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CLASSICAL MARKOV PROCESSES FROM
QUANTUM LÉVY PROCESSES

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par

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Classical Markov Processes from Quantum Lévy Processes

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Abstract

We show how classical Markov processes can be obtained from quantum Lévy processes. It is shown that quantum Lévy processes are quantum Markov processes, and sufficient conditions for restrictions to subalgebras to remain quantum Markov processes are given. A classical Markov process (which has the same time-ordered moments as the quantum process in the vacuum state) exists whenever we can restrict to a commutative subalgebra without losing the quantum Markov property[Küm88]. Several examples, including the Azéma martingale, with explicit calculations are presented. In particular, the action of the generator of the classical Markov processes on polynomials or their moments are calculated using Hopf algebra duality.

1 Introduction

It is an interesting question which quantum stochastic processes admit classical versions, i.e. for what families of operators $(X_t)_{t \in I}$ there exists a classical stochastic process $(\tilde{X}_t)_{t \in I}$ on some probability space $(\Omega, \mathcal{F}, P)$ such that all time-ordered moments agree, i.e.

$$\Phi(X_{t_1}^{k_1} \dots X_{t_n}^{k_n}) = \mathbb{E}(\tilde{X}_{t_1}^{k_1} \dots \tilde{X}_{t_n}^{k_n}), \quad (1)$$

for all $n, k_1, \dots, k_n \in \mathbb{N}$, $t_1 \leq \dots \leq t_n \in I$, for the (vacuum) state Φ .

A famous example of a classical version of a quantum Lévy process is the Azéma martingale [Azé85, Eme89, Par90, Sch91, Sch93]. In this case we can choose an element x of the bialgebra on which the process is defined such that the process $(X_t = j_t(x))$ is actually commutative, and thus it is clear that it is equivalent to a classical process. This process turn out to have several surprising properties, it was the first example of a process that has the chaotic representation property, but is not a (classical) Lévy process, see the references cited above.

We will use the theory of quantum Markov processes to give new conditions that guarantee the existence of classical versions of Lévy processes on $*$ -bialgebras. We construct conditional expectations and show that Lévy processes on $*$ -bialgebras are Markovian. It is well known that the quantum Markov property is sufficient for the existence of classical versions, see e.g. [BKS96]. Therefore it is sufficient to find commutative $*$ -subalgebras such that the restriction of our process is still Markovien. We will show that we can obtain quantum Markov processes on commutative subalgebras from quantum Lévy processes that are not commutative, and that there exist also commutative processes that are not Markovian.

We also show that our approach leads to a powerful tool for the explicit calculation of the classical generators or measures.

In Section 2 we recall the definitions of quantum Lévy processes and some of the fundamental results about these processes. We also recall a criterium by M. Schürmann for the existence of a classical version.

In the next section (Section 3) we construct a family of conditional expectations on the inductive limit representation of a quantum Lévy process and show that these processes are always Markovian. We also give nessary and sufficient conditions for restrictions of Lévy processes to be also Markovian.

In the following sections (Sections 4, 5, and 6) we give a detailed study of several examples, including the Azéma martingale and a symmetrized Poisson process, and show how Hopf algebra duality can be used for explicite calculations.

In this paper Lévy processes are defined on $*$ -bialgebras, but we do not introduce any topology. Therefore we can calculate the moments of the associated classical processes as well as the action of their generators on polynomials, but we will not adress any analytical questions such as uniqueness.

2 Preliminaries

In this section we introduce the basic notions and results about Lévy processes on bialgebras that we will need later on.

A quantum probability space is a pair $(\mathcal{A}, \Phi)$ consisting of a $*$ -algebra $\mathcal{A}$ and a state (i.e. a normalized positive linear functional) Φ on $\mathcal{A}$ and a quantum random variable j over a quantum probability space $(\mathcal{A}, \Phi)$ on a $*$ -algebra $\mathcal{B}$ is simply a $*$ -algebra homomorphism $j : \mathcal{B} \rightarrow \mathcal{A}$. A quantum stochastic process is an indexed family of random variables $(j_t)_{t \in I}$. The index set I will be $\mathbb{R}_+$ or an interval of the form $[0, T]$. For a quantum random variable $j : \mathcal{B} \rightarrow \mathcal{A}$ we will call $\varphi_j = \Phi \circ j$ its distribution in the state Φ .

The term ‘quantum stochastic process’ is sometimes also used for a indexed family $(X_t)_{t \in I}$ of operators on a Hilbert space or more generally of elements of a quantum probability space. We will reserve the name ‘operator process’ for this case. An operator process $(X_t)_{t \in I} \subseteq \mathcal{A}$ defines a quantum stochastic process $(j_t : \mathbb{C}\langle a, a^* \rangle \rightarrow \mathcal{A})_{t \in I}$ on the free $*$ -algebra with one generator, if we set $j_t(a) = X_t$ and extend as a $*$ -algebra homomorphism. On the other hand operator processes can be obtained from quantum stochastic processes $(j_t : \mathcal{B} \rightarrow \mathcal{A})_{t \in I}$ by simply choosing an element x of the algebra $\mathcal{B}$ and setting $X_t = j_t(x)$. Our operator processes will generally arise in this way.

Classical versions of operator processes are defined by condition (1).

There are several different notions of independence in quantum probability, and thus there exist several different notions of Lévy processes associated to these, cf. [Sch95b]. The notion of independence used here is that of (braided) tensor independence.

Definition 2.1 *Let $(\mathcal{A}, \Phi)$ be a quantum probability space, $\mathcal{B}$ a $*$ -algebra, and $\Psi : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ a linear map. An n -tuple $(j_1, \dots, j_n)$ of quantum random variables $j_i : \mathcal{B} \rightarrow \mathcal{A}$, $i = 1, \dots, n$, over $(\mathcal{A}, \Phi)$ on $\mathcal{B}$ is Ψ -independent or braided independent, if*

$$(i) \quad \Phi(j_{\sigma(1)}(b_1) \cdots j_{\sigma(n)}(b_n)) = \Phi(j_{\sigma(1)}(b_1)) \cdots \Phi(j_{\sigma(n)}(b_n)) \text{ for all permutations } \sigma \in \mathcal{S}(n) \text{ and all } b_1, \dots, b_n \in \mathcal{B}, \text{ and}$$

$$(ii) \quad m_{\mathcal{A}} \circ (j_l \otimes j_k) = m_{\mathcal{A}} \circ (j_k \otimes j_l) \circ \Psi \text{ for all } 1 \leq k < l \leq n.$$

Remark: In the applications we have in mind Ψ is actually a braiding, cf. [JS93].

If $\mathcal{B}$ is also a (braided) $*$ -bialgebra, i.e. if there exist $*$ -homomorphisms $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ and $\varepsilon : \mathcal{B} \rightarrow \mathbb{C}$ satisfying

$$\begin{aligned} (\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta \\ (\varepsilon \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \varepsilon) \circ \Delta = \text{id} \end{aligned}$$

then we can define a convolution of two linear maps $j_1, j_2 : \mathcal{B} \rightarrow \mathcal{A}$ by

$$j_1 \star j_2 = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta.$$

If j_1, j_2 are Ψ -independent quantum random variables, then $j_1 \star j_2$ is again a quantum random variable. The multiplication and the involution in $\mathcal{B} \otimes \mathcal{B}$ are defined by $m^{\otimes} = (m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id})$ and $(a \otimes b)^* = \Psi(b^* \otimes a^*)$ for $a, b \in \mathcal{B}$. We will refer to the case where Ψ is the flip automorphism $\tau : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$, $\tau : a \otimes b \mapsto b \otimes a$ as the **symmetric case**.

A Lévy process on $\mathcal{B}$ is a process with independent and stationary increments, where independence is now Ψ -independence, and the increments are with respect to the convolution introduced above.

Definition 2.2 [Sch93] *Let $\mathcal{B}$ be a (braided) $*$ -bialgebra. A quantum stochastic process $(j_{st})_{0 \leq s \leq t \leq T}$, $T \in \mathbb{R}_+ \cup \{\infty\}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ is called a **Lévy process**, if the following conditions are satisfied.*

1. (increment property)

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \text{for all } 0 \leq r \leq s \leq t \leq T, \\ j_{tt} &= e \circ \varepsilon \quad \text{for all } 0 \leq t \leq T, \end{aligned}$$

2. (independence of increments) the family $(j_{st})_{0 \leq s \leq t \leq T}$ is independent,

3. (stationarity of increments) the distribution $\varphi_{st} = \Phi \circ j_{st}$ of j_{st} depends only on the difference $t - s$,

4. (weak continuity) j_{st} converges to $j_{ss} (= e \circ \varepsilon)$ in distribution for $t \searrow s$.

Remark: In all our examples $\mathcal{B}$ will actually be a (braided) Hopf algebra, i.e. there exists a (unique) linear map $S : \mathcal{B} \rightarrow \mathcal{B}$, called antipode, that satisfies

$$m_{\mathcal{B}} \circ (S \otimes \text{id}) \circ \Delta = e_{\mathcal{B}} \circ \varepsilon = m_{\mathcal{B}} \circ (\text{id} \otimes S) \circ \Delta.$$

Then we can define increments of a process $(j_t)_{0 \leq t \leq T}$ by

$$j_{st} = (j_s \circ S) \star j_t,$$

this automatically satisfies the increment property. We call $(j_t)_{0 \leq t \leq T}$ a Lévy process, if $(j_{st})_{0 \leq s \leq t \leq T}$ is one. The functional defined by

$$L(a) = \left. \frac{d}{dt} \right|_{t=0} \Phi(j_t(a))$$

is called the generator of the Lévy process $(j_t)_{t \in \mathbb{R}_+}$, it characterizes the process. It is hermitian and conditionnally positive, i.e. positive on the kernel of ε . On the other hand, given a hermitian, conditionnally positive, Ψ -invariant (i.e. $\Psi \circ (\text{id} \otimes L) = L \otimes \text{id}$) functional L , there exists a Lévy process with L as generator.

We recall the following criterium for the existence of a classical version.

