

$$
\rho[\varphi_1, \varphi_2] = N_G \exp \left( -i \langle \bar{\pi} | \varphi_1 - \varphi_2 \rangle - \langle \varphi_1 - \varphi | (\frac{1}{4G} - i \Sigma) | \varphi_1 - \varphi \rangle 
\right.
$$

$$
- \langle \varphi_2 - \bar{\varphi} | (\frac{1}{4G} + i \Sigma) | \varphi_2 - \bar{\varphi} \rangle + \frac{1}{2} \langle \varphi_1 - \bar{\varphi} | \frac{1}{\sqrt{G}} \zeta \frac{1}{\sqrt{G}} | \varphi_2 - \bar{\varphi} \rangle \right),
$$

(A-1)

where $N_G$ is a normalization factor. Here $\zeta$ is a mixing parameter and we divide it into real and imaginary parts as $\zeta = \xi + 4i\sqrt{G}\sigma \sqrt{G}$. The real part, $\xi$, is a symmetric kernel $(\xi(x, y) = \xi(y, x))$ and the imaginary part $\sigma$ is an antisymmetric one. If $\zeta = 0$, the Gaussian density matrix is expressed in terms of pure states by $\rho[\varphi_1, \varphi_2] = \Psi[\varphi_2] \Psi^*[\varphi_1]$. Averaged values are easily calculated by

$$
\langle O \rangle = \text{Trace}(\rho O) = \int \int D\varphi_1 D\varphi_2 \rho[\varphi_1, \varphi_2] O[\varphi_2, \varphi_1],
$$

(A-2)

where $O[\varphi_2, \varphi_1] \equiv \langle \varphi_2 | O | \varphi_1 \rangle$.

The energy density calculated for such a Gaussian density matrix is found to be

$$
\langle H(x) \rangle = E_0(x) + \frac{1}{8} \text{Trace}(x) G^{-\frac{1}{2}} (1 + \xi) G^{-\frac{1}{2}} | x \rangle
$$

$$
+ \frac{1}{2} \text{Trace}(x) (\Delta + m_0^2) \left( G^{\frac{1}{2}} - \frac{1}{\sqrt{1 - \xi}} G^{\frac{1}{2}} \right) | x \rangle
$$

$$
+ 2 \text{Trace}(x) (\Sigma + \sigma) G^{\frac{1}{2}} - \frac{1}{\sqrt{1 - \xi}} G^{\frac{1}{2}} (\Sigma - \sigma) | x \rangle + \langle V(x) \rangle,
$$

(A-3)
where
\[ E_0 = \frac{1}{2\bar{\pi}_a^2} + \frac{1}{2}(\nabla \bar{\varphi}_a)^2 + \frac{1}{2} m_0^2 \bar{\varphi}_a^2 + \frac{1}{24} \lambda (\bar{\varphi}_a^2)^2 \]  
(A.4)
is the classical energy density and the Trace is to be taken over the isospin index \( a \). The last term in this equation is expressed as
\[
\langle V(x) \rangle = \frac{1}{24} \lambda (2 \bar{n}\langle \hat{\varphi}_a(x) \hat{\varphi}_a(x) \rangle + \langle \hat{\varphi}_a(x) \hat{\varphi}_a(x) \rangle \langle \hat{\varphi}_b(x) \hat{\varphi}_b(x) \rangle)
+ 4 \bar{\varphi}_a \langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle \bar{\varphi}_b \langle \hat{\varphi}_b(x) \hat{\varphi}_b(x) \rangle,\]
where \( \langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle \equiv [G^{1/2}(1 - \xi)^{-1}G^{1/2}]_{ab}(x,x) \) and repeated indices are supposed to be summed over.

The classical equations of motion for \( \bar{\varphi}_a(x, t) \) and \( \bar{\pi}_a(x, t) \) in quantum and thermal fluctuations are written as
\[
\dot{\bar{\varphi}}_a(x, t) = -\bar{\pi}_a(x, t),
\]
\[
\dot{\bar{\pi}}_a(x, t) = (-\Delta + m_0^2 + \frac{1}{6} \bar{\varphi}_a^2 + \frac{1}{6} \langle \hat{\varphi}_c(x) \hat{\varphi}_c(x) \rangle) \bar{\varphi}_a + \frac{1}{3} \langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle \bar{\varphi}_b.
\]
(A.6)

Let us introduce the generalized density matrix in O(N) model as is similar to that of \( \phi^4 \)-theory:
\[
\mathcal{M}^{ab}(x, y; t) = \langle x | \mathcal{M}^{ab}(t) | y \rangle = \left( \begin{array}{cc} -i \langle \hat{\varphi}_a(x) \hat{n}_b(y) \rangle & \langle \hat{\varphi}_a(x) \hat{\varphi}_b(y) \rangle \\ \langle \hat{n}_a(x) \hat{\varphi}_b(y) \rangle & i \langle \hat{n}_a(x) \hat{n}_b(y) \rangle \end{array} \right) - \frac{1}{2},
\]
(A.7)
where \( \hat{\varphi} = \varphi - \bar{\varphi} \), \( \hat{\pi} = \pi - \bar{\pi} \), 1 is the \( 2N \times 2N \) unit matrix and expectation values are calculated with the Gaussian density matrix \( \rho[\varphi_1, \varphi_2] \). As in the case of zero temperature, eigenvectors of this matrix allow one to build mode functions. These functions define new creation and annihilation operators via Eq.(3.8) which can be used to bring the density matrix (A.1) in canonical form.

