CERN-TH/98-388

KUL-TF-98/51

December 1998

On short and long $SU(2, 2/4)$ multiplets in the AdS/CFT correspondence ¹

Laura Andrianopoli

*Institute for Theoretical Physics, KULeuven, Celestijnenlaan 200 D,
B-3001 Leuven, Belgium*

Sergio Ferrara

CERN Theoretical Division, CH 1211 Geneva 23, Switzerland.

Abstract

We analyze short and long multiplets which appear in the OPE expansion of “chiral” primary operators in $N = 4$ Super Yang–Mills theory. Among them, higher spin long and new short multiplets appear, having the interpretation, in the AdS/CFT correspondence, of string states and supergravity multiparticle states respectively.

We also analyze the decomposition of long multiplets under $N = 1$ supersymmetry, as a possible tool to explore other supersymmetric deformations of IIB string on $AdS_5 \times S_5$.

¹Work supported in part by EEC under TMR contract ERBFMRX-CT96-0045 (LNF Frascati and K.U.Leuven) and by DOE grant DE-FG03-91ER40662

1 Introduction

The recent advances in putting forward the correspondence between superstring theories on AdS_{d+1} backgrounds and d -dimensional superconformal field theories [1] deeply rely on the supergravity approximation of string theory which, on the field theory side, roughly corresponds to retain a certain subclass of conformal operators which have “finite” conformal dimension in the limit of large t’Hooft parameter g^2N (for the case of 3-branes with AdS_5 geometry).

On the other hand, if one would like to explore further this connection at finite N , then stringy corrections to the supergravity approximation must be taken into account, as for example the inclusion of R^4 terms or D-instanton mediated processes [2].

On the side of the superconformal sector, stringy corrections may correspond to the appearance, in the OPE algebra, of conformal primary operators with g^2N dependent, i.e. not quantized, dimension [1, 3].

It is in fact a known fact that these operators do indeed occur, in perturbative supersymmetric Yang–Mills theory [4], in the OPE of the stress tensor multiplet, superconformal symmetry then requiring the extension of this analysis to the entire class of operators related by superconformal transformations [5].

Another context in which such multiplets play a role is the analysis of deformation of a given superconformal field theory [6, 7] or, on the supergravity side, choosing the vacuum of the corresponding theory on AdS_5 [8, 9, 10].

In this analysis, the notion of conformal dimension of a given operator is a property of the supergravity vacuum, since in the AdS/CFT correspondence the conformal dimension is mapped into the AdS energy E_0 , the latter depending on the particular extremum of the AdS supergravity potential.

However in the conformal field theory framework it is possible to explore vacuum solutions beyond the AdS_5 supergravity analysis because it is possible to add conformal deformations corresponding to massive K–K states or even string states which are not present in the supergravity potential which only involves “massless scalars”, i.e. those scalars which belong to the $n = 8$ “massless” supergravity multiplet in the AdS bulk.

It is therefore relevant to classify sequences of multiplets and their corresponding scalar content in the wider context of $n = 4$ superconformal field theory, which allows one to consider general classes of operators other than the K–K tower. In this context $n = 4$ superconformal symmetry plays an

important role because it allows to separate short and long multiplets in a rather simple way.

The former correspond to the K–K excitations of type II string theory on $AdS_5 \times S_5$, the latter should correspond to string states, since they cannot have a supergravity interpretation. It should be emphasised that such separation makes sense for $n = 8$ supergravity but not for $n = 2$. Indeed in the latter case it is possible to have K–K states with anomalous dimensions, i.e. long multiplets in the pure supergravity context [11]. This is for instance what generally happens for the K–K recurrences of theories on backgrounds of the type $AdS_5 \times X$ where X is a manifold preserving $n = 2$ supersymmetry [7]. In these cases the only necessarily short multiplets are the hypermultiplets but not the graviton, gravitino or vectors K–K recurrences.

2 A class of long supermultiplets and their scalar content

The classification of UIR reps. of highest weights can be done, in certain cases, with the oscillator method developed by Günaydin et al. [12].

This construction is especially powerful to classify the so-called “chiral primary” $n = 4$ operators in the AdS_5/CFT_4 correspondence.

The interpretation of the K–K spectrum of type IIB on $AdS_5 \times S_5$ in terms of UIR of the $SU(2, 2/4)$ superalgebra with the oscillator construction was obtained by Günaydin and Marcus [13]. The correspondence of these “short multiplets” to a particular class of superfield operators of the $n = 4$ superconformal field theory, introduced by Howe and West [14], was elucidated in ref. [15]. These short superfields correspond to massless ($p = 2$) and massive ($p > 2$) reps. of the $SU(2, 2/4)$ superalgebra with a lowest component scalar in the $(0, p, 0)$ rep. of $SU(4)$ and with a total number $\frac{1}{12}p^2(p^2 - 1)2^8$ of states, with highest spin 2 in the $(0, p - 2, 0)$ $SU(4)$ rep.

The shortening corresponds to the fact that only half of the 16 θ ’s in the superfield expansion produces states. These short multiplets only exist for “quantized” dimensions i.e. for lowest component scalars with energy $E_0 = p$.

On the other hand long multiplets, in which all θ ’s components produce states although one can obtain them by multiplying short multiplets, also exist for values of the energy which are not quantized, but rather an arbitrary

real number, only subject to a certain unitarity bound ($E_0 \geq 2 + J_1 + J_2$ for the lowest component of the class of long supermultiplets we are going to consider) [16, 17].

These multiplets have multiplicity proportional to 2^{16} , the proportionality factor being related to a finite dimensional representation of the maximal compact subgroup $SO(4) \times SO(2) \times SU(4)$ of the bosonic subalgebra $SO(4, 2) \times SU(4)$ in the $SU(2, 2/4)$ superalgebra.

The simplest of these long multiplets, discussed in ref. [3], is the real scalar superfield, with maximum spin 4 in a $SU(4)$ singlet.

This multiplet is contained in the tensor product of “two” singleton supermultiplets as the first component is a scalar state

$$s = \text{Tr}(\phi_\ell \phi^\ell) \quad (1)$$

where ϕ_ℓ is the $\theta = 0$ component of the Yang–Mills “singleton” superfield [18].

This is the $n = 4$ version of the so-called Konishi-multiplet.

In the free field theory limit, i.e. when the singleton superfield is abelian, this multiplet becomes short and in fact can be obtained as the product of two conjugate singleton reps. with maximum spin $(2, 0)$ and $(0, 2)$ respectively.

These are the singleton supermultiplets described in ref. [19], with lowest spin $(J_L - 1, 0), (0, J_R - 1)$, each with multiplicity $(2J_L + 1)2^4$. Multiplying two conjugated singletons $(J_L - 1, 0) \times (0, J_R - 1)$ one obtains the massless reps. with minimum spin $(J_L - 1, J_R - 1)$ and maximum spin $(J_L + 1, J_R + 1)$ and energy, for the lowest component, $E_0 = J_L + J_R$. The number of states is $2^8(2(J_L + J_R) + 1)$. All (J_1, J_2) representations with $J_1 \cdot J_2 \neq 0$ inside these multiplets correspond to “conserved operators” on the boundary [3].