Theorem 2.3 [Sch93, Proposition 4.2.3] *Let $(j_t)_{t \in [0, T]}$ be a Lévy process on a (braided) bialgebra $\mathcal{B}$, and $b_1, \dots, b_n \in \mathcal{B}$ be commuting elements of $\mathcal{B}$, i.e. $b_i b_k = b_k b_i$ for all $i, k = 1, \dots, n$. If $b_i \otimes 1$ and Δb_k commute in $\mathcal{B} \otimes \mathcal{B}$ for all $i, k = 1, \dots, n$, then the family $(j_t(b_1), \dots, j_t(b_n))_{t \in [0, T]}$ is a family of commuting operators.*

Remark: In [Sch93] this result is only stated for the symmetric case and for $n = 1$, but the generalisation is straightforward.

Proof: Let $0 \leq s \leq t \leq T$, and let $\Delta(b_k) = \sum b_k^{(1)} \otimes b_k^{(2)}$. Then we have

$$\begin{aligned}
j_t(b_k) j_s(b_i) &= \sum j_s(b_k^{(1)}) j_{st}(b_k^{(2)}) j_s(b_i) \\
&= \sum m_{\mathcal{A}} \circ (j_s \otimes j_{st} \otimes j_s)(b_k^{(1)} \otimes b_k^{(2)} \otimes b_i) \\
&= \sum m_{\mathcal{A}} \circ (j_s \otimes j_s \otimes j_{st})(b_k^{(1)} \otimes \Psi(b_k^{(2)} \otimes b_i)) \\
&= m_{\mathcal{A}} \circ (j_s \otimes j_{st})(m_{\mathcal{B} \otimes \mathcal{B}}(\Delta b_k \otimes (b_i \otimes e))) \\
&= m_{\mathcal{A}} \circ (j_s \otimes j_{st})(m_{\mathcal{B} \otimes \mathcal{B}}((b_i \otimes e) \otimes \Delta(b_k))) \\
&= \sum m_{\mathcal{A}} \circ (j_s \otimes j_{st})(b_i b_k^{(1)} \otimes b_k^{(2)}) \\
&= \sum j_s(b_i b_k^{(1)}) j_{st}(b_k^{(2)}) = \sum j_s(b_i) j_s(b_k^{(1)}) j_{st}(b_k^{(2)}) \\
&= j_s(b_i) j_t(b_k).
\end{aligned}$$

■

For more details on Hopf algebras and braidings see also [Maj95b], concerning quantum probability and Lévy processes on bialgebra see [Sch93, Mey93].

3 Classical versions of quantum Lévy processes

3.1 From quantum Lévy processes to quantum Markov processes

In this paper we will always assume that $(\mathcal{A}, \Phi)$ is the quantum probability space $(\tilde{\mathcal{B}}, \tilde{\Phi})$ obtained as inductive limit from finite tensor products of $\mathcal{B}$ and the marginal distribution $\varphi_t = \Phi \circ j_t$ by the construction described in [Sch93, pp. 38-40] or in [FFS97, Section 8]. For an interval $I \subseteq [0, T]$ we denote by $\mathcal{A}_I$ the *-subalgebra of $\mathcal{A}$ generated by $\bigcup_{\tau \in I} j_{\tau}(\mathcal{B})$. For singletons $\{t\} = [t, t]$ the algebra $\mathcal{A}_t = \mathcal{A}_{[t, t]}$ is isomorphic to $\mathcal{B}$, and we have the shift isomorphisms $T_{s, t} : \mathcal{A}_s \rightarrow \mathcal{A}_t$, $T_{s, t} = j_t \circ j_s^{-1}$. Let $\mathcal{F}_I$ be the *-subalgebra of $\mathcal{A}$ generated by the increments of subintervals of I , i.e. the algebra generated by $\bigcup_{[\tau_1, \tau_2] \subseteq I} j_{\tau_1, \tau_2}(\mathcal{B})$. Since $j_{t'} = j_t \star j_{t, t'}$ for $0 \leq t \leq t' \leq T$, we see that $\mathcal{A}_{[t_1, t_2]}$ is exactly the algebra generated by $\mathcal{A}_{t_1} \cup \mathcal{F}_{[t_1, t_2]}$. In particular, we obviously have $\mathcal{F}_{[0, t]} = \mathcal{A}_{[0, t]}$ for all $t \in [0, T]$.

It is not essential to use this realisation, we could just as well use the realisation of the Lévy process on Bose Fock space defined in [Sch93, Theorem 2.5.3], here the decomposition $\Gamma(L^2([0, T], H)) \cong \Gamma(L^2([0, t_1], H)) \otimes \Gamma(L^2([t_1, t_2], H)) \otimes \Gamma(L^2([t_2, T], H))$ for $[t_1, t_2] \subseteq [0, T]$ can be used to define conditional expectations $P_{[t_1, t_2]}$. But since we want to treat the symmetric and the braided case simultaneously, this would require extending Schürmann's results to the braided case.

We want to define conditional expectations $P_I : \mathcal{A} \rightarrow \mathcal{F}_I$ for $I \subseteq [0, T]$, and show that a Lévy process satisfies $P_{[0, s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$ for all $0 \leq s \leq t \leq T$, i.e. that it is a quantum Markov process. We use the notation of [FFS97, Section 8]. Remember that the factors in an element $b_1 \otimes \cdots \otimes b_n$ of $\mathcal{B}_{(t_1, \dots, t_n)}$ get mapped to $j_{t_i}(b_i) \in \mathcal{A}_{t_i}$, and that we can use the coproduct to 'decompose' them into their increments. This was done by the map

$$\psi_{(t_1, \dots, t_n)} = (\psi_2 \otimes \text{id}_{\mathcal{B}} \otimes \cdots \otimes \text{id}_{\mathcal{B}}) \circ (\text{id}_{\mathcal{B}} \otimes \psi_2 \otimes \text{id}_{\mathcal{B}} \otimes \cdots \otimes \text{id}_{\mathcal{B}}) \circ \cdots \circ (\text{id}_{\mathcal{B}} \otimes \cdots \otimes \text{id}_{\mathcal{B}} \otimes \psi_2),$$

$\psi_{(t_1, \dots, t_n)} : \mathcal{B}_{(t_1, \dots, t_n)} \rightarrow \mathcal{B}^{\otimes n}$, where $\psi_2 = (m \otimes \text{id}_{\mathcal{B}}) \circ (\text{id}_{\mathcal{B}} \otimes \Delta)$. This motivates the following definition.

Let $\mathcal{B}_{(s_1, \dots, s_n)}$ be the 'approximation' of $\mathcal{A} = \tilde{\mathcal{B}}$ corresponding to $(s_1, \dots, s_k, \dots, s_n)$, $s_1 < \cdots < s_n$. Suppose $s_k = s$ for some k . Then we define

$$\begin{aligned} P_{[0, s]}^{(s_1, \dots, s_k, \dots, s_n)} &: \mathcal{B}_{(s_1, \dots, s_n)} \rightarrow \mathcal{B}_{(s_1, \dots, s_k=s)} \subseteq \mathcal{A}_{[0, s]} \\ P_{[0, s]}^{(s_1, \dots, s_k, \dots, s_n)} &= \psi_{(s_1, \dots, s_k)}^{-1} \circ (\text{id}_{\mathcal{B}^{\otimes k}} \otimes e \circ \varphi_{s_{k+1}-s_k} \otimes \cdots \otimes e \circ \varphi_{s_n-s_{n-1}}) \circ \psi_{(s_1, \dots, s_n)} \end{aligned}$$

If we introduce $\Pi_{(s_1, \dots, s_k)}^{(s_1, \dots, s_k, \dots, s_n)} = \text{id}_{\mathcal{B}^{\otimes k}} \otimes e \circ \varphi_{s_{k+1}-s_k} \otimes \cdots \otimes e \circ \varphi_{s_n-s_{n-1}}$, then we can also write $P_{[0, s]}^{(s_1, \dots, s_k, \dots, s_n)} = \psi_{(s_1, \dots, s_k)}^{-1} \circ \Pi_{(s_1, \dots, s_k)}^{(s_1, \dots, s_k, \dots, s_n)} \circ \psi_{(s_1, \dots, s_n)}$.

Proposition 3.1 *The maps $P_{[0, s]}^{(s_1, \dots, s_k, \dots, s_n)} : \mathcal{B}_{(s_1, \dots, s_n)} \rightarrow \mathcal{A}_{[0, s]}$ extend to a unique linear map $P_{[0, s]} : \mathcal{A} \rightarrow \mathcal{A}_{[0, s]}$.*

Proof: Let $i_{S', S} : \mathcal{B}_{S'} \rightarrow \mathcal{B}_S$ with $S = (s_1, \dots, s_n)$ and $S' \subseteq S$ be the inclusion maps from [FFS97, Proposition 8.2] (i.e. the maps that adds the unit element in the factors that are missing in S'). To prove the proposition it is sufficient to show $P_{[0, s]}^S \circ i_{S', S} = P_{[0, s]}^{S'}$, where we can assume that s occurs in S' (and therefore also in S). We show the proof for the case $S = (s_1, \dots, s_k = s, \dots, s_n)$ and $S' = (s_1, \dots, s_k = s, \dots, s_{l-1}, s_{l+1}, \dots, s_n)$, the general case is similar. In this case the inclusion map acts as $i_{S', S}(b_1 \otimes \cdots \otimes b_{l-1} \otimes b_{l+1} \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes b_{l-1} \otimes e \otimes b_{l+1} \otimes \cdots \otimes b_n$ for all $b_1 \otimes \cdots \otimes b_{l-1} \otimes b_{l+1} \otimes \cdots \otimes b_n \in \mathcal{B}_{S'}$. Therefore $\psi_S \circ i_{S', S} = (\text{id}_{\mathcal{B}^{\otimes (l-1)}} \otimes \Delta \otimes \text{id}_{\mathcal{B}^{\otimes (n-l-1)}}) \circ \psi_{S'}$, and thus

$$\begin{aligned} P_{[0, s]}^S \circ i_{S', S} &= \\ &= \psi_{S \cap [0, s]}^{-1} \circ (\text{id}_{\mathcal{B}^{\otimes k}} \otimes e \circ \varphi_{s_{k+1}-s_k} \otimes \cdots \otimes e \circ (\varphi_{s_l-s_{l-1}} \star \varphi_{s_{l+1}-s_l}) \otimes \cdots \otimes e \circ \varphi_{s_n-s_{n-1}}) \circ \psi_{S'} \\ &= P_{[0, s]}^{S'}. \end{aligned}$$

■

Conditional expectations are usually studied on von Neumann algebras with a fixed faithful normal state (or on C^* -algebras), where they are defined as completely positive maps that preserve the unit and the state. Here, in our purely algebraic setting, we will call an element x of a $*$ -algebra $\mathcal{A}$ positive if and only if it can be written in the form $x = \sum_{i=1}^n a_i^* a_i$ for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}$. The set $M_n(\mathcal{A})$ of all $n \times n$ matrices with entries in $\mathcal{A}$ is again an involutive algebra with the obvious matrix multiplication and the involution $(a^*)_{ij} = a_{ji}^*$. A linear map $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ between two involutive algebras is called completely positive, if, for all $n \in \mathbb{N}$, the linear maps $\varphi_n : M_n(\mathcal{A}_1) \rightarrow M_n(\mathcal{A}_2)$ defined by $(\varphi_n(a))_{ij} = \varphi(a_{ij})$ preserve positivity.