In the mean field approximation, we postulate that at each time \( t \) the state of the system is described by a Gaussian density matrix of the form (A.1). Evolution equations in this approximation scheme are obtained by considering the Heisenberg evolution equations for the operators \( \varphi \varphi \), \( \varphi \pi \) and \( \pi \pi \) and by calculating expectation values in the Gaussian state (A.1). As a result we find
\[
i \dot{\mathcal{M}}^{ab}(t) = [\mathcal{H}, \mathcal{M}(t)]^{ab},
\]
(A.8)
where we define \( \mathcal{H} \) as
\[
\mathcal{H} = \begin{pmatrix} 0 & 1 \\ \Gamma & 0 \end{pmatrix},
\]
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\[ \Gamma_{ab} \equiv \left( -\Delta + m_0^2 + \frac{\lambda}{6} \sum_{c=1}^{N} \langle \bar{\varphi}_c(x) \bar{\varphi}_c(x) \rangle + \frac{\lambda}{6} \sum_{c=1}^{N} \langle \bar{\varphi}_c(x) \varphi_c(x) \rangle \right) \delta_{ab} \\
+ \frac{\lambda}{3} \left( \langle \bar{\varphi}_a(x) \bar{\varphi}_b(x) \rangle + \varphi_a(x) \bar{\varphi}_b(x) \right) . \] 

Thus, we can get the equations of motion governing the time-evolution of quantum fluctuation in thermal fluctuation from the above equation of motion. Noting the explicit expression for the generalized density matrix,

\[ M_{ab} \equiv \left( -2i \left[ \sqrt{G} \frac{1}{1-\xi} \sqrt{G}(\Sigma - \sigma) \right]_{ab} \right) \left( \frac{1}{4} \frac{1}{\sqrt{G}} (1+\xi) \frac{1}{\sqrt{G}} + 4(\Sigma + \sigma) \sqrt{G} \frac{1}{1-\xi} \sqrt{G}(\Sigma - \sigma) \right)_{ab} \left( 2i \left[ (\Sigma + \sigma) \sqrt{G} \frac{1}{1-\xi} \sqrt{G} \right]_{ab} \right) , \]

we can obtain the equations of motion for \( G, \Sigma, \xi \) and \( \sigma \) as

\[ \dot{\Sigma} = \frac{1}{8} (G^{-2} - \eta^2) - 2(\Sigma^2 + \sigma^2) - \frac{1}{2} \Gamma , \]
\[ \dot{G}^{-1} = -2(G^{-1} \Sigma + \Sigma G^{-1}) + 2(\eta \sigma - \sigma \eta) , \]
\[ \dot{\eta} = -2(\eta \Sigma + \Sigma \eta) + 2(G^{-1} \sigma - \sigma G^{-1}) , \]
\[ \dot{\sigma} = -2(\sigma \Sigma + \Sigma \sigma) + \frac{1}{8} (\eta G^{-1} - G^{-1} \eta) , \]

where we define \( \eta \equiv G^{-1/2} \xi G^{-1/2} \).

---

**Appendix B**

Equations of motion in isospin-rotating frame at finite temperature

In this Appendix we show how to construct solutions which correspond to rotations of the components \( \varphi_1 \) and \( \varphi_2 \) in isospin space with an angular frequency \( \omega \). These solutions are similar to those developed by Thouless and Valatin \(^{22}\) in the context of mean field theory for deformed nuclei and the restoration of rotational symmetry when it is spontaneously broken in the mean field ground state at zero temperature. Related work concerning the case of a multicomponent scalar field can be found in references \(^{23},^{24}\).

To construct these solutions we first consider the equations of motion in the isospin rotating frame. In this frame the Hamiltonian is

\[ H_{\text{rot}} = H_{O(N)} - \omega_{ab} \int d^3 \mathbf{x} \varphi_a(\mathbf{x}) \pi_b(\mathbf{x}) , \]

where we define \( \omega_{ab} \) as \( \omega_{12} = -\omega_{21} \equiv \omega \) and the otherwise are equal to 0. We can also derive the equation of motion for \( \mathcal{M} \) in the isospin rotating frame :