The free field Konishi-multiplet corresponds to $J_L = J_R = 1$ and indeed contains a singlet scalar only, with $E_0 = J_L + J_R = 2$.

Note that all massless higher spin supermultiplets of interest, given in the table 12 of ref. [19], are obtained by taking $J_L = J_R > 1$, and none of them contains scalar fields.

Let us now consider long multiplets. These are obtained by multiplying two singleton interacting multiplets, i.e. two singleton non-abelian Yang–Mills multiplets.

In this case one gets 2^{16} states for the “massive” Konishi-multiplet. All (J_L, J_R) reps. with $J_L \cdot J_R \neq 0$ occurring in this multiplet are no longer conserved, as it was in the free field case. The analysis of this multiplet in component notation was given in ref. [3].

It is useful to report here the components with their $SU(4)$ assignment and the AdS energy, i.e. the conformal weight. The analysis for each field in the supermultiplet is described in the tables 1-15. It is immediate to see, from table 1, that this multiplet contains many scalars with different $SU(4)$ and conformal weight assignment.

The important point is that the spectrum of this multiplet is independent from the value of E_0 of its lowest component, which can then be lifted to any value $E_0 \geq 2$. We see that there are new scalar states with $E_0 + 1, E_0 + 2, \dots, E_0 + 8$. All these scalars are analogous of F and D terms which would otherwise vanish in the free field theory.

Let us now consider higher spin superfields of the type appearing in the OPE of the stress tensor multiplet in $n = 4$ SYM theory [5]. These superfields are believed to be superfields whose highest spin component is a $(J+1, J+1)$ $SU(4)$ singlet (non conserved) operator whose dimension, in the free field theory limit, would be $E_0 = 2(2 + J)$.

It is obvious that this is the massive generalization of the massless supermultiplet obtained by multiplying two conjugate singletons with lowest spin component $(J-1, 0)$, $(0, J-1)$ and integral $J > 1$. These multiplets are simply obtained by tensoring the scalar supermultiplet (the massive spin 4 superfield) with a rep. $(J-1, J-1)$ of $SL(2, C)$.

The even spin $J > 4$ are then obtained by tensoring with $(1, 1)$, $(2, 2)$ etc.

Since the highest spin in the scalar superfield is $(2, 2)$, one will obtain scalars only up to a superfield which transforms as a $(2, 2)$ of $SL(2, C)$.

The scalars of the spin 6 and spin 8 superfields are all “irrelevant” from a conformal point of view in the sense that their naive dimension is $\ell \geq 6$. They all vanish in the free field theory limit, where these multiplets become “massless”.

The superfield which contains relevant and marginal operators is the Konishi-multiplet, other than the stress tensor multiplet and the $p = 3, 4$ massive short multiplets. There are other superfields that contain scalars with naive dimension corresponding to relevant or marginal deformations. These are the long multiplets contained in the lowest reps. of the symmetric tensor product of p singletons with $p \leq 4$. For $p = 2$ this is exactly the Konishi superfield since

$$(6 \times 6)_S = 20_R + 1 \quad (2)$$

For $p = 3$ we have:

$$(6 \times 6 \times 6)_S = 50 + 6 \quad (3)$$

so there is a long multiplet whose lowest component has naive dimension $E_0 = 3$ in the 6 of $SU(4)$. It has a scalar partner with $E_0 = 4$ in the $15 + 15 + 45 + \overline{45}$.

For $p = 4$ we have

$$(6 \times 6 \times 6 \times 6)_S = 105 + 20_R + 1 \quad (4)$$

so we have two long multiplets with lowest component $E_0 = 4$ in the 20_R and 1 of $SU(4)$.

All these multiplets, having as lowest component a scalar field, have the same structure of the Konishi-multiplet (max spin 4) where all states have $SU(4)$ reps. tensored with the rep. of the lowest component, i.e. the 6 for $p = 3$ and the $20 + 1$ for $p = 4$.

Note that the long multiplets of the type of Konishi (max spin 4) but maked up with more than two ϕ_ℓ 's in the free field theory limit do not correspond to massless higher spin fields, but rather to massive ones. This is because $E_0 = 2 + J_L + J_R$ is not satisfied for these multiplets [16, 17]. This of course also implies that these multiplets will have the same structure irrespectively wheter the theory is abelian or not. From the AdS point of view of UIR reps. of $O(4, 2)$, this has to do with the fact that the product of more than two singletons gives rise to massve representations.

In the $SU(N)$ Yang–Mills theory these multiplets with higher power of ϕ 's are also distinguished by simple traces or multiple traces in the YM gauge group. We will return to this in section 4.

2.1 Spectrum of scalars in $J_{max} = 4, 6, 8$ multiplets

These are the multiplets which contain, as maximum spin, the spin 4, 6, 8 $SU(4)$ singlets. The scalar spectrum is ²:

- $J_{max} = 4$ (ℓ_4 denotes the dimension of its lowest weight component):

$SU(4)$	$E_0, (\ell_4^0 = 2)$
1	$\ell_4, \ell_4 + 2, 3(\ell_4 + 4), \ell_4 + 6, \ell_4 + 8$
10	$\ell_4 + 1, 2(\ell_4 + 3), 2(\ell_4 + 5), \ell_4 + 7$
$\overline{10}$	$\ell_4 + 1, 2(\ell_4 + 3), 2(\ell_4 + 5), \ell_4 + 7$
20_R	$2(\ell_4 + 2), 3(\ell_4 + 4), 2(\ell_4 + 6)$
15	$\ell_4 + 2, \ell_4 + 4, \ell_4 + 6$
84	$\ell_4 + 2, \ell_4 + 4, \ell_4 + 6$

²We denote by ℓ^0 the “naive” dimension of these operators, since in general they have anomalous dimension

- $J_{max} = 6$:

$SU(4)$	$E_0, (\ell_6^0 = 4)$
1	$\ell_6 + 2, 3(\ell_6 + 4), \ell_6 + 6$
15	$\ell_6 + 2, 4(\ell_6 + 4), \ell_6 + 6$
20_R	$\ell_6 + 2, 3(\ell_6 + 4), \ell_6 + 6$
6	$2(\ell_6 + 3), 2(\ell_6 + 5)$
10	$2(\ell_6 + 3), 2(\ell_6 + 5)$
$\overline{10}$	$2(\ell_6 + 3), 2(\ell_6 + 5)$
64	$2(\ell_6 + 3), 2(\ell_6 + 5)$
45	$\ell_6 + 4$
$\overline{45}$	$\ell_6 + 4$
84	$\ell_6 + 4$

- $J_{max} = 8$:

$SU(4)$	$E_0, (\ell_8^0 = 6)$
1	$\ell_8 + 4$

The only multiplet containing scalars with $\ell^0 \leq 4$ is the Konishi multiplet.