We can now show that the maps $P_{[0,s]}$ really have the desired properties.

Proposition 3.2 *Let $s \in [0, T]$. The linear map $P_{[0,s]} : \mathcal{A} \rightarrow \mathcal{A}_{[0,s]}$ is completely positive and satisfies $P_{[0,s]}^2 = P_{[0,s]}$, $\Phi \circ P_{[0,s]} = \Phi$, $P_{[0,s]}(1) = 1$, and $P_{[0,s]}(xyz) = xP_{[0,s]}(y)z$ for all $x, z \in \mathcal{A}_{[0,s]}$, $y \in \mathcal{A}$, i.e. it is a **conditional expectation**. Furthermore, we have $P_{[0,s_1]} \circ P_{[0,s_2]} = P_{[0,s_2]} \circ P_{[0,s_1]} = P_{[0, \min(s_1, s_2)]}$ for all $s_1, s_2 \in [0, T]$.*

Proof: $P_{[0,s]}^2 = P_{[0,s]}$ is clear, because $P_{[0,s]}^S = \psi_S^{-1} \circ \text{id}_{\mathcal{B} \otimes |S|} \circ \psi_S = \text{id}$ for all $S \subseteq [0, s]$. $\Phi \circ P_{[0,s]} = \Phi$ and $P_{[0,s]}(1) = 1$ follow also immediately from the definition.

We will show $P_{[0,s]}(xyz) = xP_{[0,s]}(y)z$ for the case $x, z \in \mathcal{A}_s$, $y \in \mathcal{A}_t$, $0 < s < t < T$, the general case is similar. For this case it is sufficient to take $S = (s, t)$, and we can write $\psi_S(x) = x \otimes e$, $\psi_S(y) = \Delta(y) = \sum y_i^{(1)} \otimes y_i^{(2)}$, $\psi_S(z) = z \otimes e$. Therefore $\psi_S(xyz) = \sum xy_i^{(1)} \Psi(y_i^{(2)} \otimes z)$, and $P_{[0,s]}(xyz) = \psi_{(s)}^{-1}((\text{id} \otimes \varphi_{t-s})(\sum xy_i^{(1)} \Psi(y_i^{(2)} \otimes z))) = \sum xy_i^{(1)} z \varphi_{t-s}(y_i^{(2)}) = xP_{[0,s]}(y)z$, where we used the Ψ -invariance of φ_{t-s} .

In order to prove the complete positivity, it is sufficient to prove that $\Pi_{(s_1, \dots, s_k)}^{(s_1, \dots, s_k, \dots, s_n)}$ is completely positive, since the ψ_S are $*$ -algebra homomorphisms. But this is obvious from the way it is defined via the identity map and the states $\varphi_{s_{i+1}-s_i}$. ■

Theorem 3.3 *We have $P_{[0,s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$ for all $0 \leq s \leq t \leq T$, i.e. $(j_t)_{t \in [0, T]}$ is a quantum Markov process.*

Proof: Since the one-dimensional distributions form a convolution semigroup, all $P_{[0,s]}^{(s_1, \dots, s_k, \dots, s_n)}$ for $n \geq 2$, $0 < s_k = s < s_n = t$ act in the same way on elements of $\mathcal{A}_t$, namely $P_{[0,s]}^{(s_1, \dots, s_k, \dots, s_n)}|_{\mathcal{A}_t} = j_s \circ (\text{id}_{\mathcal{B}} \otimes \varphi_{t-s}) \circ \Delta \circ j_t^{-1}$, and therefore this is also true for $P_{[0,s]}|_{\mathcal{A}_t}$. In particular, it follows $P_{[0,s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$. ■

In this proof we have seen how $P_{[0,s]}$ can be calculated on $\mathcal{A}_t$.

Corollary 3.4 *For the Markov semigroup $P_{s,t} = P_{[0,s]} \circ T_{s,t}$ we have $P_{s,t}(j_s(b)) = j_s(p_{t-s}(b))$ for all $b \in \mathcal{B}$, where $p_{t-s} = (\text{id} \otimes \varphi_{t-s}) \circ \Delta$.*

This result allows us to do all calculations directly on the Hopf algebra, without using any particular realisation of the process (j_t) . The map $p_t : \mathcal{B} \rightarrow \mathcal{B}$ is also known as the **right dual** or **right regular representation** $\rho(\varphi_t)$ of the functional φ_t .

We will now recall a few facts about duals of Hopf algebras and dual representations that will be used later.

The dual space of a Hopf algebra forms an algebra with the multiplication defined by $(xy)(b) = (x \otimes y)\Delta b$ for all $b \in \mathcal{B}$. The dual, or at least a subspace of the dual, can also be equipped with a coalgebra structure, so that it becomes a bialgebra (or even a Hopf algebra, if there is also an antipode). Two bialgebras $(\mathcal{B}_i, m_i, e_i, \Delta_i, \varepsilon_i)$, $i = 1, 2$, are called dually paired, if there exists a linear map $\langle \cdot, \cdot \rangle : \mathcal{B}_1 \otimes \mathcal{B}_2 \rightarrow \mathbb{C}$ such that

$$\begin{aligned} \langle a_1 \otimes b_1, \Delta_2 a_2 \rangle_{\otimes} &= \langle m_1(a_1 \otimes b_1), a_2 \rangle, \\ \langle \Delta_1 a_1, a_2 \otimes b_2 \rangle_{\otimes} &= \langle a_1, m_2(a_2 \otimes b_2) \rangle \\ \langle e_1, a_2 \rangle &= \varepsilon_2(a_2), \quad \langle a_1, e_2 \rangle = \varepsilon_1(a_1) \end{aligned}$$

for all $a_i, b_i \in \mathcal{B}_i$. For Hopf algebras there is the additional condition

$$\langle S_1(a_1), a_2 \rangle = \langle a_1, S_2(a_2) \rangle, \quad \text{for all } a_i \in \mathcal{B}_i.$$

There is also a condition for involutive Hopf algebras,

$$\langle a_1^*, a_2 \rangle = \overline{\langle a_1, S_2(a_2)^* \rangle}, \quad \text{for all } a_i \in \mathcal{B}_i,$$

but we will not need it here.

Unfortunately there are two different conventions for the definition of $\langle \cdot, \cdot \rangle_{\otimes}$, in the symmetric case (i.e. if $\Psi = \tau$) one uses

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{\otimes} = \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle, \quad \text{for all } a_i, b_i \in \mathcal{B}_i \quad (2)$$

whereas

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{\otimes} = \langle a_1, b_2 \rangle \langle b_1, a_2 \rangle, \quad \text{for all } a_i, b_i \in \mathcal{B}_i$$

is often preferred in the braided case (i.e. if $\Psi \neq \tau$). In this paper we will **always** use the **first** convention (Formula (2)), in this way the symmetric case appears as a special case of the general braided case.

Lemma 3.5 *a) The right dual representation is an anti-homomorphism, i.e.*

$$\rho(xy) = \rho(y)\rho(x)$$

for all $x, y \in \text{Hom}(\mathcal{B}, \mathbb{C})$.

b) If x is in a bialgebra dually paired with $\mathcal{B}$, then $\rho(x)$ satisfies a generalised Leibniz formula, which has the form

$$\rho(x)(ab) = \sum_i m \circ (\rho \otimes \text{id})(a \otimes \Psi^{-1}(\rho(x_i^{(2)})b \otimes x_i^{(1)}))$$

where $\Delta x = \sum_i x_i^{(1)} \otimes x_i^{(2)}$. In the symmetric case we can also write

$$\rho(x)(ab) = \sum_i (\rho(x_i^{(1)})a) (\rho(x_i^{(2)})b).$$

Proof:

- a) $\rho(xy)a = (\text{id} \otimes xy) \circ \Delta(a) = (\text{id} \otimes x \otimes y) \circ (\text{id} \otimes \Delta) \circ \Delta(a) = (\text{id} \otimes x \otimes y) \circ (\Delta \otimes \text{id}) \circ \Delta(a) = \rho(x)\rho(y)a.$
- b) $\rho(x)(ab) = (\text{id} \otimes x)\Delta \circ m(a \otimes b) = (\text{id} \otimes x)(m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)(a \otimes b) = \sum_i (m \otimes x_i^{(1)} \otimes x_i^{(2)}) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)(a \otimes b) = \sum_i (m \otimes x_i^{(1)}) \circ (\text{id} \otimes \Psi)(\Delta(a) \otimes \rho(x_i^{(2)})b) = \sum_i m \circ (\rho \otimes \text{id})(a \otimes \Psi^{-1}(\rho(x_i^{(2)})b \otimes x_i^{(1)})).$

■

3.2 From quantum Markov processes to classical Markov processes

We know (see e.g. [BKS96]) that for every quantum Markov process on a commutative $*$ -algebra (e.g. the subalgebra generated by one self-adjoint element) there exists a classical version, i.e. a classical stochastic process that has the same (time-ordered) joint moments as the quantum Markov process. The quantum Markov property is sufficient (not necessary) for the joint density associated to the joint time-ordered moments to be positive, and then the classical process exists by Kolmogorov's construction.