\[ i \mathcal{M}^{ab} = [\mathcal{H}(\omega), \mathcal{M}(t)]^{ab} , \]
where we define $\mathcal{H}(\omega)$ as
\[
\mathcal{H}(\omega) \equiv \begin{pmatrix}
-\omega \tau_y & 1 \\
\Gamma & -\omega \tau_y
\end{pmatrix}, \quad \tau_y \equiv \begin{pmatrix}
0 & -i & 0 & \cdots \\
i & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\] (B.3)
and $\Gamma_{ab}$ is the same as (A.9). Hereafter, we consider a situation that the time-evolution of $G$, $\Sigma$ etc. is only originated from the rotation in the isospin space. In this case, the generalized density matrix $\mathcal{M}$ is “static” in the isospin-rotating frame, namely, we obtain
\[
[\mathcal{H}(\omega), \mathcal{M}] = 0 .
\] (B.4)
Thus, $\mathcal{H}(\omega)$ and $\mathcal{M}$ are simultaneously diagonalizable. The equations for $Q \equiv G^{-1}$, $\Sigma$, $\eta$ ($\xi$) and $\sigma$ are obtained by replacing $\dot{Q}$ into $i\omega[Q, \tau_y]$ in Eq.(A.11). For the condensate or mean field, we obtain the following equation in the isospin-rotating frame:
\[
\omega^2 \bar{\varphi}_a = \left\{ -\Delta + m_0^2 + \frac{\lambda}{6} \bar{\varphi}^2 + \frac{\lambda}{6} \langle \hat{\varphi}_c(x) \hat{\varphi}_c(x) \rangle \right\} \delta_{ab} + \frac{\lambda}{3} \langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle \bar{\varphi}_b.
\] (B.5)

Appendix C

--- Spectral decomposition of generalized density matrix ---

In this Appendix, we give a spectral decomposition for the generalized density matrix $\mathcal{M}$ in the $\sigma = 0$ case in terms of mode functions, namely, the solution of the Hartree-Bogoliubov equations.

We will start with a situation that the condensate or mean field $\varphi$ does not depend on space and time in the isospin-rotating frame, pointing in the direction of the first isospin:
\[
\varphi = \begin{pmatrix}
\varphi_0 \\
0 \\
\vdots \\
0
\end{pmatrix} = \varphi_0 \delta_{a1} .
\] (C.1)

As is shown later the averaged values $\langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle$ have a diagonal form in the isospins space. The mass matrix $M_{ab}^2$ from $\Gamma$ in Eq.(A.9) reads
\[
\Gamma_{ab} = \Delta \delta_{ab} + M_{ab}^2 ,
\]
\[
M_{ab}^2 \equiv \left( m_0^2 + \frac{\lambda}{6} \sum_{c=1}^N (\bar{\varphi}_c(x))^2 + \frac{\lambda}{6} \sum_{c=1}^N (\hat{\varphi}_c(x) \hat{\varphi}_c(x)) \right) \delta_{ab} + \frac{\lambda}{3} \left( \langle \hat{\varphi}_a(x) \hat{\varphi}_b(x) \rangle + \bar{\varphi}_a \bar{\varphi}_b \right).
\]
\[ M_2^a \equiv m_0^2 + \frac{\lambda}{6} \varphi_0^2 + \frac{\lambda}{6} \sum_{c=1}^{N} (\langle \hat{\phi}_c(x)\hat{\phi}_c(x) \rangle + \langle \hat{\phi}_a(x)\hat{\phi}_a(x) \rangle + \varphi_0^2 \delta_{a1}) . \] (C.2)

Further, we will show later that \( \langle \hat{\phi}_c(x)\hat{\phi}_c(x) \rangle \) for \( c \geq 3 \) have same values because isospin rotation occurs only in \( c = 1 \)- and \( 2 \)-components. Thus, we can set up the masses as

\[
\begin{align*}
M_1^2 &\equiv M_0^2 + \mu_0^2 , \\
M_2^2 &\equiv M_0^2 - \mu_0^2 , \\
M_3^2 &= M_4^2 = \cdots = M_N^2 \equiv \mu^2 ,
\end{align*}
\] (C.3)

where we define \( M_0^2 \) and \( \mu_0^2 \) as

\[
\begin{align*}
M_0^2 &\equiv m_0^2 + \frac{\lambda}{3} \varphi_0^2 + \frac{\lambda}{3} \left( \langle \hat{\phi}_1(x)\hat{\phi}_1(x) \rangle + \langle \hat{\phi}_2(x)\hat{\phi}_2(x) \rangle \right) + \frac{\lambda}{6} \sum_{c=3}^{N} \langle \hat{\phi}_c(x)\hat{\phi}_c(x) \rangle , \\
\mu_0^2 &\equiv \frac{\lambda}{6} \varphi_0^2 + \frac{\lambda}{6} \left( \langle \hat{\phi}_1(x)\hat{\phi}_1(x) \rangle - \langle \hat{\phi}_2(x)\hat{\phi}_2(x) \rangle \right) .
\end{align*}
\] (C.4)