The total spectrum of scalar reps. is from $E_0 = 2$ to $E_0 = 10$.

For $J_{max} > 8$ no scalars exist.

Short multiplets

	$SU(4)$
Super-Konishi deformations:	
$E_0 = 2$	1
$E_0 = 3$	10 ($\overline{10}$)
$E_0 = 4$	1, $2 \times 20_R$, 15, 84
$p = 2$ Supercurrent deformations:	
$E_0 = 2$	20_R
$E_0 = 3$	10 ($\overline{10}$)
$E_0 = 4$	$1 + \overline{1}$
$p = 3$ Supercurrent deformations:	
$E_0 = 3$	50
$E_0 = 4$	45 ($\overline{45}$)
$p = 4$ Supercurrent deformations:	
$E_0 = 4$	105

3 $n = 1$ multiplet analysis for scalar operators with naive dimension $\ell = 3, 4$

The analysis of scalar operators with $E_0 \leq 4$ ³ in $n = 4$ Yang–Mills theory can be obtained in a rather straightforward way by knowing the relevant, massless and massive reps. of the $SU(2, 2/4)$ algebra.

We first remind the result in the analysis of the short K–K multiplets: These multiplets are classified by a quantum number p , such that the lowest component scalar has $E_0 = p$ and is in the $(0, p, 0)$ of $SU(4)$.

The analysis therefore includes the 3 multiplets with $p = 2, 3, 4$. The first multiplet is the supergravity multiplet and therefore its 42 scalars are:

$$42 = 20_R (E_0 = 2), 10 + \overline{10} (E_0 = 3), 1 + \overline{1} (E_0 = 4) \quad (5)$$

These are the scalars which appear in the gauged supergravity potential. It is understood, according to the previous analysis, that the values of E_0 in the above formula refer to the $SU(4) \times AdS_5$ invariant vacuum in which $\langle 20_R \rangle = \langle 10 \rangle = 0$.

There are two extra scalar operators in the $p = 3$ sector:

$$50 (E_0 = 3), 45 (E_0 = 4) \quad (6)$$

and finally $105 (E_0 = 4)$ in the $p = 4$ sector.

Note that all scalars, except the $SU(4)$ singlet, necessarily break $n = 4$ supersymmetry.

The scalars which preserve $n = 1$ supersymmetry can be found by decomposing $SU(4) \rightarrow SU(3) \times U(1)$ and looking at scalars which are the highest component of $n = 1$ superfields, according to the classification of ref. [3]. For the relevant representations we have:

$$\begin{aligned} 20_R &\rightarrow 6\left(\frac{4}{3}\right) + \overline{6}\left(-\frac{4}{3}\right) + 8(0) \\ 10 &\rightarrow 1(2) + 3\left(\frac{2}{3}\right) + 6\left(-\frac{2}{3}\right) \\ 50 &\rightarrow 15\left(\frac{2}{3}\right) + \overline{15}\left(-\frac{2}{3}\right) + 10(2) + \overline{10}(-2) \\ 45 &\rightarrow 3\left(\frac{8}{3}\right) + \overline{3}\left(\frac{4}{3}\right) + 6\left(\frac{4}{3}\right) + 8(0) + 10(0) + 15\left(-\frac{4}{3}\right) \end{aligned} \quad (7)$$

³We consider here the naive canonical dimension

The analysis of these operators was performed in ref. [3]. They contain, in particular, the chiral superfields corresponding to a symmetric superpotential $10(2)$, a singlet superpotential $1(2)$, a supersymmetric mass term $6(\frac{4}{3})$.

Let us now consider the long multiplets. The only interesting multiplets are the ones up to spin 4, since higher spin supermultiplets have scalars with too high dimension.

We consider the basic Konishi superfield with $\ell^0 = 2, 3, 4$, corresponding to the traces in the $Tr(\phi_{\ell_1} \times \phi_{\ell_2} \cdots)$ product up to fourth order polynomial. This multiplet has scalars with:

$$\begin{aligned} E_0 &= \ell & SU(4) \text{ singlet} \\ E_0 &= \ell + 1 & 10 + \overline{10} \\ E_0 &= \ell + 2 & 1 + 2 \times 20_R + 15 + 84 \end{aligned} \tag{8}$$

For $\ell^0 = 2$ we have both relevant and marginal.

For $\ell^0 = 4$ we have only one marginal deformation in the $20_R + 1$.

For the scalar superfield (with $\ell^0 = 2$) there is a supersymmetric deformation in the $10, \overline{10}$, corresponding to a superpotential which reduces to a $SU(3)$ singlet $f_{\alpha\beta\gamma}\phi_\ell^\alpha\phi_m^\beta\phi_n^\gamma$. The higher dimensional operators correspond to the following objects:

$$f_{\alpha\beta\gamma}\psi_A^\alpha\psi_B^\beta\phi_{[CD]}^\gamma, \quad f_{\alpha\beta\gamma}\phi_\ell^\beta\phi_m^\gamma f_{\alpha\epsilon\delta}\phi_p^\epsilon\phi_q^\delta \tag{9}$$

which give rise precisely to the rep. content listed above.

There is another long multiplet which starts with scalars in the 6 ($\ell^0 = 3$), but this does not give chiral $n = 1$ multiplets among its components.

Finally, there is also a massive $J_{max} = 5$ multiplet with scalars with $\ell = 4$, but this multiplet does not appear in the OPE of the stress tensor, because of symmetry reasons.

3.1 $n = 1$ massive representations

The long multiplets of the $SU(2, 2/1)$ algebra can be obtained from the lowest dimensional representation, by tensoring with a (J_1, J_2) $SL(2, C)$ representation.

Since $n = 4$ long multiplets have anomalous dimension, in their decomposition to $n = 1$ they cannot contain “short” $n = 1$ “chiral” multiplets.

