Supernova Remnant in a Stratified Medium: Explicit, Analytical Approximations for Adiabatic Expansion and Radiative Cooling

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ABSTRACT

We propose simple, explicit, analytical approximations for the kinematics of an adiabatic blast wave propagating in an exponentially stratified ambient medium, and for the onset of radiative cooling, which ends the adiabatic era. Our method, based on the Kompaneets implicit solution and the Kahn approximation for the radiative cooling coefficient, gives straightforward estimates for the size, expansion velocity, and progression of cooling times over the surface, when applied to supernova remnants (SNRs). The remnant shape is remarkably close to spherical for moderate density gradients, but even a small gradient in ambient density causes the cooling time to vary substantially over the remnant’s surface, so that for a considerable period there will be a cold dense expanding shell covering only a part of the remnant. Our approximation provides an effective tool for identifying the approximate parameters when planning 2-dimensional numerical models of SNRs, the example of W44 being given in a subsequent paper.

Subject headings: shock waves — methods: analytical — ISM: supernova remnants — ISM: bubbles

1. Introduction

A good model of a supernova remnant’s (SNR’s) blast wave propagating through the ambient medium allows us to derive, from the observed quantities, important properties of the medium, as well as the explosion energy. The Sedov solution (1959) for adiabatic expansion into an isotropic medium usually serves as the prototype, and further, more realistic models also assume isotropy (see e.g. Jun, Jones & Norman 1996). Nevertheless, in most cases of astrophysical interest (stellar winds, Garcia-Segura & Mac Low 1995; SNRs in vicinity of molecular clouds, Dohm-Palmer & Jones 1996; galactic supershells, Maciejewski et al. 1996; and starburst galaxies, Jogee, Kenney & Smith 1998), the ambient medium is not isotropic. At the next level of approximation, Kompaneets (1960) found an implicit solution for a blast wave propagating in an exponentially stratified ambient medium, under the assumption that the post-shock pressure is uniform over the surface of the remnant. An explicit form can be obtained when one additional assumption is made, that the shape of the shock wave can be approximated by a prolate ellipsoid. After an alternative presentation of the Kompaneets solution in §2, we derive explicit analytical formulae for the size, velocity and the onset of cooling in §3. A prolate ellipsoid is a good approximation for all evolutionary times prior to blowout (Kompaneets 1960), but we show in §4 that the Kompaneets approximation of uniform post-shock pressure loses validity much earlier. In §5 we show how our explicit approximation can be used in exploring the parameter space to identify values of interest in making 2-dimensional numerical models of SNRs.
2. A simple method to derive basic Kompaneets conclusions

Before we introduce our assumption, let us show a simple way of re-deriving the basics of the original Kompaneets solution, using one of his conclusions, namely that in the direction perpendicular to the density gradient, the remnant is widest halfway between the high and low ambient density extremes of the shock, a location we shall refer to as the equator. With the resulting formulae for the remnant size parallel and perpendicular to the gradient, our approximation for the shock shape immediately provides the volume and pressure needed to learn the time evolution.

We consider the shock expansion in cylindrical coordinates \((z,r)\): the explosion occurs at \((z,r) = (0,0)\) (Fig.1). The ambient density \(\rho_0\) is an exponential function of \(z\):

\[
\rho_0 = \rho_* e^{-z/h},
\]

(1)

where \(\rho_*\) is the density at the explosion site and \(h\) is the stratification scaleheight. Denoting \(z_H\) and \(z_L\) as the location of the dense and the tenuous end of the remnant, respectively, we have \(z_L - z_H = 2a\), where \(2a\) is the extent of the remnant in the direction along the density gradient. For a strong, non-radiative shock in a \(\gamma = 5/3\) gas, the post-shock pressure \(P_S\), pre-shock density \(\rho_0\) and shock velocity \(v\) are related by

\[
\rho_0 v^2 = \frac{4}{3} P_S. \tag{2}
\]

Assuming after Kompaneets that the post-shock pressure \(P_S\) is constant over the surface of the remnant, the shock speed at a particular time can be written as a function of \(z\):

\[
v(z,t) = v_*(t) e^{z/2h}. \tag{3}
\]

Comparing the values of shock velocities at the ambient density extremes and at the equator, we can eliminate the unknown \(v_*(t)\) and calculate the evolution of the remnant shape as a function of its semi-major axis \(a\).