Then, the matrix \( \Gamma \) has a simple form:

\[
\Gamma_{ab} = (-\Delta + m_a^2 + \mu_0^2 \tau_z) \delta_{ab} ,
\] (C.5)

where \( \tau_{11} = -(\tau_{22}) = 1 \) and the others are equal to 0. Here we define \( m_1^2 = m_2^2 \equiv M_0^2 \) and \( m_a^2 \equiv \mu^2 \) for \( a \geq 3 \). Using the above simple expressions, we can easily get the eigenvalues for \( \mathcal{H}(\omega) \) in momentum representation from the following eigenvalue equations:

\[
\begin{pmatrix}
-\omega \tau_y & 1 \\
\Gamma(k) & -\omega \tau_y
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix} = E(k)
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix},
\] (C.6)

where \( (u_k, v_k) \) and \( E(k) \) are eigenvectors and eigenvalues for \( \mathcal{H}(\omega) \) in momentum representation, respectively. As a result, we get

\[
\begin{align*}
E^2(k, a = \pm) &= k^2 + M_0^2 + \omega^2 \pm \sqrt{\mu_0^4 + 4\omega^2(k^2 + M_0^2)} , \\
E^2(k, a \geq 3) &= k^2 + \mu^2 ,
\end{align*}
\] (C.7)

where +, (−) means \( a = 1, (2) \).

In order to get the spectral decomposition for the generalized density matrix \( \mathcal{M} \) in terms of \( (u_k, v_k) \), it should be noted that the matrix \( \mathcal{M} \) is expressed with the help of three matrices as

\[
\mathcal{M} = V_\sigma \begin{pmatrix}
\frac{1}{2\sqrt{1-\xi}} & 0 \\
0 & -\frac{1}{2\sqrt{1-\xi}}
\end{pmatrix} W_\sigma ,
\] (C.8)
where

\[
V_\sigma = \begin{pmatrix}
G^\frac{1}{2} & 2i(\Sigma + \sigma)G^\frac{1}{2} + G^{-\frac{1}{2}}\sqrt{1-\xi^2} \\
2i(\Sigma + \sigma)G^\frac{1}{2} + G^{-\frac{1}{2}}\sqrt{1-\xi^2} & 2i(\Sigma + \sigma)G^\frac{1}{2} - G^{-\frac{1}{2}}\sqrt{1-\xi^2}
\end{pmatrix},
\]

\[
W_\sigma = \begin{pmatrix}
\frac{1}{2}G^{-\frac{1}{2}} - 2i\frac{1}{\sqrt{1-\xi^2}}G^\frac{1}{2}(\Sigma - \sigma) & \frac{1}{\sqrt{1-\xi^2}}G^\frac{1}{2} \\
\frac{1}{2}G^{-\frac{1}{2}} + 2i\frac{1}{\sqrt{1-\xi^2}}G^\frac{1}{2}(\Sigma - \sigma) & -\frac{1}{\sqrt{1-\xi^2}}G^\frac{1}{2}
\end{pmatrix}.
\]  

(C.9)

We will now look for solutions in which the imaginary part of the mixing parameter, \(\sigma\), is equal to 0. This corresponds to the “static” case in the rotating frame. Then, the matrix \(\mathcal{M}\) is simply expressed as

\[
\mathcal{M} = V \begin{pmatrix}
\frac{1}{2}\sqrt{1+\xi} & 0 \\
0 & -\frac{1}{2}\sqrt{1+\xi}
\end{pmatrix} V^{-1},
\]  

(C.10)

where \(V \equiv V_{\sigma=0}\) and \(V^{-1}(= W_{\sigma=0})\) is an inverse operator matrix of \(V\). Then, for positive eigenvalues of \(\mathcal{M}\), we obtain

\[
\mathcal{M}\begin{pmatrix}
|u_c(k)\rangle \\
|v_c(k)\rangle
\end{pmatrix} = \frac{1}{2}\sqrt{1+\xi} \begin{pmatrix}
1+\xi_c(k) & |u_c(k)\rangle \\
1-\xi_c(k) & |v_c(k)\rangle
\end{pmatrix}, \quad \begin{pmatrix}
1\rangle |u_c(k)\rangle \\
n_c\rangle |v_c(k)\rangle
\end{pmatrix} = V \begin{pmatrix}
|w_c(k)\rangle \\
0
\end{pmatrix},
\]  

(C.11)

where \(\xi_c(k)\) and \(|w_c(k)\rangle\) are eigenvalue and eigenvector for the operator \(\xi(t)\), respectively. Here, \(n_c\) are normalization factors. It should be noted that the \(|v_c(k)\rangle\) is related to \(|u_c(k)\rangle\) as follows:

\[
|v_c(k)\rangle = \left(\frac{1}{2}\sqrt{1-\xi^2} \frac{1}{\sqrt{G}} + 2i\Sigma\right)|u_c(k)\rangle,
\]  