Therefore the content of a generic massive $n = 1$ AdS_5 multiplet is:

$$\begin{aligned}
& \mathcal{D}(E_0, J_1, J_2, q), \mathcal{D}(E_0 + \frac{1}{2}, J_1 + \frac{1}{2}, J_2, q + 1), \\
& \mathcal{D}(E_0 + \frac{1}{2}, J_1 - \frac{1}{2}, J_2, q + 1), \mathcal{D}(E_0 + \frac{1}{2}, J_1, J_2 + \frac{1}{2}, q - 1), \\
& \mathcal{D}(E_0 + \frac{1}{2}, J_1, J_2 - \frac{1}{2}, q - 1), \mathcal{D}(E_0 + 1, J_1, J_2, q + 2), \\
& \mathcal{D}(E_0 + 1, J_1, J_2, q - 2), \mathcal{D}(E_0 + 1, J_1 + \frac{1}{2}, J_2 + \frac{1}{2}, q), \\
& \mathcal{D}(E_0 + 1, J_1 + \frac{1}{2}, J_2 - \frac{1}{2}, q), \mathcal{D}(E_0 + 1, J_1 - \frac{1}{2}, J_2 + \frac{1}{2}, q), \\
& \mathcal{D}(E_0 + 1, J_1 - \frac{1}{2}, J_2 - \frac{1}{2}, q), \mathcal{D}(E_0 + \frac{3}{2}, J_1 + \frac{1}{2}, J_2, q + 1), \\
& \mathcal{D}(E_0 + \frac{3}{2}, J_1 - \frac{1}{2}, J_2, q - 1), \mathcal{D}(E_0 + \frac{3}{2}, J_1, J_2 + \frac{1}{2}, q + 1), \\
& \mathcal{D}(E_0 + \frac{3}{2}, J_1, J_2 - \frac{1}{2}, q - 1), \mathcal{D}(E_0 + 2, J_1, J_2, q)
\end{aligned} \tag{10}$$

This multiplet is described, in the AdS/CFT correspondence, by a “super-field” (of conformal dimension $\ell = E_0$):

$$V_{\alpha_1, \dots, \alpha_{2J_1}, \dot{\alpha}_1, \dots, \dot{\alpha}_{2J_2}}^{q, E_0}(x, \theta, \bar{\theta}) \tag{11}$$

with $2J_1, 2J_2$ symmetrized $SL(2, C)$ indices.

For massless representations, which occur when $E_0 = \ell = 2 + J_1 + J_2$, then V is “conserved” ($J_1 \cdot J_2 \neq 0$):

$$D^{\alpha_1 \dot{\alpha}_1} V_{\alpha_1, \dots, \alpha_{2J_1}, \dot{\alpha}_1, \dots, \dot{\alpha}_{2J_2}}^{q, E_0} = 0. \tag{12}$$

When $J_1 \cdot J_2 = 0$, the multiplet can obey another “shortening condition”, i.e. the chirality constraint:

$$\bar{D}^{\dot{\alpha}_1} V_{\alpha_1, \dots, \alpha_{2J}, \dot{\alpha}_1, \dots, \dot{\alpha}_{2J}}^{q, E_0}(x, \theta, \bar{\theta}) = 0. \tag{13}$$

(This demands $\ell = q$ where q is the $U(1)$ R-charge). For $\ell = q = 1 + J$ the chiral multiplet describes a singleton representation in AdS_5 .

For $\ell = q > 1 + J$ one gets both massless and massive conformal “chiral” excitations in the bulk.

3.2 $n = 1$ decomposition of long $n = 4$ multiplets

In order to analyze the $N = 1$ content of $n = 4$ super Yang–Mills theories it is important to analyze the UIR’s of the $SU(2, 2/4)$ algebra in terms of $n = 1$ superconformal representations.

For a generic $n = 4$ long multiplet, this amounts to decompose it in terms of $n = 1$ representations as discussed in the previous section.

In this section we will report such a decomposition for the $n = 4$ Konishi-multiplet, which corresponds to the smallest $n = 4$ long multiplet and is of physical interest since it appears in the OPE of $n = 4$ Yang–Mills theory.

On general grounds, this multiplet has a maximum spin content $J_1 = J_2 = 2$, with $\ell_{J_1=J_2=2} = \ell + 4$, where ℓ is the conformal dimension of the superfield (i.e. the dimension of its lowest θ component).

This immediately implies that there will be a highest $n = 1$ massive rep. of the type $V_{J_1=J_2=\frac{3}{2}}(x, \theta, \bar{\theta})$, i.e. a spin 3 $n = 1$ massive superfield. Then the $N = 4$ Konishi-multiplet will decompose in a hierarchy of $n = 1$ superfields $\sum V_{J_1, J_2}^{R_{J_1 J_2}}$, with $J_1 \leq \frac{3}{2}$, $J_2 \leq \frac{3}{2}$, where $R_{J_1 J_2}$ will be the suitable $SU(3) \times U(1)$ representation which appears in the decomposition of “highest weight” $SU(4)$ representations.

Recently different supergravity vacua on AdS_5 have been interpreted as possible conformal deformations of $n = 4$ Yang–Mills theory [7, 8, 9].

We will only refer on those $n = 1$ multiplets containing scalars with naive dimension $\ell \leq 4$. The only $n = 1$ massive multiplets containing scalars are those for which the lowest component is $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, so only these types of multiplets will be analyzed.

The lowest dimensional state of the $n = 4$ Konishi-multiplet is a real scalar with $E_0 = \ell$. It then follows that the $n = 1$ multiplet with the lowest value of E_0 will be a $V_{J_1=J_2=0}$ multiplet with $E_0 = \ell$.

The next lowest E_0 multiplet will then be a $V_{J_1=\frac{1}{2}, J_2=0}^{R_{\frac{1}{2}, 0}}$ multiplet with $E_0 = \ell + \frac{1}{2}$ and $R_{\frac{1}{2}, 0} = 3(-\frac{1}{3})$, and so on. In this way we get a unique decomposition of the $n = 4$ long multiplets in terms of $n = 1$ ones, that is we have, for the $n = 1$ multiplets with lowest component up to $E_0 = \ell + 2$:

E_0	$n = 1$ multiplet
ℓ	$V_{(0,0)}^{1(0)}$
$\ell + \frac{1}{2}$	$V_{(\frac{1}{2},0)}^{3(-\frac{1}{3})}; V_{(0,\frac{1}{2})}^{\bar{3}(\frac{1}{3})}$
$\ell + 1$	$V_{(0,0)}^{6(-\frac{2}{3})}; V_{(0,0)}^{\bar{6}(\frac{2}{3})}; V_{(\frac{1}{2},\frac{1}{2})}^{1(0)}; V_{(\frac{1}{2},\frac{1}{2})}^{8(0)}$
$\ell + \frac{3}{2}$	$V_{(\frac{1}{2},0)}^{\bar{3}(\frac{1}{3})}; V_{(\frac{1}{2},0)}^{\bar{15}(\frac{1}{3})}; V_{(\frac{1}{2},0)}^{8(-1)}; V_{(0,\frac{1}{2})}^{3(-\frac{1}{3})}; V_{(0,\frac{1}{2})}^{15(-\frac{1}{3})}; V_{(0,\frac{1}{2})}^{8(1)}$
$\ell + 2$	$V_{(0,0)}^{\bar{6}(-\frac{4}{3})}; V_{(0,0)}^{6(\frac{4}{3})}; V_{(0,0)}^{1(0)}; V_{(0,0)}^{8(0)}; V_{(0,0)}^{27(0)}$

4 Comments on the $n = 4$ OPE Expansion

General properties of superconformal covariant OPE expansions have been investigated in several papers [4, 5, 14, 20].