Thus we need to look for self-adjoint elements of $\mathcal{B}$ who generate a subalgebra such that the restriction of the quantum Lévy process to this algebra remains Markov.

We give two criteria that guarantee that the restriction of (j_t) to a subalgebra $\mathcal{B}_0$ remains Markovian.

Theorem 3.6

- (a) Let $\mathcal{B}_0$ be a $*$ -subalgebra of $\mathcal{B}$. If $\Delta(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes \mathcal{B}$ (i.e. $\mathcal{B}_0$ is a right coideal), then the restriction $(j_t|_{\mathcal{B}_0})$ to $\mathcal{B}_0$ of every Lévy process (j_t) on $\mathcal{B}$ is a quantum Markov process.
- (b) Let L be the generator of (j_t) , and $\mathcal{B}_0 \subseteq \mathcal{B}$ a $*$ -subalgebra of $\mathcal{B}$. Then the restriction of (j_t) to $\mathcal{B}_0$ is a quantum Markov process if and only if $\rho(L) = (\text{id} \otimes L) \circ \Delta$ leaves $\mathcal{B}_0$ invariant, i.e. if $\rho(L)(\mathcal{B}_0) \subseteq \mathcal{B}_0$.

Proof:

- a) If $\mathcal{B}_0$ is a right coideal, then $\rho(L)(\mathcal{B}_0) \subseteq \mathcal{B}_0$ for any functional L , and consequently $P_{[0,s]}(j_t(\mathcal{B}_0)) = j_s(\rho(e^{(t-s)L})(\mathcal{B}_0)) \subseteq j_s(\mathcal{B}_0)$.
- b) That the condition is sufficient is clear, since $P_{[0,s]} = j_s \circ e^{(t-s)\rho(L)} \circ j_t^{-1}$ on $\mathcal{A}_t$. To see that it is necessary it is enough to differentiate the relation $\rho(e^{(t-s)L})(\mathcal{B}_0) \subseteq \mathcal{B}_0$ with respect to t and then set $t = s$.

■

In the symmetric case an abelian subalgebra is a right coideal only if all its elements satisfy the condition of Proposition 2.3, so that Part a) of the preceding theorem does not enable us to find any classical versions that could not have been obtained with Schürmann's result.

Proposition 3.7 *Let $\mathcal{B}$ be a symmetric bialgebra. If $\mathcal{B}_0$ is an abelian subalgebra and a right coideal of $\mathcal{B}$, then $a \otimes 1$ and Δb commute for all $a, b \in \mathcal{B}_0$.*

Proof: Let $\Delta b = \sum b_i^{(1)} \otimes b_i^{(2)}$, then $(a \otimes 1)\Delta b = \sum ab_i^{(1)} \otimes b_i^{(2)} = \sum b_i^{(1)}a \otimes b_i^{(2)} = \Delta a(b \otimes 1)$ since the $b_i^{(1)}$ are in $\mathcal{B}_0$ and commute therefore with a . ■

In Subsection 5.1, we will see that there exist commutative operator processes of the form $(j_t(x))$ where the quantum stochastic process $(j_t|_{\mathfrak{C}[x]})$ is not Markovian, and also restrictions of Lévy processes to commutative *-subalgebras that are Markovian but not commutative, i.e. we can get processes that have a classical version with Part b) of Theorem 3.6, but that do satisfy the conditions of Theorem 2.3.

3.3 Martingales

Definition 3.8 *Let $(\mathcal{A}, \Phi)$ be a quantum probability space equipped with a family of conditional expectations $(P_{[0,s]})_{s \in [0,T]}$. An operator process $(X_t)_{t \in [0,T]} \subseteq \mathcal{A}$ is called a $(P_{[0,s]})_{s \in [0,T]}$ -**martingale**, if*

$$P_{[0,s]}(X_t) = X_s$$

for all $0 \leq s \leq t \leq T$.

It is often useful to find a function $f(x, t)$ for a given initial value $g(x)$ and a given stochastic process (X_t) such that $f(x, 0) = g(x)$ and $f(X_t, t)$ is a martingale. For example, if $(X_t)_{t \in \mathbb{R}_+}$ is a classical Lévy process (with characteristic exponent Ψ , i.e. if $\mathbb{E}(e^{iuX_t}) = e^{t\Psi(u)}$) and $g(x) = e^{iux}$ for some $u \in \mathbb{R}$, then $(e^{iuX_t - t\Psi(u)})_{t \in \mathbb{R}_+}$ is a martingale (w.r.t. the filtration of $(X_t)_{t \in \mathbb{R}_+}$), the so-called exponential martingale of X_t .

We will show how solutions of this problem for quantum Lévy processes can be given.

Theorem 3.9 *Let $(j_t)_{t \in [0, T]}$ be a Lévy process on a $\ast$ -bialgebra $\mathcal{B}$ (realized on the inductive limit quantum probability space $\tilde{\mathcal{B}}$) with generator L and let $b \in \mathcal{B}$. Then $(j_t(e^{-t\rho(L)}b))_{t \in [0, T]}$ is a martingale.*

Proof: With Corollary 3.4 we get

$$\begin{aligned} P_{[0, s]}(j_t(e^{-t\rho(L)}b)) &= j_s(e^{(t-s)\rho(L)}e^{-t\rho(L)}b) \\ &= j_s(e^{-s\rho(L)}b), \end{aligned}$$

for all $0 \leq s \leq t \leq T$. ■

4 Examples of classical versions of Lévy processes on $\mathbb{C}_q\langle a, a^\ast \rangle$

Let us first consider the bialgebra that leads to the Azéma martingale (cf. [Sch91, Sch93]). It can be viewed as the free algebra with two generators $a, a^\ast$ (and the obvious $\ast$ -structure), we shall denote it here by $\mathbb{C}_q\langle a, a^\ast \rangle$. The set $\mathcal{B} = \{1, a, a^\ast, aa, aa^\ast, a^\ast a, a^\ast a^\ast, \dots\}$ of all words in the two letters $a, a^\ast$ forms a basis of $\mathbb{C}_q\langle a, a^\ast \rangle$. The coproduct and counit are defined as

$$\Delta(a) = a + a', \quad \Delta(a^\ast) = a^\ast + a^{\ast'}, \quad \varepsilon(a) = \varepsilon(a^\ast) = 0$$

on the generators and extended as algebra homomorphisms (Notation: $a^{(\ast)} = a^{(\ast)} \otimes 1$, $a^{(\ast)'} = 1 \otimes a^{(\ast')}$). But the algebra structure of $\mathbb{C}_q\langle a, a^\ast \rangle \otimes \mathbb{C}_q\langle a, a^\ast \rangle$ is determined by the braid relations

$$\begin{aligned} a'a &= qaa', & a^{\ast'}a &= qaa^{\ast'}, \\ a'a^\ast &= q^{-1}a^\ast a & a^{\ast'}a^\ast &= q^{-1}a^\ast a^{\ast'}, \end{aligned}$$

where we assume $q \in \mathbb{R} \setminus \{0\}$.

We need to construct the dual $\mathcal{U}$ of $\mathbb{C}_q\langle a, a^\ast \rangle$. For $q = 1$ this is the shuffle algebra, see e.g. [SS93, Section 3.8]. In the general case the dual might be called a q -shuffle algebra, the formulas of the shuffle algebra only have to be modified by some q -dependent combinatorial coefficients.

A functional on $\mathbb{C}_q\langle a, a^\ast \rangle$ is determined by its action on the basis chosen above, i.e. on the words in $a, a^\ast$. Thus it can be written as

$$u = \sum_{\chi \in \mathcal{X}} c_\chi \chi,$$

where $\mathcal{X}$ is the dual basis of $\mathcal{B}$, i.e. the set of all words in two letters (x, x^*) , and $c_\chi \in \mathbb{C}$. As dual of a coalgebra this is an algebra, and if we restrict to finite linear combinations, then it is even a bialgebra, and the dual pairing

$$\langle \chi, \beta \rangle = \begin{cases} 1 & \text{if } \chi, \beta \text{ are identical modulo the substitution } a \leftrightarrow x, a^* \leftrightarrow x^*, \\ 0 & \text{else,} \end{cases}$$

$\chi \in \mathcal{X}, \beta \in \mathcal{B}$, is still non-degenerate.

The coproduct of a word in x, x^* is just the sum over the different ways to split the word in two, i.e. for a word with n letters there are exactly $n + 1$ terms. Thus we have e.g. $\Delta x = x \otimes 1 + 1 \otimes x$, $\Delta x^* = x^* \otimes 1 + 1 \otimes x^*$, or $\Delta x x^* x = x x^* x \otimes 1 + x x^* \otimes x + x \otimes x^* x + 1 \otimes x x^* x$.

To compute the product¹ of two basis elements χ_1, χ_2 in the q -shuffle algebra, we have to look at all the ways in which the first word χ_1 can be ‘shuffled’ into the second. The first term is simply the concatenation $\chi_1 \chi_2$. Now we form all combinations of the letters of χ_1 and χ_2 where the order of the letters of χ_1 (and χ_2 , resp.) remains unchanged, and add a factor q every time an x (from χ_1) is moved to the right across a letter of χ_2 , and a factor q^{-1} every time an x^* (from χ_1) is moved to the right across a letter of χ_2 . Thus, e.g.

$$\begin{aligned} x \cdot x &= x x + q x x = (1 + q) x x, & x \cdot x^* &= x x^* + q x^* x, \\ x^* \cdot x &= x^* x + q^{-1} x x^*, & x^* \cdot x^* &= x^* x^* + q^{-1} x^* x^* = (1 + q^{-1}) x^* x^*, \end{aligned}$$

or

$$\begin{aligned} \overbrace{x x^*} \cdot \overbrace{x x^*} &= \overbrace{x x^*} \overbrace{x x^*} + q^{-1} \overbrace{x x x^*} x^* + \overbrace{x x x^*} x^* + q^{-2} \overbrace{x x x^*} x^* + q^{-1} \overbrace{x x x^*} x^* + \overbrace{x x^*} \overbrace{x x^*} \\ &= 2 x x^* x x^* + (1 + 2 q^{-1} + q^{-2}) x x x^* x^*. \end{aligned}$$

WARNING: $x x, x x^*, x x^* x x^*$, etc. means concatenation, in this section the multiplication in $\mathcal{U}$ is **always** indicated by a dot.