(C.12)

where \(|u_c(k)\rangle = n_cG^\frac{1}{2}|w_c(k)\rangle\). From (C.11), we obtain the spectral decomposition for the generalized density matrix \(\mathcal{M}\):

\[
\mathcal{M} = \sum_{k,(E>0)} \sum_{\epsilon=1}^N \begin{pmatrix}
|u_c(k)\rangle \\
|v_c(k)\rangle
\end{pmatrix} \frac{1}{2}\sqrt{1+\xi_c(k)} \begin{pmatrix}
1+\xi_c(k) & \langle v_c(k),u_c(k)\rangle \\
1-\xi_c(k) & \langle u_c(k),v_c(k)\rangle
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
|u_c^*(k)\rangle \\
|v_c^*(k)\rangle
\end{pmatrix} \frac{1}{2}\sqrt{1+\xi_c(k)} \begin{pmatrix}
1+\xi_c(k) & \langle u_c^*(k),v_c^*(k)\rangle \\
1-\xi_c(k) & \langle v_c^*(k),u_c^*(k)\rangle
\end{pmatrix},
\]  

(C.13)

where we used the relations

\[
\mathcal{M}\begin{pmatrix}
|u_c^*(k)\rangle \\
|v_c^*(k)\rangle
\end{pmatrix} = -\frac{1}{2}\sqrt{1+\xi_c(k)} \begin{pmatrix}
|u_c^*(k)\rangle \\
|v_c^*(k)\rangle
\end{pmatrix},
\]  

(C.14)

for the eigenvectors of negative eigenvalues. The notation \(E > 0\) means that the sum runs over the eigenvectors with positive eigenvalues. It should be noted that \(\mathcal{H}(\omega)\) and \(\mathcal{M}\) are
simultaneously diagonalizable because the relation $[\mathcal{H}(\omega), \mathcal{M}] = 0$ is satisfied in the rotating frame. Applying adequate normalization factors $n_c$, we get the ortho-normalization and completeness relations as

$$\int d\mathbf{x} \{ v_c(\mathbf{k}, \mathbf{x})^* u_c(\mathbf{k}', \mathbf{x}) + u_c(\mathbf{k}, \mathbf{x})^* v_c(\mathbf{k}', \mathbf{x}) \} = \pm \delta_{kk'}$$  \hspace{1cm} (C.15)

with $+$ ($-$) sign for positive (negative) eigenvalues, and

$$\sum_{\mathbf{k}, (E>0)} \sum_{c} \left\{ \begin{array}{c} |u_c(\mathbf{k})| \\ |v_c(\mathbf{k})| \end{array} \right\} \left( \begin{array}{c} \langle v_c(\mathbf{k}), u_c(\mathbf{k}) \rangle \\ -\langle v_c^*(\mathbf{k}), u_c^*(\mathbf{k}) \rangle \end{array} \right) = 1 \hspace{1cm} (C.16)$$

Finally, it is necessary to determine $\xi_c(\mathbf{k})$ in $\mathcal{M}$, while the eigenvectors are directly calculable. Since we treat the case $\sigma$ (the imaginary part of the mixing parameter) is equal to 0, the generalized density matrix has a simple form in Eq. (C.10). Thus, we can write the entropy from this density matrix in the unit $k_B = 1$ as

$$S = \sum_{\mathbf{k}, c} [(1 + n_{k,c}) \ln(1 + n_{k,c}) - n_{k,c} \ln n_{k,c}] ,$$  \hspace{1cm} (C.17)

where the $n_{k,c}$'s are defined by

$$n_{k,c} = \frac{1}{2} \left( \frac{1 + \xi_c(\mathbf{k})}{1 - \xi_c(\mathbf{k})} - 1 \right) .$$  \hspace{1cm} (C.18)

Thus, $\xi_c(\mathbf{k})$ or $n_{k,c}$ occurring in the generalized density matrix can be obtained by imposing that the free energy $F(\omega)$ in the rotating frame

$$F(\omega) = \langle \mathcal{H}(\omega) \rangle - \frac{S}{\beta}$$  \hspace{1cm} (C.19)

is stationary because all variables are time-independent in the rotating frame. Here, $\beta \equiv 1/T$ is inverse of temperature. The variation of $\langle \mathcal{H}(\omega) \rangle$ with respect to $n_{k,c}$ is easily obtained by noting that the change $\delta \mathcal{M}$ is given by

$$\delta \mathcal{M} = \sum_{\mathbf{k}, c} \left( \begin{array}{c} u_c(\mathbf{k}) \\ v_c(\mathbf{k}) \end{array} \right) \delta n_{k,c} (u_c(\mathbf{k})^*, v_c(\mathbf{k})^*).$$  \hspace{1cm} (C.20)

Since mode functions $(u_c(\mathbf{k}), v_c(\mathbf{k}))$ are eigenstates of the matrix $\mathcal{H}(\omega)$ due to $[\mathcal{H}(\omega), \mathcal{M}] = 0$, we have

$$\delta \langle \mathcal{H}(\omega) \rangle = \text{Trace}(\mathcal{H}(\omega) \delta \mathcal{M}) = \sum_{\mathbf{k}, c} E_{k,c} \delta n_{k,c} .$$  \hspace{1cm} (C.21)

Moreover, since we obtain $\delta S = \sum_{k,c} \delta n_{k,c} \ln((1 + n_{k,c})/n_{k,c})$, thus $\delta F(\omega) = 0$ gives us

$$n_{k,c} = \frac{1}{e^{\beta E_{k,c}} - 1} , \hspace{0.5cm} \text{or} \hspace{0.5cm} \xi_c(\mathbf{k}) = \frac{1}{\cosh \beta E_{k,c}} .$$  \hspace{1cm} (C.22)

One can understand that $n_{k,c}$ are nothing but boson distribution functions.