From the general results on the $n = 4$ case one can draw some conclusions. If one denotes by O_{SG} , O_{KK} , O_{ST} , operators in the $n = 4$ super Yang-Mills theory that correspond to, respectively, supergravity (i.e. AdS -massless), K-K, and string states, then the 3-point functions of the type $\langle O_{SG} O_{SG} O_{ST} \rangle$, $\langle O_{KK} O_{KK} O_{ST} \rangle$ are nonvanishing. In ref. [5] an analog of the class of O_{ST} operators contributing to the OPE of the stress tensor was given.

By $n = 4$ superconformal symmetry, this analysis can be further generalized by stating the following result:

The OPE expansion of the $n = 4$ O_{SG} multiplet contains the O_{SG} multiplet itself, together with all long multiplets whose maximum spin is an (s, s) representation of $SL(2, C)$ with $s \geq 1$ [5].

The lowest-energy (dimension) component of this AdS massive representation is $E_0 = \delta + 2(s - 1)$, with spin $(s - 2, s - 2)$. For $\delta = 0$ this representation has a massless limit, for which the (s, s) tensors (with AdS energy $E_0 = 2(s + 1)$) are conserved. This is the case of the OPE expansion in the free field theory.

The Konishi-multiplet just corresponds to the $s = 2$ case.

Notice that in the free field theory, sequences of such representations exist for integral δ , corresponding to multilinear (rather than bilinear) composites in the Yang-Mills superfield. These multilinear operators, from an AdS point of view, correspond to massive (rather than massless) AdS representations, obtained by tensoring more than two singleton representations. Let us call such sequences of higher- δ level O_{ST}^δ ($\delta = 0$ corresponds to the Konishi-multiplet). These sequences are expected to appear naturally in

$\langle O_{KK} O_{KK} O_{ST}^\delta \rangle$, as it is implied by the free field theory result. In particular, all K–K operators, corresponding to the same p -level, will contain the $\delta = 0$ multiplet O_{ST} , which is just the Konishi superfield, in their OPE, i.e. $\langle O_{KK}^p O_{KK}^p O_{ST} \rangle \neq 0$.

Note that in the Yang–Mills theory one can get other sequences of multilinear operators by taking not single traces.

For example $(Tr(\phi_\ell \phi_m) - traces)(Tr(\phi_p \phi_q) - traces) \rightarrow 105 + 84 + 20 + 1$ would give rise both to short (105) and long ($84 + 20 + 1$) multiplets which are neither K–K nor string states.

These states should correspond to multiparticle states of pure supergravity [20].

The existence of short multiplets not corresponding to K–K states can be seen directly by working in harmonic superspace [21]. In this case the K–K states are (G-)analytic (F-)holomorphic fields $Tr W^p$ [14].

Now it is obvious that if we take, at level p , $\Pi_{i=1}^r Tr(W^{q_i})$, $(\sum_r q_r = p)$ this superfield is also G-analytic, F-holomorphic, i.e. a short “chiral primary” in the $n = 4$ superconformal language. In the $n = 1$ formalism, this is related to the fact that chiral primary superfields form a closed algebraic ring [20].

The non-vanishing superfields for which $\{q_r\} \neq p$ will be called multiparticle states.

For a $SU(N)$ gauge theory, single trace operators exist up to level $p = N$. Therefore if $p > N$ we would be in a situation in which only multiparticle states would exist.

Since in supergravity theory K–K states exist for arbitrary p , this is another reason why the supergravity limit of the AdS/CFT correspondence works only in the limit of large N [20].

It is worthwhile to mention that, for finite N , the number of single-trace chiral primary $n = 4$ superfields is in one to one correspondence with the odd de Rham cohomology classes H^ℓ ($3 \leq \ell \leq 2N - 1$) of the group manifold $SU(N)$.⁴

If this would be the case also for the multitrace operators, then there would be only a finite number of additional short multiplets coming from the cohomology classes H^ℓ with $2N - 1 < \ell \leq N^2 - 1$.

Incidentally we remark that $N^2 - 1$ is essentially the central charge of the $N = 4$ superconformal algebra [4, 5] so the bound would be similar to the case of two dimensional superconformal field theories [22].

⁴We would like to thank Raymond Stora for a discussion on this point.

The window of “multiparticle states” chiral primaries would then be $\Delta s = N(N - 2)$ and it would grow, for large N , as N^2 , faster than single particle states which grow like N .

For finite N , the fact that the number of “single trace chiral primaries” is finite may be related to a stringy effect that is not seen in the supergravity approximation. It is analogous to the “stringy exclusion principle” discussed by Maldacena and Strominger [22].

Acknowledgements

We would like to thank D. Anselmi, L. Girardello, M. Porrati, R. Stora and A. Zaffaroni for enlightening conversations.

References

- [1] J.M. Maldacena, hep-th/9711200; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428** (1998) 105, hep-th/9802109; E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.
- [2] T. Banks and M.B. Green, J.High Energy Phys. 05 (1998) 002, hep-th/9804170; M. Bianchi, M.B. Green, Stefano Kovacs and G. Rossi, J.High Energy Phys. 9808 (1998) 013, hep-th/9807033; J.H. Brodie and M. Gutperle, hep-th/9809067.
- [3] S. Ferrara, and A. Zaffaroni, hep-th/9807090; L. Andrianopoli and S. Ferrara, hep-th/9807150; S. Ferrara, M.A. Lledo and A. Zaffaroni, Phys. Rev. **D58** (1998) 10.
- [4] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Phys. Lett **B394** (1997) 329, hep-th/9608125.
- [5] D. Anselmi, hep-th/9808004; hep-th/9809192; hep-th/9811149.
- [6] R.G. Leigh and M.J. Strassler, Nucl. Phys. **B447** (1995) 95, hep-th/9503121.
- [7] I.R. Klebanov and E. Witten, hep-th/9807080; S.S. Gubser, hep-th/9807164.

- [8] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, hep-th/9810126.
- [9] J. Distler and F. Zamora, hep-th/9810206.
- [10] A. Khavaev, K. Pilch and N.P. Warner, hep-th/9812035.
- [11] A. Ceresole, P. Fré and H. Nicolai, Class. Quant. Grav. 2 (1985) 133.
- [12] M. Günaydin and C. Saclioglu, Commun. Math. Phys. 87 (1982) 159.
- [13] M. Günaydin and N. Marcus, Class. Quant. Grav. 2 (1985) L11.
- [14] P.S. Howe and P.C. West, Phys. Lett. **B389** (1996) 273; Phys. Lett. **B400** (1997) 307.
- [15] L. Andrianopoli and S. Ferrara, Phys. Lett. **B430** (1998) 248, hep-th/9803171.
- [16] S. Ferrara and C. Fronsdal, Phys. Lett. **B433** (1998) 19; S. Ferrara, C. Fronsdal and A. Zaffaroni, Nucl. Phys. **B532** (1998) 153, hep-th/9802203.
- [17] S. Ferrara, R. Gatto and A.F. Grillo, Phys. Rev. **D9** (1974) 3564.
- [18] P. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. **B192** (1981) 332.
- [19] M. Gunaydin, D. Minic and M. Zagermann, Nucl. Phys. **B534** (1998) 96, hep-th/9806042.
- [20] H. Osborn, hep-th/9808041; H. Liu, hep-th/9811152; S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, hep-th/9806074; H. Liu and A.A. Tseytlin, hep-th/9807097; D.Z. Freedman, S.D. Mathur, A Matusis and L. Rastelli, hep-th/9808006; T. Banks, M.R. Douglas, G.T. Horowitz and E. Martinec, hep-th/9808016; B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, hep-th/9811172.
- [21] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetskii, E. Sokatchev, Class. Quant. Grav. 1 (1984) 469
- [22] J.M. Maldacena and A. Strominger, hep-th/9804085.