The rate of expansion along the density gradient is an average of shock velocities at two density extremes: \(v_H\) and \(v_L\)

\[
\dot{a} = \frac{v_H + v_L}{2} = v_*(t) \frac{e^{z_H/2h} + e^{z_L/2h}}{2}, \tag{4}
\]

where \(v_H\) and \(v_L\) are velocity magnitudes, \(z_H\) is negative, and \(v_H = -\dot{z}_H\). The center of the remnant is displaced from the explosion site by \(s = 0.5(z_L + z_H)\). At \(z = s\) (the equator), the remnant expands only in the lateral direction and the expansion rate is

\[
v_S = v_*(t) e^{s/2h}. \tag{5}
\]

Although the equator moves along the \(z\) axis (Fig.1) as the remnant expands, the rate of change of the remnant’s semi-minor axis \(b\) is exactly equal to \(v_S\): \(\dot{b} = v_S\). The evolution of the displacement \(s\) can be written as

\[
\dot{s} = \frac{\dot{z}_L + \dot{z}_H}{2} = \frac{v_L - v_H}{2} = v_*(t) \frac{e^{z_L/2h} - e^{z_H/2h}}{2}. \tag{6}
\]

After substituting \(z_H = s - a\) and \(z_L = s + a\) and simple manipulations, we can eliminate the constant \(v_*(t)\) and get

\[
\frac{da}{db} = \cosh \frac{a}{2h}; \quad \frac{ds}{da} = \tanh \frac{a}{2h}. \tag{7}
\]
Fig. 1.— (left) Representation of SNR characteristics used in the text. The ellipsoid is axially symmetric about the $z$ axis. Fig. 2.— (right) The ratio of cooling times on the tenuous and dense ends of the remnant expanding in a stratified medium, as a function of the ratio of extreme ambient densities of an equivalent remnant at its cooling time. The equivalent remnant is defined as a remnant with the same explosion energy, but expanding in a uniform density equal to the density at the explosion site. For the radius $r_{cool}$ of the equivalent remnant at its cooling time, the ratio of extreme ambient densities is defined as $W = \exp\left(2r_{cool}/h\right)$, where $h$ is the density scaleheight of the ambient medium.

Integrating the equations above, we can readily find $b$ and $s$ as functions of $a$

$$\tan\frac{b}{2h} = \sinh\frac{a}{2h} \quad ; \quad \exp\frac{s}{2h} = \cosh\frac{a}{2h}. \quad (8)$$

These results are independent of the assumed shape of the blast wave and the same as found by Kompaneets (1960), after his auxiliary variable $x$ is recognized as $\tanh(a/2h)$. In particular, the equations above allow us to estimate the flattening of the remnant $b/a$ and the relative displacement of the center $s/a$. It appears that even remnants extending several scale-heights retain a nearly spherical shape: for example, if $a = 1.7h$, the ambient density contrast between the ends of the shock is almost 30, but the remnant remains barely flattened (size ratio $b/a=0.9$). On the other hand, the ellipsoid’s center is displaced from the explosion site by $s = 0.38a$ in the low density direction. The same conclusions about sphericity and the center shift can be drawn from the analytical solution based on sectoral approximation (Gnatyk 1988). Hydrodynamical models constructed by Dohm-Palmer & Jones (1996) arrive to these conclusions as well. One can also
notice that \( \exp(z_H/2h) = \exp((s - a)/2h) = 0.5 (1 + \exp(-a/h)) \): in the blowout case, when \( a \to +\infty \), it is 0.5. This is Kompaneets' famous result, that as blowout occurs and \( z_L \) becomes large, \( z_H \) goes to a constant \(-2h \log 2\). The lateral size at blowout also agrees with Kompaneets' prediction: for \( a \to +\infty \), we get \( b/2h \to \pi/2 \).