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Appendix D

— Gap equation in the rotating frame —

In this Appendix, we give solutions for \( \langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle \) in terms of the mode functions. In a first step we consider the special case of a Gaussian density matrix with zero momentum. To treat the most general case of a Gaussian density matrix with finite momentum we exploit in a second step the evolution equations for mode functions to generate in a natural way solutions with a finite value of the momentum. We can get the results with the finite momentum by following the method developed in §5. The gap equation to determine the chiral order parameter \( \varphi_0 \) is derived in the closed set of equations.

From the spectral decomposition of \( M \), we can write \( \langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle \) explicitly as

\[
\langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle = M^{ab}_{12}(x,x) = \sum_{k, (E>0)} \sum_{\sigma} \frac{1}{2} \left[ \frac{1 + \xi_\sigma(E_{k,\sigma})}{1 - \xi_\sigma(E_{k,\sigma})} \right] \times [u^{a}_{\sigma k}(x)u^{b*}_{\sigma k}(x) + u^{a*}_{\sigma k}(x)u^{b}_{\sigma k}(x)].
\]

We can obtain \( \langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle \) by calculating directly the mode functions \( u^{a}_{\sigma k}(x) \) for \( H(\omega) \) such as

\[
\langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle = \delta_{ab} \sum_{k} \frac{1}{\sqrt{k^2 + \mu^2}} \left( n_{k,a} + \frac{1}{2} \right) \quad \text{for } a, b = 3, 4, \ldots, N,
\]

where \( \mu^2 \) is defined by Eqs. (C.2) and (C.3). For \( a \geq 3 \), we can easily obtain the above expression by calculating the mode functions directly. Although the isospin-rotating components, \( \langle \hat{\varphi}_a(x)\hat{\varphi}_b(x) \rangle \) \( (a, b = 1, 2) \), are easily calculable through mode functions, we give another derivation here. The two methods, of course, give identical results.

Let us introduce the following \( N \times N \) matrix operator :

\[
\langle x, x_0 | S_{ab}(y, y_0) = \theta(x_0 - y_0) \sum_{k, (E>0)} \sum_{\sigma} \frac{1 + \xi_\sigma(k)}{1 - \xi_\sigma(k)} u^{a}_{\sigma k}(x, x_0)u^{b*}_{\sigma k}(y, y_0)
\]

\[
+ \theta(y_0 - x_0) \sum_{k, (E>0)} \sum_{\sigma} \frac{1 + \xi_\sigma(k)}{1 - \xi_\sigma(k)} u^{a*}_{\sigma k}(x, x_0)u^{b}_{\sigma k}(y, y_0)
\]

\[
\equiv \langle x | S_{ab}(x_0, y_0) | y \rangle,
\]

where the mode functions \( u^{a}_{\sigma k}(x, x_0) \), together with \( u^{a}_{\sigma k}(x, x_0) \), satisfy the following equations of motion in the isospin-rotating frame :

\[
i \partial_{x_0} \begin{pmatrix} u^{a}_{\sigma k}(x, x_0) \cr v^{a}_{\sigma k}(x, x_0) \end{pmatrix} = \begin{pmatrix} -\omega & 1 \\ \Gamma(k) & -\omega \end{pmatrix} \begin{pmatrix} u^{a}_{\sigma k}(x, x_0) \\ v^{a}_{\sigma k}(x, x_0) \end{pmatrix},
\]

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where $\Gamma$ has been already defined in Eq. (C.5). Eliminating $v_{\sigma k}(x, x_0)$, we get the following Klein-Gordon-type equations of motion:

$$(-\partial_x^2 + \Gamma + \omega^2 + 2i\omega\tau_y\partial_x)u_{\sigma k}(x, x_0) = 0.$$  \hspace{1cm} (D.5)