A Field content of the $n = 4$ Konishi-multiplet

Table 1: $(0,0)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta^0\bar{\theta}^0)_\ell; (\theta^2\bar{\theta}^2)_{\ell+2}; (\theta^8)_{\ell+4}; (\theta^4\bar{\theta}^4)_{\ell+4}; (\bar{\theta}^8)_{\ell+4}; (\theta^6\bar{\theta}^6)_{\ell+6}; (\theta^8\bar{\theta}^8)_{\ell+8}$
10	$(\theta^2)_{\ell+1}; (\theta^2\bar{\theta}^4)_{\ell+3}; (\bar{\theta}^6)_{\ell+3}; (\theta^4\bar{\theta}^6)_{\ell+5}; (\theta^2\bar{\theta}^8)_{\ell+5}; (\theta^8\bar{\theta}^6)_{\ell+7}$
$\bar{10}$	$(\bar{\theta}^2)_{\ell+1}; (\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^6)_{\ell+3}; (\theta^6\bar{\theta}^4)_{\ell+5}; (\theta^8\bar{\theta}^2)_{\ell+5}; (\theta^6\bar{\theta}^8)_{\ell+7}$
20_R	$(\theta^4)_{\ell+2}; (\bar{\theta}^4)_{\ell+2}; (\theta^6\bar{\theta}^2)_{\ell+4}; (\theta^4\bar{\theta}^4)_{\ell+4}; (\theta^2\bar{\theta}^6)_{\ell+4}; (\theta^4\bar{\theta}^8)_{\ell+6}; (\theta^8\bar{\theta}^4)_{\ell+6}$
15	$(\theta^2\bar{\theta}^2)_{\ell+2}; (\theta^4\bar{\theta}^4)_{\ell+4}; (\theta^6\bar{\theta}^6)_{\ell+6}$
84	$(\theta^2\bar{\theta}^2)_{\ell+2}; (\theta^4\bar{\theta}^4)_{\ell+4}; (\theta^6\bar{\theta}^6)_{\ell+6}$
64	$(\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^2\bar{\theta}^4)_{\ell+3}; (\theta^6\bar{\theta}^4)_{\ell+5}; (\theta^4\bar{\theta}^6)_{\ell+5}$
126	$(\theta^2\bar{\theta}^4)_{\ell+3}; (\theta^4\bar{\theta}^6)_{\ell+5}$
$\bar{126}$	$(\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^6\bar{\theta}^4)_{\ell+5}$
35	$(\theta^2\bar{\theta}^6)_{\ell+4}$
$\bar{35}$	$(\theta^6\bar{\theta}^2)_{\ell+4}$
45	$(\theta^2\bar{\theta}^6)_{\ell+4}$
$\bar{45}$	$(\theta^6\bar{\theta}^2)_{\ell+4}$
105	$(\theta^4\bar{\theta}^4)_{\ell+4}$
175	$(\theta^4\bar{\theta}^4)_{\ell+4}$

Table 2: $(1, 0)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
6	$(\theta^2)_{\ell+1}; (\theta^6)_{\ell+3}; (\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^2\bar{\theta}^4)_{\ell+3}; (\theta^6\bar{\theta}^8)_{\ell+5}; (\theta^6\bar{\theta}^4)_{\ell+5}; (\theta^4\bar{\theta}^6)_{\ell+5}; (\theta^2\bar{\theta}^8)_{\ell+7}$
15	$(\theta^4)_{\ell+2}; (\theta^2\bar{\theta}^2)_{\ell+2}; (\theta^6\bar{\theta}^2)_{\ell+4}; (\theta^4\bar{\theta}^4)_{\ell+4}; (\theta^2\bar{\theta}^6)_{\ell+4}; (\theta^6\bar{\theta}^6)_{\ell+6}; (\theta^4\bar{\theta}^8)_{\ell+6}$
$\overline{45}$	$(\theta^2\bar{\theta}^2)_{\ell+2}; (\theta^6\bar{\theta}^2)_{\ell+4}; (\theta^4\bar{\theta}^4)_{\ell+4}$
45	$(\theta^4\bar{\theta}^4)_{\ell+4}; (\theta^2\bar{\theta}^6)_{\ell+4}; (\theta^6\bar{\theta}^6)_{\ell+6}$
$\overline{10}$	$(\theta^4\bar{\theta}^2)_{\ell+3}$
10	$(\theta^4\bar{\theta}^6)_{\ell+5}$
64	$(\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^2\bar{\theta}^4)_{\ell+3}; (\theta^6\bar{\theta}^4)_{\ell+5}; (\theta^4\bar{\theta}^6)_{\ell+5}$
$\overline{70}$	$(\theta^4\bar{\theta}^2)_{\ell+3}$
70	$(\theta^4\bar{\theta}^6)_{\ell+5}$
50	$(\theta^2\bar{\theta}^4)_{\ell+3}; (\theta^6\bar{\theta}^4)_{\ell+5}$
20_R	$(\theta^4\bar{\theta}^4)_{\ell+4}$
175	$(\theta^4\bar{\theta}^4)_{\ell+4}$