3. Our approximations

The implicit solution by Kompaneets shows that the shock has the shape of a somewhat boxy ellipsoid. In fact, its shape is virtually indistinguishable from a true ellipsoid within limits of accuracy set by Kompaneets assumption of uniform post-shock pressure. We use this fact in order to introduce time as an independent variable instead of \( a \), when we assume that the post-shock pressure \( P_S \) has the same volume relationship as in the 1D Sedov solution (see e.g. Bisnovatyi-Kogan & Silich 1995)

\[
P_S = \frac{4\pi \zeta^5 E}{25 V},
\]

where \( \zeta = 2.025 \), \( E \) is the explosion energy and \( V = \frac{4}{3} \pi ab^2 \) is the volume of the remnant approximated by an ellipsoid. Combining equations (1), (2) and (9), and identifying \( z = s \) along the minor axis of the ellipsoid we have

\[
\rho_* e^{-s/h} b^2 = \frac{4\zeta^5}{25} E ab^2.
\]

After expressing \( s \) and \( b \) as functions of \( a \), we get

\[
\frac{da}{dt} = \frac{\zeta^{5/2}}{5h} \sqrt{\frac{E}{\rho_*} \cosh^2(a/2h) \arctanh \sinh(a/2h)},
\]

which, after integrating gives

\[
t(a) = \zeta^{-5/2} \left( \frac{E}{\rho_*} \right)^{-1/2} (2h)^{5/2} I(a/2h) \equiv t_{2h} I(a/2h),
\]

where \( t_{2h} \) is the nominal time at which a Sedov remnant in a homogeneous medium would reach a radius of \( 2h \), and the integral \( I(x) \) is defined in Table 1 for numerical evaluation. The numerical form of \( t_{2h} \) is

\[
t_{2h} = 17.32 \text{ yr} \left(2h_{pc}\right)^{5/2} n_*^{1/2} / E_{51}^{1/2}
\]

where \( h_{pc} \) is \( h \) in parsecs, \( E_{51} \) is \( E \) in units of \( 10^{51} \text{ ergs} \), and \( n_* \) is the nuclear number density of the ambient medium in \( \text{cm}^{-3} \). In this notation, the mass density is \( \rho = mn \), where the average mass per nucleus is \( m = (1.4/1.1)m_H \), \( m_H \) being the mass of hydrogen.

The shock velocity at any \( z \) can be easily calculated by combining equations (2) and (9) and substituting the local ambient density for \( \rho_0 \). In the particular case of top, bottom and equator expansion velocities, one can use equation (4), which after combining with equation (3) gives

\[
\dot{a} = v_H \exp(a/2h) \cosh(a/2h) = v_L \exp(-a/2h) \cosh(a/2h).
\]

After substituting \( \dot{a} \) from equation (11), one gets explicit formulae for shock velocities at two density extremes: \( v_H \) and \( v_L \). Obviously, the shock expansion velocity on the equator is the geometrical average of \( v_H \) and \( v_L \) — these results are given in Table 1. The ratio of expansion velocities at the two ends is \( \exp(a/h) \).

In SNR evolution in a uniform medium, the adiabatic era is brought to close by the onset of significant radiative cooling, at remnant radius \( r_{cool}(E, \rho_*), \) closely followed by formation of a dense shell. When a
remnant is evolving in a density gradient of scaleheight $h$, cooling occurs first on the dense end, and then spreads over the remnant surface. The evolution has two characteristic scales: the kinematic $h$ and the radiative $r_{cool}$, with a history of shell formation that is dependent on their ratio.

In order to estimate cooling times, we used Kahn’s cooling law approximation (1976), in which the cooling coefficient takes the form $L = aT^{-5/2}$. It is a reasonable approximation to the actual cooling coefficient for the temperature range of cooling SNRs (Smith et al.1996), and it has the special property that in the absence of thermal conduction, the time for a parcel of hot gas to cool from an initial state is independent of its history and equals

$$\Delta t_{cool} = \frac{P_S}{Ln^2}. \tag{14}$$

For a newly shocked parcel of gas, the post-shock values of pressure $P_S$, temperature $T$, and density $n$ are known, so we can find $\Delta t_{cool}$. This incremental cooling time $\Delta t_{cool}$ is added to the time at which a parcel is shocked, to find the actual cooling time of that parcel, and then the sum minimized over all parcels to find the earliest parcel to cool. For a homogeneous medium, $\Delta t_{cool}$ is proportional to $P_S^n$ and therefore to $r^{-9/2}$ or $t^{-9/5}$. Numerically, the minimum of $t + Ct^{-9/5}$ occurs at $t_1 = [9C/5]^{5/14} \approx 10$ such that the first gas to cool was initially a distance