The matrix operator $S$ is related to $\langle \hat{\phi}_a(x)\hat{\phi}_b(x) \rangle$ through the relation

$$\langle x, x_0|S|y, y_0 \rangle = \mathcal{M}_{12}(x, x)$$  \hspace{1cm} (D.6)

and Eq. (D.1). Here we note that $u_{\sigma k}(x, x_0)$ are also eigenvectors for the operator $\xi(x_0)$ from the relation $|u_c\rangle \propto G^{1/2} |u_c\rangle$ because $\xi(x_0)$ and $G(x_0)$ commute due to the equation of motion for the imaginary part of the mixing parameter, $\sigma$, in the case $\sigma = 0$. We introduce the Feynman propagator

$$\langle x, x_0|\bar{S}_{ab}|y, y_0 \rangle \equiv \langle x| \left( \frac{1 + \xi(x_0)}{1 - \xi(x_0)} \right)^{-\frac{i}{4}} S_{ab}(x, y_0) \left( \frac{1 + \xi(y_0)}{1 - \xi(y_0)} \right)^{-\frac{i}{4}} |y \rangle$$

$$= \theta(x_0 - y_0) \sum_{k, (E > 0)} \sum_{\sigma} u_{ab}^\sigma(x, x_0) u_{ab}^{\sigma^*}(y, y_0)$$

$$+ \theta(y_0 - x_0) \sum_{k, (E > 0)} \sum_{\sigma} u_{ab}^{\sigma^*}(x, x_0) u_{ab}^\sigma(y, y_0) ,$$  \hspace{1cm} (D.7)

which satisfies

$$(\partial_x^2 + \Gamma - \omega^2 - 2i\omega\tau_y\partial_x)\langle x, x_0|\bar{S}|y, y_0 \rangle = -i\delta^4(x - y)$$  \hspace{1cm} (D.8)

by using the equations of motion for $u_{\sigma k}(x, x_0)$ and $v_{\sigma k}(x, x_0)$ and the completeness relation. Thus, it is symbolically written by

$$\langle x, x_0|\bar{S}|y, y_0 \rangle = \frac{-i}{\partial_x^2 + \Gamma - \omega^2 - 2i\omega\tau_y\partial_x - i\epsilon}$$  \hspace{1cm} (D.9)

with infinitesimal positive constant $\epsilon$. From the Fourier-transformation for the propagator $\langle x, x_0|\bar{S}|y, y_0 \rangle$, namely

$$\tilde{S}(x, y) = \int \frac{d^4p}{(2\pi)^4} \tilde{S}(p)e^{-ip(x-y)} ,$$  \hspace{1cm} (D.10)

we obtain

$$\tilde{S}(p) = \left( \frac{1 + \xi(p_0)}{1 - \xi(p_0)} \right)^{-\frac{i}{4}} S(p) \left( \frac{1 + \xi(p_0)}{1 - \xi(p_0)} \right)^{-\frac{i}{4}}$$

$$= \frac{1 + \xi(p_0)}{\mu_0^2 - p^2 - m^2 - \mu_0^2\tau_z + \omega^2 - 2\omega p_0\tau_y + i\epsilon} .$$  \hspace{1cm} (D.11)

In this way, we get the Fourier transformation of $\langle x, x_0|S|y, y_0 \rangle$:

$$S(p) = \frac{1 + \xi(p_0)}{(p_0 - \omega\tau_y)^2 - p^2 - m^2 - \mu_0^2\tau_z + i\epsilon} ,$$  \hspace{1cm} (D.12)
where

\[ n(p_0) = \frac{1}{e^{\beta p_0} - 1} \quad (D.13) \]

with the help of Eqs.\((C.18)\) and \((C.22)\).

Let us exploit in the second step the evolution equations for mode functions to generate solutions with a finite value of the three-momentum \(q\) in addition to \(\omega\). This corresponds to using the mode function \(u_{\omega k}^q(x, x_0) = e^{i q \cdot x_0} u_{\omega k}(x, x_0)\). We obtain the following general expression:

\[ S(p) = (2n(p_0) + 1) \frac{i}{(p_0 - \omega \tau_y)^2 - (p - q \tau_y)^2 - m^2 - \mu_0^2 \tau_z + i\epsilon} \quad (D.14) \]

We can get \(M_{12}^{ab}(x, x)\) or \(M_{12}^{ab}(k)\), which are introduced by \(M_{12}^{ab}(x, x) = \sum_k M_{12}^{ab}(k)\), performing the inverse Fourier transformation from \(S(p)\) by the use of the relation of Eq.\((D.6)\):

\[ M_{12}(k) = - \int \frac{dk_0}{2\pi i} \frac{1}{(2n(k_0) + 1)} \frac{1}{(k_0 - \omega \tau_y)^2 - (k - q \tau_y)^2 - m^2 - \mu_0^2 \tau_z + i\epsilon} \quad (D.15) \]