Table 3: $(\frac{1}{2}, \frac{1}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta\bar{\theta})_{\ell+1}; (\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}; (\theta^7\bar{\theta}^7)_{\ell+7}$
15	$(\theta\bar{\theta})_{\ell+1}; (\theta^5\bar{\theta})_{\ell+3}; (\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^3\bar{\theta}^3)_{\ell+3}; (\theta\bar{\theta}^5)_{\ell+3}; (\theta^7\bar{\theta}^3)_{\ell+5}; (\theta^5\bar{\theta}^5)_{\ell+5}; (\theta^5\bar{\theta}^5)_{\ell+5}; (\theta^3\bar{\theta}^7)_{\ell+5}; (\theta^7\bar{\theta}^7)_{\ell+7}$
6	$(\theta^3\bar{\theta})_{\ell+2}; (\theta\bar{\theta}^3)_{\ell+2}; (\theta^7\bar{\theta})_{\ell+4}; (\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}; (\theta\bar{\theta}^7)_{\ell+4}; (\theta^7\bar{\theta}^5)_{\ell+6}; (\theta^5\bar{\theta}^7)_{\ell+6}$
10	$(\theta^3\bar{\theta})_{\ell+2}; (\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}; (\theta\bar{\theta}^7)_{\ell+4}; (\theta^7\bar{\theta}^5)_{\ell+6};$
$\overline{10}$	$(\theta\bar{\theta}^3)_{\ell+2}; (\theta^7\bar{\theta})_{\ell+4}; (\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}; (\theta^5\bar{\theta}^7)_{\ell+6}$
64	$(\theta^3\bar{\theta})_{\ell+2}; (\theta\bar{\theta}^3)_{\ell+2}; (\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}; (\theta^7\bar{\theta}^5)_{\ell+6}; (\theta^5\bar{\theta}^7)_{\ell+6}$
20_R	$(\theta^5\bar{\theta})_{\ell+3}; (\theta^3\bar{\theta}^3)_{\ell+3}; (\theta\bar{\theta}^5)_{\ell+3}; (\theta^7\bar{\theta}^3)_{\ell+5}; (\theta^5\bar{\theta}^5)_{\ell+5}; (\theta^3\bar{\theta}^7)_{\ell+5}$
45	$(\theta^3\bar{\theta}^3)_{\ell+3}; (\theta\bar{\theta}^5)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}; (\theta^3\bar{\theta}^7)_{\ell+5}$
$\overline{45}$	$(\theta^5\bar{\theta})_{\ell+3}; (\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^7\bar{\theta}^3)_{\ell+5}; (\theta^5\bar{\theta}^5)_{\ell+5}$
84	$(\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}$
175	$(\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}$
50	$(\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}$
70	$(\theta^5\bar{\theta}^3)_{\ell+4}$
$\overline{70}$	$(\theta^3\bar{\theta}^5)_{\ell+4}$
126	$(\theta^3\bar{\theta}^5)_{\ell+4}$
$\overline{126}$	$(\theta^5\bar{\theta}^3)_{\ell+4}$

Table 4: $(2, 0)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta^4)_{\ell+2}$
$\overline{10}$	$(\theta^4\overline{\theta}^2)_{\ell+3}$
10	$(\theta^4\overline{\theta}^6)_{\ell+5}$
20_R	$(\theta^4\overline{\theta}^4)_{\ell+4}$

Table 5: $(\frac{3}{2}, \frac{1}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
6	$(\theta^3\overline{\theta})_{\ell+2}; (\theta^5\overline{\theta}^3)_{\ell+4}; (\theta^3\overline{\theta}^5)_{\ell+4}; (\theta^5\overline{\theta}^7)_{\ell+6}$
$\overline{10}$	$(\theta^3\overline{\theta})_{\ell+2}; (\theta^5\overline{\theta}^3)_{\ell+4}$
10	$(\theta^3\overline{\theta}^5)_{\ell+4}; (\theta^5\overline{\theta}^7)_{\ell+6}$
1	$(\theta^5\overline{\theta})_{\ell+3}; (\theta^3\overline{\theta}^7)_{\ell+5}$
15	$(\theta^5\overline{\theta})_{\ell+3}; (\theta^3\overline{\theta}^3)_{\ell+3}; (\theta^5\overline{\theta}^5)_{\ell+5}; (\theta^3\overline{\theta}^7)_{\ell+5}$
20_R	$(\theta^3\overline{\theta}^3)_{\ell+3}; (\theta^5\overline{\theta}^5)_{\ell+5}$
$\overline{45}$	$(\theta^3\overline{\theta}^3)_{\ell+3}$
45	$(\theta^5\overline{\theta}^5)_{\ell+5}$
64	$(\theta^5\overline{\theta}^3)_{\ell+4}; (\theta^3\overline{\theta}^5)_{\ell+4}$

Table 6: $(1, 1)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta^2\overline{\theta}^2)_{\ell+2}; (\theta^6\overline{\theta}^2)_{\ell+4}; (\theta^4\overline{\theta}^4)_{\ell+4}; (\theta^2\overline{\theta}^6)_{\ell+4}; (\theta^6\overline{\theta}^6)_{\ell+6}$
15	$(\theta^2\overline{\theta}^2)_{\ell+2}; (\theta^6\overline{\theta}^2)_{\ell+4}; (\theta^4\overline{\theta}^4)_{\ell+4}; (\theta^4\overline{\theta}^4)_{\ell+4}; (\theta^2\overline{\theta}^6)_{\ell+4}; (\theta^6\overline{\theta}^6)_{\ell+6}$
20_R	$(\theta^2\overline{\theta}^2)_{\ell+2}; (\theta^6\overline{\theta}^2)_{\ell+4}; (\theta^4\overline{\theta}^4)_{\ell+4}; (\theta^2\overline{\theta}^6)_{\ell+4}; (\theta^6\overline{\theta}^6)_{\ell+6}$
6	$(\theta^4\overline{\theta}^2)_{\ell+3}; (\theta^2\overline{\theta}^4)_{\ell+3}; (\theta^6\overline{\theta}^4)_{\ell+5}; (\theta^4\overline{\theta}^6)_{\ell+5}$
10	$(\theta^4\overline{\theta}^2)_{\ell+3}; (\theta^2\overline{\theta}^4)_{\ell+3}; (\theta^6\overline{\theta}^4)_{\ell+5}; (\theta^4\overline{\theta}^6)_{\ell+5}$
$\overline{10}$	$(\theta^4\overline{\theta}^2)_{\ell+3}; (\theta^2\overline{\theta}^4)_{\ell+3}; (\theta^6\overline{\theta}^4)_{\ell+5}; (\theta^4\overline{\theta}^6)_{\ell+5}$
64	$(\theta^4\overline{\theta}^2)_{\ell+3}; (\theta^2\overline{\theta}^4)_{\ell+3}; (\theta^6\overline{\theta}^4)_{\ell+5}; (\theta^4\overline{\theta}^6)_{\ell+5}$
45	$(\theta^4\overline{\theta}^4)_{\ell+4}$
$\overline{45}$	$(\theta^4\overline{\theta}^4)_{\ell+4}$
84	$(\theta^4\overline{\theta}^4)_{\ell+4}$

Table 7: $(2, 1)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
6	$(\theta^4\bar{\theta}^2)_{\ell+3}; (\theta^4\bar{\theta}^6)_{\ell+5}$
15	$(\theta^4\bar{\theta}^4)_{\ell+4}$

Table 8: $(\frac{3}{2}, \frac{3}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}$
15	$(\theta^3\bar{\theta}^3)_{\ell+3}; (\theta^5\bar{\theta}^5)_{\ell+5}$
6	$(\theta^5\bar{\theta}^3)_{\ell+4}; (\theta^3\bar{\theta}^5)_{\ell+4}$
10	$(\theta^5\bar{\theta}^3)_{\ell+4}$
$\overline{10}$	$(\theta^3\bar{\theta}^5)_{\ell+4}$