The dual action or right regular representation eliminates the letters corresponding to those of χ , if they are in the same order, with possibly additional letters in between, and adds a factor q (q^{-1} , resp.) for every letter to the left of an a (a^* , resp.) that is suppressed. E.g., on an element $a_1 a_2 \cdots a_n \in \mathbb{C}_q \langle a, a^* \rangle$, with $a_i \in \{a, a^*\}$ for $i = 1, \dots, n$,

$$\begin{aligned} \rho(x) : a_1 \cdots a_n &\mapsto \sum_{i: a_i = a} q^{i-1} a_1 \cdots \check{a}_i \cdots a_n \\ \rho(x^*) : a_1 \cdots a_n &\mapsto \sum_{i: a_i = a^*} q^{-i+1} a_1 \cdots \check{a}_i \cdots a_n \\ \rho(x x^*) : a_1 \cdots a_n &\mapsto \sum_{i < j: a_i = a, a_j = a^*} q^{i-j+1} a_1 \cdots \check{a}_i \cdots \check{a}_j \cdots a_n. \end{aligned}$$

¹Remember that we use the convention $\langle u \cdot v, \beta \rangle = \sum \langle u, \beta^{(1)} \rangle \langle v, \beta^{(2)} \rangle$, i.e. we define the dual pairing of the tensor product algebra with the flip automorphism $\tau: \langle \cdot, \cdot \rangle_{\otimes} = (\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle) \circ (\text{id} \otimes \tau \otimes \text{id})$.

4.1 The Azéma martingale

If we choose the generator $L = c_1xx^* + c_2x^*x$, with $c_1, c_2 \in \mathbb{R}_+$, then the Azéma martingale M_t (cf. [Sch91, Sch93]) is a classical version of $\{j_t(a + a^*)\}$, i.e. the (finite) joint moments of $(j_t(a + a^*))_{t \in \mathbb{R}_+}$ agree with those of the (classical) Azéma martingale: $\mathbb{E}(M_{t_1}^{n_1} \cdots M_{t_k}^{n_k}) = \Phi((j_{t_1}(a + a^*))^{n_1} \cdots (j_{t_k}(a + a^*))^{n_k})$ for all $t_1, \dots, t_k \in \mathbb{R}_+$, $n_1, \dots, n_k \in \mathbb{N}$, $k \in \mathbb{N}$. In order to find the generator of the Azéma martingale (as a classical Markov process) and to compute the moments, we need to know how L acts on the subalgebra generated by $z = a + a^*$.

Proposition 4.1 (Leibniz formulas). *Let $f = \sum_{k=0}^n f_k z^k$ with $f_k \in \mathbb{C}$ and $z = a + a^*$. Then*

$$\begin{aligned} \rho(x)(zf) &= f + qz\rho(x)f, \\ \rho(x^*)(zf) &= f + q^{-1}z\rho(x^*)f, \\ \rho(xx^*)(zf) &= \rho(x^*)f + z\rho(xx^*)f, \\ \rho(x^*x)(zf) &= \rho(x)f + z\rho(x^*x)f. \end{aligned}$$

Proof: This follows from the Lemma 3.5. We have $\rho(u)(\alpha\beta) = \sum \alpha^{(1)} \cdot (\text{id} \otimes u^{(1)})(\Psi(\alpha^{(2)} \otimes \rho(u^{(2)})\beta))$ (where we used Sweedler's notation for the coproduct). Applying the braid relations and recalling that z, x, x^* are primitive, while $\Delta(xx^*) = xx^* \otimes 1 + x \otimes x^* + 1 \otimes xx^*$, $\Delta(x^*x) = x^*x \otimes 1 + x^* \otimes x + 1 \otimes x^*x$, we get the desired formulas. $\blacksquare$

Setting $f = z^n$, we obtain recurrence relations that allow us to determine the operators on polynomials in z ,

$$\begin{aligned} \rho(x) : z^n &\mapsto q_n z^n &\Rightarrow (\rho(x)f)(z) &= \delta_q f(z) = \frac{f(qz) - f(z)}{z(q-1)} \\ \rho(x^*) : z^n &\mapsto (q^{-1})_n z^n &\Rightarrow (\rho(x^*)f)(z) &= \delta_{1/q} f(z) = \frac{f(q^{-1}z) - f(z)}{z(q^{-1}-1)} \\ \rho(xx^*) : z^n &\mapsto \begin{cases} \frac{(q^{-1})_{n-1}}{q^{-1}-1} z^{n-2} & n \geq 2, \\ 0 & n = 0, 1 \end{cases} &\Rightarrow (\rho(xx^*)f)(z) &= \frac{\delta_{1/q} f(z) - f'(z)}{z(q^{-1}-1)} \\ \rho(x^*x) : z^n &\mapsto \begin{cases} \frac{q_{n-1}}{q-1} z^{n-2} & n \geq 2, \\ 0 & n = 0, 1 \end{cases} &\Rightarrow (\rho(x^*x)f)(z) &= \frac{\delta_q f(z) - f'(z)}{z(q-1)} \end{aligned}$$

for $q \neq 1$, and $\rho(x)f = \rho(x^*)f = f'$, $\rho(xx^*)f = \rho(x^*x)f = \frac{1}{2}f''$ for $q = 1$. Here we used the q -numbers $q_n = \sum_{\nu=0}^{n-1} q^\nu = \frac{q^n-1}{q-1}$ for $q \neq 1$ and $q_n = n$ for $q = 1$.

Using these operators to calculate the Appell polynomials

$$h_n(z; t) = e^{t\rho(L)} z^n,$$

we can also calculate the moments of the Azéma martingale, $\mathbb{E}(M_t^n) = h_n(0, t)$. We summarize the results in the following theorem.

Theorem 4.2 *Let $(j_t)_{t \in \mathbb{R}_+}$ be the quantum Lévy process on $\mathbb{C}_q\langle a, a^* \rangle$ with generator $L = c_1 x x^* + c_2 x^* x$, $c_1, c_2 \in \mathbb{R}_+$, and $(M_t)_{t \in \mathbb{R}_+}$ a classical version of $j_t(a + a^*)$ (i.e. the Azéma martingale). Then the generator L_{a+a^*} of M_t (as a classical Markov process) act as*

$$L_{a+a^*} f(z) = \begin{cases} \frac{c_2 \delta_q f(z) - q c_1 \delta_{1/q} f(z) - (c_2 - q c_1) f'(z)}{z^{(q-1)}} & q \neq 1 \\ \frac{c_1 + c_2}{2} f''(z) & q = 1 \end{cases}$$

on polynomials $f(z) = \sum_{k=0}^n f_k z^k$, $f_0, f_1, \dots, f_n \in \mathbb{C}$. The moments of this process are

$$\mathbb{E}(M_t^n) = \begin{cases} \frac{k_{2m} k_{2m-2} \dots k_2}{m!} t^m & n = 2m \text{ even,} \\ 0 & n \text{ odd,} \end{cases}$$

where $k_n = c_1 \frac{(q^{-1})_{n-1}}{q^{-1}-1} + c_2 \frac{q_{n-1}}{q-1}$ for $q \neq 1$, and $k_n = (c_1 + c_2) \frac{n(n-1)}{2}$ for $q = 1$.

4.2 Other processes on $\mathbb{C}_q\langle a, a^* \rangle$

If we want to obtain other classical processes we can either change the quantum process, i.e. choose a different generator, or use a different commutative subalgebra of $\mathbb{C}_q\langle a, a^* \rangle$ that satisfies the conditions discussed in Subsection 3.2. Let us briefly look at the second possibility. We want an element of $u \in \mathcal{B} = \mathbb{C}_q\langle a, a^* \rangle$ such that $\Delta \mathbb{C}[u] \subseteq \mathbb{C}[u] \otimes \mathbb{C}_q\langle a, a^* \rangle$. A possible choice is $u = a^* a + q a a^*$. In fact, $\Delta u^n = \sum_{\nu=0}^n \binom{n}{\nu} u^\nu (u')^{n-\nu}$, so that u actually generates a Hopf subalgebra.

But, since this Hopf subalgebra is isomorphic to the Hopf algebra of polynomials in $\mathbb{R}$, we find exactly the classical Lévy processes whose moments are finite.