Let us consider the propagator in the subspace of the first two isospins. First, we concentrate on the time-like \(q^2\) as \(q^2 = \omega^2\) with \(q = 0\). After integration with respect to \(k_0\), we have \(M_{12}^{ab}(k)\) for \(a, b = 1, 2\):

\[
\begin{align*}
\langle \hat{\phi}_1(x) | \hat{\phi}_1(x) \rangle &= M_{12}^{11}(x, x) = S_{11}(x, x) \\
&= \int \frac{d^3k}{(2\pi)^3} \left\{ (2n(E_+) + 1) \left[ \frac{1}{4E_+} + \frac{2\omega^2 + \mu_0^2}{2E_+(E_+^2 - E_-^2)} \right] \\
&\quad + (2n(E_-) + 1) \left[ \frac{1}{4E_-} - \frac{2\omega^2 + \mu_0^2}{2E_- (E_+^2 - E_-^2)} \right] \right\},  \\
\langle \hat{\phi}_2(x) | \hat{\phi}_2(x) \rangle &= M_{12}^{22}(x, x) = S_{22}(x, x) \\
&= \int \frac{d^3k}{(2\pi)^3} \left\{ (2n(E_+) + 1) \left[ \frac{1}{4E_+} + \frac{2\omega^2 - \mu_0^2}{2E_+(E_+^2 - E_-^2)} \right] \\
&\quad + (2n(E_-) + 1) \left[ \frac{1}{4E_-} - \frac{2\omega^2 - \mu_0^2}{2E_- (E_+^2 - E_-^2)} \right] \right\},  \\
\langle \hat{\phi}_a(x) | \hat{\phi}_b(x) \rangle &= 0 \quad \text{for } a \neq b,
\end{align*}
\]

where we denote \(\langle x, x_0 | S | y, y_0 \rangle\) as \(S(x, y)\). Here, \(E_\pm\) are defined in \((C.7)\) and

\[ n(E) = \frac{1}{e^{\beta E} - 1} \quad (D.17) \]

Also, \(\langle \hat{\phi}_c(x) | \hat{\phi}_c(x) \rangle\) for \(c \geq 3\) are given in Eq.\((D.2)\). Thus, our assumption that \(M_{ab}^{ab}\) has a diagonal form is valid because \(\langle \hat{\phi}_a(x) | \hat{\phi}_b(x) \rangle = 0\) for \(a \neq b\).
Secondly, let us consider the space-like \( q^2 \) as \( q^2 = -q^2 \) with \( \omega = 0 \). In this case, we can get \( M_{12}^{st}(x, x) \) in the same way as the time-like case:

\[
\langle \hat{\phi}_1(x) \hat{\phi}_1(x) \rangle = M_{12}^{\text{st}}(x, x) = S_{11}(x, x) = \int \frac{d^3k}{(2\pi)^3} \left\{ (2n(E_+) + 1) \left[ \frac{1}{4E_+} + \frac{\mu_0^2}{2E_+(E_+^2 - E_-^2)} \right] 
+ (2n(E_-) + 1) \left[ \frac{1}{4E_-} - \frac{\mu_0^2}{2E_-(E_+^2 - E_-^2)} \right] \right\},
\]

\[
\langle \hat{\phi}_2(x) \hat{\phi}_2(x) \rangle = M_{12}^{\text{st}}(x, x) = S_{22}(x, x) = \int \frac{d^3k}{(2\pi)^3} \left\{ (2n(E_+) + 1) \left[ \frac{1}{4E_+} - \frac{\mu_0^2}{2E_+(E_+^2 - E_-^2)} \right] 
+ (2n(E_-) + 1) \left[ \frac{1}{4E_-} + \frac{\mu_0^2}{2E_-(E_+^2 - E_-^2)} \right] \right\},
\]

\[
\langle \hat{\phi}_a(x) \hat{\phi}_b(x) \rangle = 0 \quad \text{for} \quad a \neq b . \quad (D.18)
\]

Here, \( E_{\pm} (> 0) \) are obtained by the poles of the integrand in (D-15) with \( \omega = 0 \):

\[
E_{\pm}^2(k) = k^2 + q^2 + M_0^2 \pm \sqrt{\mu_0^4 + 4(k \cdot q)^2} . \quad (D.19)
\]

Finally, the gap equation in the isospin-rotating frame at finite temperature is derived from the equation of motion for \( \bar{\phi} \) with \( \bar{\phi}_a = \phi_0 \delta_{a1} \). We obtain the gap equation for \( \phi_0 \) in the symmetry broken phase:

\[
\frac{\lambda}{6} \phi_0^2 = \omega^2 - q^2 - m_0^2 - \frac{\lambda}{6} \sum_{c=1}^{N} \langle \hat{\phi}_c(x) \hat{\phi}_c(x) \rangle - \frac{\lambda}{3} \langle \hat{\phi}_1(x) \hat{\phi}_1(x) \rangle \quad (D.20)
\]

due to \( \langle \hat{\phi}_a(x) \hat{\phi}_b(x) \rangle = 0 \) for \( a \neq b \). By using the mass \( M_1^2 \), the above gap equation is recast into simpler form as

\[
\frac{\lambda}{3} \phi_0^2 = -\omega^2 + q^2 + M_1^2 . \quad (D.21)
\]

References

14) R. E. Peierls, Surprises in Theoretical Physics, Princeton University Press, Princeton, 1979, section 1.3
19) See e.g., J. D. Bjorken and S. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964
27) J. Zinn Justin, Field theory and critical phenomena
|T| [MeV] | |q| [MeV] |
|---|---|---|---|
|0 | | | |
|40| | | |
|60| | | |
|80| | | |