Table 9: $(2, 2)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
1	$(\theta^4\bar{\theta}^4)_{\ell+4}$

Table 10: $(\frac{1}{2}, 0)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
4	$(\theta)_{\ell+\frac{1}{2}}; (\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta\bar{\theta}^8)_{\ell+\frac{9}{2}}; (\theta^7\bar{\theta}^6)_{\ell+\frac{13}{2}}$
$\bar{4}$	$(\theta\bar{\theta}^2)_{\ell+\frac{3}{2}}; (\theta^7)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}; (\theta^7\bar{\theta}^8)_{\ell+\frac{15}{2}}$
20	$(\theta^3)_{\ell+\frac{3}{2}}; (\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta\bar{\theta}^6)_{\ell+\frac{7}{2}}; (\theta^7\bar{\theta}^4)_{\ell+\frac{11}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}; (\theta^3\bar{\theta}^8)_{\ell+\frac{11}{2}}$
$\overline{20}$	$(\theta^5)_{\ell+\frac{5}{2}}; (\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta\bar{\theta}^4)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta^7\bar{\theta}^2)_{\ell+\frac{9}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}; (\theta^5\bar{\theta}^8)_{\ell+\frac{13}{2}}$
$\overline{36}$	$(\theta\bar{\theta}^2)_{\ell+\frac{3}{2}}; (\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
36	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}; (\theta^7\bar{\theta}^6)_{\ell+\frac{13}{2}}$
140	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}$
$\overline{140}$	$(\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
60	$(\theta\bar{\theta}^4)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}$
$\overline{60}$	$(\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^7\bar{\theta}^4)_{\ell+\frac{7}{2}}$
$\overline{84'}$	$(\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}$
84'	$(\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}$
$\overline{140'}$	$(\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}$
140'	$(\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}$
20''	$(\theta\bar{\theta}^6)_{\ell+\frac{7}{2}}$
$\overline{20''}$	$(\theta^7\bar{\theta}^2)_{\ell+\frac{9}{2}}$

Table 11: $(\frac{3}{2}, 0)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
$\bar{4}$	$(\theta^3)_{\ell+\frac{3}{2}}; (\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^8)_{\ell+\frac{11}{2}}$
4	$(\theta^5)_{\ell+\frac{5}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}; (\theta^5\bar{\theta}^8)_{\ell+\frac{13}{2}}$
$\overline{20}$	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}$
20	$(\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
$\overline{20''}$	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}$
$20''$	$(\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
$\overline{36}$	$(\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}$
36	$(\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}$
$\overline{60}$	$(\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}$
60	$(\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}$

 Table 12: $(1, \frac{1}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
4	$(\theta^2\bar{\theta})_{\ell+\frac{3}{2}}; (\theta^6\bar{\theta})_{\ell+\frac{7}{2}}; (\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}; (\theta^2\bar{\theta}^5)_{\ell+\frac{7}{2}}; (\theta^6\bar{\theta}^5)_{\ell+\frac{11}{2}}; (\theta^4\bar{\theta}^7)_{\ell+\frac{11}{2}}$
$\bar{4}$	$(\theta^4\bar{\theta})_{\ell+\frac{5}{2}}; (\theta^2\bar{\theta}^3)_{\ell+\frac{5}{2}}; (\theta^6\bar{\theta}^3)_{\ell+\frac{9}{2}}; (\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}; (\theta^2\bar{\theta}^7)_{\ell+\frac{9}{2}}; (\theta^6\bar{\theta}^7)_{\ell+\frac{13}{2}}$
$\overline{20}$	$(\theta^2\bar{\theta})_{\ell+\frac{3}{2}}; (\theta^6\bar{\theta})_{\ell+\frac{7}{2}}; (\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}; (\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}; (\theta^2\bar{\theta}^5)_{\ell+\frac{7}{2}};$ $(\theta^6\bar{\theta}^5)_{\ell+\frac{11}{2}}; (\theta^4\bar{\theta}^7)_{\ell+\frac{11}{2}}$
20	$(\theta^4\bar{\theta})_{\ell+\frac{5}{2}}; (\theta^2\bar{\theta}^3)_{\ell+\frac{5}{2}}; (\theta^6\bar{\theta}^3)_{\ell+\frac{9}{2}}; (\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}; (\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}; (\theta^2\bar{\theta}^7)_{\ell+\frac{9}{2}};$ $(\theta^6\bar{\theta}^7)_{\ell+\frac{13}{2}}$
$\overline{36}$	$(\theta^4\bar{\theta})_{\ell+\frac{5}{2}}; (\theta^2\bar{\theta}^3)_{\ell+\frac{5}{2}}; (\theta^6\bar{\theta}^3)_{\ell+\frac{9}{2}}; (\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}};$
36	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}; (\theta^2\bar{\theta}^5)_{\ell+\frac{7}{2}}; (\theta^6\bar{\theta}^5)_{\ell+\frac{11}{2}}; (\theta^4\bar{\theta}^7)_{\ell+\frac{11}{2}}$
$\overline{60}$	$(\theta^2\bar{\theta}^3)_{\ell+\frac{5}{2}}; (\theta^6\bar{\theta}^3)_{\ell+\frac{9}{2}}; (\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}$
60	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}; (\theta^2\bar{\theta}^5)_{\ell+\frac{7}{2}}; (\theta^6\bar{\theta}^5)_{\ell+\frac{11}{2}}$
$\overline{20''}$	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}$
$20''$	$(\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}$
140	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}$
$\overline{140}$	$(\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}$

Table 13: $(\frac{3}{2}, 1)$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
4	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}$
$\bar{4}$	$(\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
$\bar{20}$	$(\theta^3\bar{\theta}^2)_{\ell+\frac{5}{2}}; (\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}; (\theta^3\bar{\theta}^6)_{\ell+\frac{9}{2}}$
20	$(\theta^5\bar{\theta}^2)_{\ell+\frac{7}{2}}; (\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}; (\theta^5\bar{\theta}^6)_{\ell+\frac{11}{2}}$
$\bar{36}$	$(\theta^3\bar{\theta}^4)_{\ell+\frac{7}{2}}$
36	$(\theta^5\bar{\theta}^4)_{\ell+\frac{9}{2}}$

Table 14: $(2, \frac{1}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
$\bar{4}$	$(\theta^4\bar{\theta})_{\ell+\frac{5}{2}}$
4	$(\theta^4\bar{\theta}^7)_{\ell+\frac{11}{2}}$
$\bar{20}$	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}$
20	$(\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}$

Table 15: $(2, \frac{3}{2})$ fields:

$SU(4)$	$(\theta\text{-component})_{E_0}$
4	$(\theta^4\bar{\theta}^3)_{\ell+\frac{7}{2}}$
$\bar{4}$	$(\theta^4\bar{\theta}^5)_{\ell+\frac{9}{2}}$