Let us now look at other generators. Since no algebraic relations are imposed on a and a^* , we can take any operator X acting on some Hilbert space $\mathcal{H}$ to define a representation ρ_X of $\mathbb{C}_q\langle a, a^* \rangle$. To get a positive functional on $\mathbb{C}_q\langle a, a^* \rangle$ we now simply fix an element $h \in \mathcal{H}$ and set $\tilde{\psi}_{h,X}(u) = \langle h, \rho_X(u) h \rangle$. Then $\psi_{h,X} = \tilde{\psi}_{h,X} - \tilde{\psi}_{h,X}(1)\varepsilon$ is conditionally positive, and, if $\psi_{h,X}$ also satisfies the invariance condition, then there exists a Lévy process with generator $\psi_{h,X}$. The invariance condition in this case means simply that $\psi_{h,X}$ vanishes on ‘words’ that do not have the same number of a ’s and a^* ’s. Let $\mathcal{H} = \mathbb{C}^2$,

$$X_\alpha = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then $\psi_{i,\alpha} = \psi_{h_i, X_\alpha}$ are generators of Lévy processes on $\mathbb{C}_q\langle a, a^* \rangle$. One verifies that they can be written as

$$\begin{aligned} \psi_{1,\alpha} &= |\alpha|^2 x x^* + |\alpha|^4 x x^* x x^* + |\alpha|^6 x x^* x x^* x x^* + \dots \\ \psi_{2,\alpha} &= |\alpha|^2 x^* x + |\alpha|^4 x^* x x^* x + |\alpha|^6 x^* x x^* x x^* x + \dots, \end{aligned}$$

i.e. $\psi_{1,\alpha}$ ($\psi_{2,\alpha}$, resp.) is the sum over all concatenations of xx^* (x^*x , resp.) with itself, with coefficient $|\alpha|^l$, where l is the length of the ‘word’. Introducing

$$\begin{aligned}\phi_{1,\alpha} &= |\alpha|^2 x^* + |\alpha|^4 x^* x x^* + |\alpha|^6 x^* x x^* x x^* + \cdots = |\alpha|^2 x^* (1 + \tilde{\psi}_{1,\alpha}) & (\text{Concatenation!}) \\ \phi_{2,\alpha} &= |\alpha|^2 x + |\alpha|^4 x x^* x + |\alpha|^6 x x^* x x^* x + \cdots = |\alpha|^2 x (1 + \tilde{\psi}_{2,\alpha}) & (\text{Concatenation!}),\end{aligned}$$

we can state the Leibnitz formulas that allow to determine the action of $\psi_{1,\alpha}$ and $\psi_{2,\alpha}$ on polynomials in $z = a + a^*$.

Proposition 4.3 *Let $f = \sum_{k=0}^n f_k z^k$. We have*

$$\begin{aligned}\rho(\psi_{1,\alpha})(zf) &= \rho(\phi_{1,\alpha})f + z\rho(\psi_{1,\alpha})f \\ \rho(\phi_{1,\alpha})(zf) &= |\alpha|^2 \rho(1 + \psi_{1,\alpha})f + q^{-1} z \rho(\phi_{1,\alpha})f \\ \rho(\psi_{2,\alpha})(zf) &= \rho(\phi_{2,\alpha})f + z\rho(\psi_{2,\alpha})f \\ \rho(\phi_{2,\alpha})(zf) &= |\alpha|^2 \rho(1 + \psi_{2,\alpha})f + qz\rho(\phi_{2,\alpha})f\end{aligned}$$

Proof: Similarly as in the proof of Proposition 4.1. We only need the first two terms of $\Delta(\psi_{i,\alpha})$, since the others vanish when applied to 1 or z . ■

We set $P_n = \rho(\psi_{1,\alpha})z^n$, and combine the recurrence relations above to

$$P_{n+1}(z) = (1 + q^{-1})zP_n(z) + (|\alpha|^2 - q^{-1}z^2)P_{n-1} + |\alpha|^2 z^{n-1},$$

for $n \geq 1$, $P_1(z) = P_0(z) = 0$. $Q_n = \rho(\psi_{2,\alpha})z^n$ satisfies the same relation with q instead of q^{-1} .

We use the generating function

$$G(t, z) = \sum_{n=0} t^n Q_n(z) = \sum_{n=0} \rho(\psi_{2,\alpha})z^n t^n,$$

this is (formally) the image of $\frac{1}{1-zt}$ under $\rho(\psi_{2,\alpha})$. Using the recurrence relation we get

$$G(t, z) = (1 + q)tzG(t, z) + (|\alpha|^2 - qz^2)t^2G(t, z) + \frac{|\alpha|^2 t^2}{1 - zt},$$

and thus

$$\begin{aligned}G(t, z) &= \frac{|\alpha|^2 t^2}{(1 - (1 + q)tz - (|\alpha|^2 - qz^2)t^2)(1 - zt)} \\ &= \frac{A_1}{1 - z_1 t} + \frac{A_2}{1 - z_2 t} - \frac{1}{1 - zt}\end{aligned}$$

where

$$\begin{aligned}z_1 &= \frac{1}{2} \left((1 + q)z + \sqrt{(1 - q)^2 z^2 + 4|\alpha|^2} \right), & A_1 &= \frac{\frac{1}{2} (1 - q)z + \sqrt{(1 - q)^2 z^2 + 4|\alpha|^2}}{\sqrt{(1 - q)^2 z^2 + 4|\alpha|^2}}, \\ z_2 &= \frac{1}{2} \left((1 + q)z - \sqrt{(1 - q)^2 z^2 + 4|\alpha|^2} \right), & A_2 &= \frac{\frac{1}{2} - (1 - q)z + \sqrt{(1 - q)^2 z^2 + 4|\alpha|^2}}{\sqrt{(1 - q)^2 z^2 + 4|\alpha|^2}},\end{aligned}$$

We thus have the following result.

Theorem 4.4 *The generator $\rho(\psi_{2,\alpha})$ acts on $\mathbb{C}[z]$ as*

$$Lf(z) = A_1 f(z + \Delta_1) + A_2 f(z + \Delta_2) - f(z)$$

where

$$\begin{aligned} \Delta_1 &= \frac{1}{2} \left((q-1)z + \sqrt{(1-q)^2 z^2 + 4|\alpha|^2} \right), & A_1 &= \frac{1}{2} \frac{(1-q)z + \sqrt{(1-q)^2 z^2 + 4|\alpha|^2}}{\sqrt{(1-q)^2 z^2 + 4|\alpha|^2}}, \\ \Delta_2 &= \frac{1}{2} \left((q-1)z - \sqrt{(1-q)^2 z^2 + 4|\alpha|^2} \right), & A_2 &= \frac{1}{2} \frac{-(1-q)z + \sqrt{(1-q)^2 z^2 + 4|\alpha|^2}}{\sqrt{(1-q)^2 z^2 + 4|\alpha|^2}}. \end{aligned}$$

The action of $\rho(\psi_{1,\alpha})$ is given by the same expressions, if we replace q by q^{-1} .

Remarks:

1. For $q = 1$ this generator is simply $Lf(z) = \frac{1}{2} \{f(z+|\alpha|) + f(z-|\alpha|) - 2f(z)\}$, and the process is a difference of two independent Poisson processes with jumps of size $|\alpha|$ and same intensity, $Z_t = \hat{N}_t = N_t^{(1)} - N_t^{(2)}$. From these we can build any symmetric compound Poisson process by taking the generator $L_\mu = \int_{\mathbb{R}} \psi_{1,\alpha} d\mu(\alpha)$, and thus we have ‘q-analogues’ of symmetric compound Poisson processes at hand.
2. Another special case where the description of Z_t becomes a lot simpler is $q = -1$. In this case $Z^2 = [Z, Z]_t + 2 \int_0^t Z_s_- dZ_s = [\hat{N}, \hat{N}]_t$ is a Poisson process, i.e. Z_t takes values in $\{\pm \sqrt{n}|\alpha|; n \in \mathbb{N}\}$. Jumps are only possible from $\sqrt{n}|\alpha|$ to $\pm \sqrt{n+1}|\alpha|$, the transition probabilities are given by $A_{1,1} = \frac{1}{2} \left(1 \pm \frac{\sqrt{n}}{\sqrt{n+1}} \right)$.
3. The jumps Δ_1, Δ_2 are the solutions of the quadratic equation

$$\Delta^2 = (q-1)z\Delta + |\alpha|^2,$$

this implies that Z_t is a solution of

$$[Z, Z]_t = (q-1) \int_0^t Z_s_- dZ_s + [\hat{N}, \hat{N}]_t,$$

i.e. a structure equation where t , which is the quadratic variation of a Brownian motion, has been replaced by the quadratic variation of a symmetric Poisson process.

5 Examples of classical versions of Lévy processes on real forms of $U_q(sl(2))$

The Hopf algebra $U_q(sl(2))$ is generated by three elements e, k, k^{-1}, f with the relations [Dri87]

$$kk^{-1} = k^{-1}k = 1, \quad ke = qek, \quad fk = qkf, \quad ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

$$\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f, \\ S(k) = k^{-1}, \quad S(q) = -qe, \quad S(f) = -q^{-1}f, \quad \varepsilon(k) = 1, \quad \varepsilon(e) = \varepsilon(f) = 0,$$

where q is a complex parameter, $q \neq 0, \pm 1$. The Casimir element of $U_q(sl(2))$ is

$$C = \frac{qk^2 + q^{-1}k^{-2} - 2}{(q - q^{-1})^2} + fe,$$

it is a generator of the center of $U_q(sl(2))$.

The dual of $U_q(sl(2))$ is generated by four functionals x, y, u, v which we will write as a matrix $X = \begin{pmatrix} x & u \\ v & y \end{pmatrix} : U_q(sl(2)) \rightarrow \mathbb{C}^{2 \times 2}$, see [MMNNSU90a]. On the generators they are defined by

$$X(k) = \begin{pmatrix} x(k) & u(k) \\ v(k) & y(k) \end{pmatrix} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \\ X(e) = \begin{pmatrix} x(e) & u(e) \\ v(e) & y(e) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ X(f) = \begin{pmatrix} x(f) & u(f) \\ v(f) & y(f) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and extended to general elements by

$$X(ab) = X(a)X(b) \quad \text{for } a, b \in U_q(sl(2)).$$

This implies that the Leibniz formula for the right dual representation reads

$$\rho(X)(ab) = (\rho(X)a)(\rho(X)b) \quad \text{for } a, b \in U_q(sl(2))$$

in the same matrix notation. Therefore, to calculate the right dual action of x, y, u, v on polynomials in some element $z \in U_q(sl(2))$ it is sufficient to calculate $\rho(X)z$, from this follows immediately

$$\rho(X)p(z) = p(\rho(X)z), \quad \text{for } p(z) \in \mathbb{C}[z] \subset U_q(sl(2)).$$

There exist three inequivalent *-structure on $U_q(sl(2))$.

Proposition 5.1 [MMNNSU90a, Proposition 3] *Real forms of $U_q(sl(2))$ are classified as follows:*

- $U_q(su(2))$ with $q \in (-1, 0) \cup (0, 1)$ (the compact real form): the involution is defined by

$$k^* = k, \quad e^* = f, \quad f^* = e.$$

- $U_q(su(1, 1))$ with $q \in (-1, 0) \cup (0, 1)$ (a non-compact real form): the involution is defined by

$$k^* = k, \quad e^* = -f, \quad f^* = -e.$$

- $U_q(sl(2, \mathbb{R}))$ with $|q| = 1$ (a non-compact real form): the involution is defined by

$$k^* = k, \quad e^* = -e, \quad f^* = -f.$$

5.1 The compact real form $U_q(su(2))$

Let us first consider the compact real form $U_q(su(2))$. The element $z = ek + kf$ is self-adjoint (i.e. $z^* = z$), and satisfies Schürmann's condition (see Theorem 2.3), since $\Delta z = z \otimes k^2 + 1 \otimes z$.

For the dual representation one obtains

$$\rho(X)z = \begin{pmatrix} qz & q^{-1/2} \\ q^{-1/2} & q^{-1}z \end{pmatrix}$$

and therefore

$$\rho(X)p(z) = p \begin{pmatrix} qz & q^{-1/2} \\ q^{-1/2} & q^{-1}z \end{pmatrix}$$

for all polynomials $p(z) \in \mathbb{C}[z]$. Since the functionals x, y, u, v generate the dual of $U_q(su(2))$ this implies that $\mathbb{C}[z] \subseteq U_q(su(2))$ is a right coideal and that $(j_t|_{\mathbb{C}[z]})$ is a quantum Markov process for any Lévy process (j_t) on $U_q(su(2))$.

Using $\frac{1}{1-\lambda z}$ as a generating function, we can get a more useful expression for $\rho(X)|_{\mathbb{C}[z]}$.

$$\begin{aligned} \rho(X)\frac{1}{1-\lambda z} &= \left\{ 1 - \lambda \begin{pmatrix} qz & q^{-1/2} \\ q^{-1/2} & q^{-1}z \end{pmatrix} \right\}^{-1} \\ &= \frac{1}{1 - (q + q^{-1})\lambda z + \lambda^2(z^2 - q^{-1})} \begin{pmatrix} 1 - q^{-1}\lambda z & q^{-1/2}\lambda \\ q^{-1/2}\lambda & 1 - q\lambda z \end{pmatrix} \\ &= \frac{1}{1 - \lambda z_1} \begin{pmatrix} \frac{(q^2-1)z + \sqrt{Q}}{2\sqrt{Q}} & \frac{\sqrt{q}}{\sqrt{Q}} \\ \frac{\sqrt{q}}{\sqrt{Q}} & \frac{(q^2-1)z + \sqrt{Q}}{2\sqrt{Q}} \end{pmatrix} \\ &\quad + \frac{1}{1 - \lambda z_2} \begin{pmatrix} \frac{-(q^2-1)z + \sqrt{Q}}{2\sqrt{Q}} & -\frac{\sqrt{q}}{\sqrt{Q}} \\ -\frac{\sqrt{q}}{\sqrt{Q}} & \frac{-(q^2-1)z + \sqrt{Q}}{2\sqrt{Q}} \end{pmatrix}, \end{aligned}$$

where $Q = (q^2 - 1)^2 z^2 + 4q$, and

$$z_1 = \frac{(q^2 + 1)z + \sqrt{Q}}{2q}, \quad z_2 = \frac{(q^2 + 1)z - \sqrt{Q}}{2q}.$$

For example for $q > 0$ and $L = y - \varepsilon$, we get again a jump process that tends to a symmetric Poisson process in the classical limit $q \rightarrow 1$, as in Theorem 4.4.

Another element that satisfies Schürmann's condition is the Casimir element since $C \otimes 1$ is in the center of $U_q(su(2)) \otimes U_q(su(2))$. Therefore $(j_t(C))_{t \in \mathbb{R}_+}$ is commutative and has a classical version for any Lévy process $(j_t)_{t \in \mathbb{R}_+}$ on $U_q(su(2))$. But $\mathbb{C}[C]$ is not a right coideal and $(j_t|_{\mathbb{C}[C]})$ will in general not be a quantum Markov process, as we can see from

$$\rho(X)C = \begin{pmatrix} qC + \frac{(q^{-2}-1)k^{-2}+2(q-1)}{(q-q^{-1})^2} & q^{-1/2}k^{-1}f \\ q^{-1/2}ek^{-1} & q^{-1}C + \frac{(q^2-1)k^{-2}+2(q^{-1}-1)}{(q-q^{-1})^2} \end{pmatrix},$$

since $\rho(X)C \not\subseteq M_2(\mathbb{C}[C])$.

But if we take, e.g., $L = y - \varepsilon$, then the restriction of the corresponding process to the commutative *-subalgebra $\mathbb{C}[C, k]$ is a quantum Markov process, and has thus a classical version. The action of L on $\mathbb{C}[C, k]$ is determined by

$$\rho(y) \frac{p(k)}{1 - \lambda C} = \frac{p(q^{-1/2}k) \left\{ 1 - \lambda \left(qC + \frac{(1-q^2)k^{-2} + 2(q-1)}{(q-q^{-1})^2} \right) \right\}}{1 - \lambda \left((q + q^{-1})C + 2 \frac{(q^{1/2} - q^{-1/2})^2}{(q-q^{-1})^2} \right) + \lambda^2 \left(C - \frac{(q^{1/2} - q^{-1/2})^2}{(q-q^{-1})^2} \right)^2}$$

for $p(k) \in \mathbb{C}[k]$. But the process $(j_t|_{\mathbb{C}[C,k]})_{t \in \mathbb{R}_+}$ is not commutative, since $[k \otimes 1, \Delta(C)] = (q^{-1} - 1)f \otimes ke + (1 - q^{-1})e \otimes fk \neq 0$ for $q \neq 1$, and

$$\begin{aligned} \Phi(j_s(f)[j_s(k), j_t(C)]j_{ts}(e)) &= \varphi_s \otimes \varphi_{t-s} \left((q^{-1} - 1)f^2 \otimes ke^2 + (1 - q^{-1})fe \otimes kfe \right) \\ &= (1 - q^{-1})\varepsilon^{\otimes 2} \left(e^{s\rho(y-\varepsilon)}(fe) \otimes e^{(t-s)\rho(y-\varepsilon)}(kfe) \right) \\ &= \frac{\left(e^{(q^{-2}-1)s} - e^{(q^{-1}-1)s} \right) \left(e^{(q^{-5/2}-1)(t-s)} - e^{(q^{-3/2}-1)(t-s)} \right)}{1 - q^{-4}} \\ &\neq 0 \quad (\text{for } q \neq 1). \end{aligned}$$

In the limit $q \rightarrow 1$, if we set $k = q^H$, we get,

$$\begin{aligned} \rho(L)p(C, H) &= \frac{\sqrt{C} - (H - \frac{1}{2})}{2\sqrt{C}} p\left(C + \sqrt{C} + \frac{1}{4}, H - \frac{1}{2}\right) \\ &\quad + \frac{\sqrt{C} + (H - \frac{1}{2})}{2\sqrt{C}} p\left(C - \sqrt{C} + \frac{1}{4}, H - \frac{1}{2}\right) - p(C, H) \end{aligned}$$

for $p(C, H) \in \mathbb{C}[C, H] \subseteq U(su(2))$. But $j_0 = e \circ \varepsilon$ implies $(\hat{C}_0, \hat{H}_0) = (1/4, 0)$ for a classical version of $(j_t(C), j_t(H))$, and we get the following result.

Proposition 5.2 *Let $(j_t)_{t \in \mathbb{R}_+}$ be the Lévy process on $U(su(2))$ with generator $L = y - \varepsilon$, and N_t a standard Poisson process. Then $((N_t + 1)^2/4, -N_t/2)_{t \in \mathbb{R}_+}$ is a classical version of $(j_t(C), j_t(H))_{t \in \mathbb{R}_+}$.*

5.2 The real form $U_q(su(1, 1))$

The element z is not self-adjoint for the *-structure of $U_q(su(1, 1))$, but we can take, e.g., $\bar{z} = ek - kf$ instead. We get

$$\rho(X)\bar{z} = \begin{pmatrix} q\bar{z} & q^{-1/2} \\ -q^{-1/2} & q^{-1}\bar{z} \end{pmatrix},$$

and therefore $\mathbb{C}[\bar{z}]$ is also right coideal.

To get conditionally positive functionals on $U_q(su(1, 1))$ and to express them in terms of x, y, u, v the formulas for the matrix elements of the irreducible unitary

representation of $U_q(su(1,1))$ given in [MMNNSU90b] can be used. Take for example $w_{00}^{(l)}$ with $0 < l < \frac{1}{2}$, as a diagonal element of a unitary representation (see [MMNNSU90b, Proposition 4]), it is a positive functional. Furthermore we can see that it converges weakly to ε as l goes to 0. Therefore $L = \lim_{l \searrow 0} \frac{w_{00}^{(l)} - \varepsilon}{l}$ is a conditionally positive functional. From [MMNNSU90b, Proposition 3] we get the following expression for L ,

$$L = \sum_{n=1}^{\infty} \frac{2h(uv)^n}{q^n - q^{-n}},$$

where $h = \ln(q)$. To calculate $\rho(uv)$ we introduce

$$X^{(2)} = \begin{pmatrix} x^2 & xu & ux & u^2 \\ xv & xy & uv & uy \\ vx & vu & yx & yu \\ v^2 & vy & yv & y^2 \end{pmatrix},$$

on $\bar{z}$ we get

$$\rho(X^{(2)})\bar{z} = \begin{pmatrix} q^2\bar{z} & q^{1/2} & q^{-1/2} & 0 \\ q^{1/2} & \bar{z} & 0 & q^{-1/2} \\ q^{-1/2} & 0 & \bar{z} & q^{-3/2} \\ 0 & q^{-1/2} & q^{-3/2} & q^{-2}\bar{z} \end{pmatrix},$$

this allows to calculate $\rho(X^{(2)})p(\bar{z}) = p(\rho(X^{(2)})\bar{z})$ for arbitrary polynomials $p(\bar{z}) \in \mathbb{C}[\bar{z}]$. $\rho(uv)$ is characterised by

$$\begin{aligned} \rho(uv) \frac{1}{1 - \lambda\bar{z}} &= \frac{(1 + q^{-2})\lambda^2}{(1 - \lambda\bar{z})(1 - (q^2 + q^{-2})\lambda\bar{z} + \lambda^2(\bar{z}^2 - q^{-1}(q + q^{-1})^2))} \\ &= \frac{q(q^2 + 1)}{\bar{z}^2 q(q^2 - 1)^2 + (q^2 + 1)^2} \left\{ \frac{-(q^2 - 1)^2\bar{z} + \sqrt{S}}{2\sqrt{S}(1 - \lambda\bar{z}_1)} + \frac{(q^2 - 1)^2\bar{z} + \sqrt{S}}{2\sqrt{S}(1 - \lambda\bar{z}_2)} - \frac{1}{1 - \lambda\bar{z}} \right\} \end{aligned}$$

where $S = (q^4 - 1)^2\bar{z}^2 + 4q(q^2 + 1)^2$ and

$$\bar{z}_1 = \frac{(q^4 + 1)\bar{z} + \sqrt{S}}{2q^2}, \quad \bar{z}_2 = \frac{(q^4 + 1)\bar{z} - \sqrt{S}}{2q^2}.$$

We see that $\rho(uv)$ is a difference operator where the differences depend on z and q . The action of L on polynomials can be written in the form

$$Lf(\bar{z}) = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} A_{nk} f(\bar{z}_{nk})$$

where the z_{nk} can be obtained by iterating $\bar{z}_1$ and $\bar{z}_2$, i.e. set $\bar{z}_{01} = \bar{z}$, $\bar{z}_{11} = \bar{z}_1(\bar{z}) = \bar{z}_1 = \frac{(q^4+1)\bar{z}+\sqrt{S}}{2q^2}$, $\bar{z}_{12} = \bar{z}_2(\bar{z}) = \frac{(q^4+1)\bar{z}-\sqrt{S}}{2q^2}$, $\bar{z}_{21} = \bar{z}_1(\bar{z}_{11})$, $\bar{z}_{22} = \bar{z}_2(\bar{z}_{11})$,

$\bar{z}_{23} = \bar{z}_1(\bar{z}_{12})$, $\bar{z}_{24} = \bar{z}_2(\bar{z}_{12})$, and so forth. A classical version is given by the process (starting from 0) that jumps after an exponential time given by $-A_{01} = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} A_{nk}$ to one of the $\bar{z}_{nk}$ (with probability $-A_{nk}/A_{01}$).

For $q \rightarrow 1$ we get $\rho(uv)p(\bar{z}) = \frac{1}{4} \{p(\bar{z} + 2) + p(\bar{z} - 2) - 2p(\bar{z})\}$, and, using $e^{\lambda\bar{z}}$ as a generating function, we get for $\rho(L)$,

$$\begin{aligned} \rho(L)e^{\lambda\bar{z}} &= \sum_{n=1}^{\infty} \frac{(e^{2\lambda} + e^{-2\lambda} - 2)^n}{4^n n} e^{\lambda\bar{z}} = e^{\lambda\bar{z}} \sum_{n=1}^{\infty} \sum_{\nu=0}^{2n} \frac{(-1)^\nu}{4^n n} \binom{2n}{\nu} e^{2\lambda(n-\nu)} \\ &= -\ln(1 - \sinh^2 \lambda) e^{\lambda\bar{z}} = -\ln \left(\frac{3 - \cosh(2\lambda)}{2} \right) e^{\lambda\bar{z}}, \end{aligned}$$

from which we get

$$\rho(L)p(x) = \sum_{k=1}^{\infty} \frac{p(x+2k) + p(x-2k)}{(3+2\sqrt{2})^k k} - \log \left(\frac{3+2\sqrt{2}}{4} \right) p(x),$$

i.e. the process

$$\sum_{k \in \mathbb{Z}, k \neq 0} 2k N_{t/[k(3+2\sqrt{2})^k]}^{(k)}$$

is a classical version of $j_t(\bar{z})$, if the $N_t^{(k)}$, $k \in \mathbb{Z} \setminus \{0\}$, are mutually independent Poisson processes. The marginal law of $(j_t(\bar{z}))_{t \in \mathbb{R}_+}$ (for $q = 1$) is uniquely determined by its generating function

$$\Phi(e^{\lambda j_t(\bar{z})}) = \varepsilon \circ e^{t\rho(L)}(e^{\lambda\bar{z}}) = \left(\frac{2}{3 - \cosh(2\lambda)} \right)^t.$$

6 Lévy processes on $U_q(aff(1))$

We will now consider an example where the parameter q is not real, but of modulus 1, and not a root of unity. Instead of $U_q(sl(2, \mathbb{R}))$ we take a simpler Hopf algebra with only two generators that can also be obtained as a sub-Hopf algebra of $U_q(sl(2, \mathbb{R}))$, namely $U_q(aff(1))$. This Hopf $*$ -algebra is generated by two elements X, Y with the relations

$$\begin{aligned} XY - YX &= Y, & X^* &= -X, & Y^* &= -Y \\ \Delta(X) &= X \otimes 1 + 1 \otimes X, & \Delta(Y) &= Y \otimes q^X + 1 \otimes Y \\ \varepsilon(X) &= \varepsilon(Y) = 0, & S(X) &= -X, & S(Y) &= Yq^{-X}, \end{aligned}$$

with $q \in \mathbb{C}$, $|q| = 1$. Let $h \in \mathbb{R}$ be such that $q = e^{ih}$. Positive functionals can be obtained from the trivial one-dimensional representation $X \mapsto ix$, $Y \mapsto 0$, for $x \in \mathbb{R}$, i.e. on the elements $\varphi_x : Y^m X^n \mapsto \delta_{m,0}(ix)^n$. Differentiating φ_x with respect to x at $x = 0$ we get the two conditionally positive functionals

$\varphi^{(1)} : Y^m X^n \mapsto i\delta_{m,0}\delta_{n,1}$ and $\varphi^{(1)} : Y^m X^n \mapsto -\delta_{m,0}\delta_{n,2}$. But these functionals do not give any interesting processes because we always get $j_t(Y) = 0$.

Let X_1 and X_2 be the operators on $l^2(\mathbb{N})$ defined by $(X_1 u)_n = nu_n + \sqrt{n(n+1)}u_{n+1}$, and $(X_2 u)_n = nu_n + \sqrt{n(n-1)}u_{n-1}$. They are closed on their maximal domains and satisfy $[X_1, X_2] = X_1 + X_2$ and $X_1^* = X_2$, see [Par91].

If we set $X = \frac{1}{2}(X_1 - X_2)$ and $Y = i(X_1 + X_2)$, then this defines a $*$ -representation of $U_q(\mathfrak{aff}(1))$. Let us compute the ‘vacuum expectation’ of $Y^n X^m$ in the state $\Omega = e_1 = (1, 0, \dots)$.

Lemma 6.1 *For α, β sufficiently small we have*

$$e^{\beta Y} e^{\alpha X} = e^{bX_2} e^{aX_1}$$

with

$$a = -\log\left(\frac{e^\alpha + 1}{2} - i\beta\right) + \alpha, \quad b = -\log\left(\frac{e^\alpha + 1}{2} - i\beta\right)$$

Proof: $\{X_1, X_2\}$ and $\{X, Y\}$ both form a basis of the Lie algebra $\mathfrak{aff}(1)$, so that the preceding formula is just a coordinate change for a neighborhood of the unit element in the Lie group $\mathfrak{Aff}(1)$. $\blacksquare$

With this lemma we obtain

$$\begin{aligned} \langle e^{\beta Y} e^{\alpha X} \rangle &= \langle e_1, e^{bX_2} e^{aX_1} e_1 \rangle = \langle e^{\bar{b}X_2} e_1, e^{aX_1} e_1 \rangle \\ &= e^{a+b}, \end{aligned}$$

from which we can deduce

$$\langle Y^m e^{\alpha X} \rangle = (m+1)! \frac{(i)^m}{\cosh^{m+2}(\alpha/2)}, \quad (3)$$

which makes sense for $\alpha \in \mathbb{C}$, $\alpha \neq (2k+1)2i\pi$ for $k \in \mathbb{Z}$.

We can now define a conditionally positive functional on $U_q(\mathfrak{aff}(1))$ by

$$L(u) = \langle u \rangle - \varepsilon(u),$$

if q is not a root of unity.

The restrictions of this process to the subalgebras generated by X or Y turns out to be commutative and Markovian, so that there exist classical versions of $j_t(iX)$ and $j_t(iY)$, respectively.

Proposition 6.2 *Let $(\hat{X})_{t \geq 0}$ and $(\hat{X})_{t \geq 0}$ be classical versions of $(j_t(iX))_{t \geq 0}$ and $(j_t(iY))_{t \geq 0}$, respectively. Then the action of the generators L_X and L_Y of $(\hat{X})_{t \geq 0}$ and $(\hat{X})_{t \geq 0}$ on polynomials is determined by*

$$\begin{aligned} L_X e^{iux} &= -\tanh^2(u/2) e^{iux}, \\ L_Y y^n &= \sum_{\nu=0}^n \frac{(-1)^{n-\nu} (n-\nu+1)! \begin{bmatrix} n \\ \nu \end{bmatrix}_q}{\cos^{n-\nu+2}\left(\frac{\nu h}{2}\right)} y^\nu - y^n, \end{aligned}$$

($u \in \mathbb{R}$, $n \in \mathbb{N}$).

Proof: This follows directly from

$$\Delta Y^m X^n = \sum_{\mu=0}^m \sum_{\nu=0}^n \begin{bmatrix} m \\ \mu \end{bmatrix}_q \begin{pmatrix} n \\ \nu \end{pmatrix} Y^\mu X^\nu \otimes Y^{m-\mu} X^{n-\nu} q^{\nu X},$$

and Equation (3). ■

Remark: In the limit $\hbar \rightarrow 0$ we obtain for L_Y ,

$$L_Y(e^{iuy}) = \left\{ \frac{1}{(1 - iu)^2} - 1 \right\} e^{iuy}.$$

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