$$r_1 \equiv r(t_1) = 19.62 \text{ pc} E_{51}^{2/7}/n_{*}^{3/7} \tag{15}$$

from the explosion site, and was shocked at time $t_1$ (see Cox & Anderson 1982, Cox 1986). The expression for $C$ can be obtained by substituting equation (2) to (14), and using formulae for the Sedov radius and velocity from Table 1. One can get then $C = \frac{1}{\sqrt{\pi}} \left(\zeta \sqrt{3}/5\right)^3 \left[n^{9/10}/(\alpha \sqrt{5})\right] \left(E_{51}^{0.6}/n_{0}^{1.6}\right)$. At the minimum, $\Delta t_{cool} = \frac{5}{2}t_1$, for other radii, $\Delta t_{cool} = \frac{5}{2}t_1(r_1/r)^{9/2}$. First parcel’s cooling is complete at $t_{cool} = \frac{14}{5}t_1$, when the remnant radius is $r_{cool} = (\frac{14}{5})^{2/5}r_1$. We define $t_{cool}$ as the cooling time of the remnant.

In our approximation for the remnant in a density gradient, a combination of equations (1), (9) and (14) gives

$$\Delta t_{cool}(a, z) = \frac{5}{9} t_1(E, n_*) \left(\frac{r_1}{abc}\right)^{3/2} \left(\frac{n_{*}}{n_0}\right)^{5/2} = 1.066 \times 10^{10} \text{ yr} \frac{E_{51}^{3/2}}{(2h_{pc})^{9/2} n_{*}^{5/2}} \frac{\exp(5z/2h)}{(a/2h)^{3/2}(b/2h)^3}, \tag{16}$$

where we substituted $\alpha = 1.3 \times 10^{-19} \text{ K}^{1/2}\text{cm}^3\text{erg s}^{-1}$ in Kahn’s formula for cooling. Noting that $t_{2h} = t_1(2h/r_1)^{5/2}$ yields

$$\Delta t_{cool}(a, z) = \frac{5}{9} t_{2h} \left(\frac{r_1}{2h}\right)^7 \frac{\exp(5z/2h)}{(a/2h)^{3/2}(b/2h)^3}. \tag{17}$$

A measure of the likelihood of finding a remnant with a cold shell only on the dense end is provided by the ratio $t_{cool}^-/t_{cool}^+$ of cooling times between the temuous and dense ends. These cooling times can be obtained by minimizing the value of the sum $\Delta t_{cool} + t(a)$ of times given by equations (17) and (12) over $x = a/2h$

$$t_{cool}^+ = t_{2h} \min_{x} \left[ I(x) + \frac{5}{9} \left(\frac{r_1}{2h}\right)^7 g_\pm(x) \right], \tag{18}$$

where the function $g_\pm(x)$ is given in Table 1, and the upper (lower) sign corresponds to the dense (tenuous) end. The formula to derive values of $x$ at minimum, $x_\pm$, is given in Table 1. One can see that $x_\pm$ is a function of only one variable: $r_1/2h = (14/9)^{2/5}r_{cool}/2h$.

The ratio of cooling times between the temuous and dense ends of the remnant is numerically close to the density contrast between the two ends, $W = \exp(2r_{cool}/h)$, at the nominal cooling radius. Figure 2 displays the ratio of cooling times as a function of $W$. Note that for cooling time ratios less than 10
(essentially all for which the parameter is interesting), it is equal to the density contrast \( W \) within 20%. From Figure 2 we see that cooling takes more than twice as long on the tenuous end for \( W > 2.4 \), or roughly speaking, for \( r_{\text{cool}} > h/2 \), yielding the unsurprising result that when the density differential between the two ends is large at the nominal onset of cooling, the evolution will contain a long period with a partial shell. This result may contribute to the explanation of the fact that we usually can see only one side of the expanding HI shells (Heiles 1979, Koo & Heiles 1991).

For completeness, we now provide an approximate evaluation of the time at which the first parcel on the equator cools, as a measure of the proximity of a remnant to having half a complete shell. This is somewhat more difficult, because the location of the equator is shifting with \( z \). Any parcel cooling at the equator was not on the equator when shocked. Nevertheless, we are able to provide a plausible approximation for the nearly spherical case. We use equation (9), which relates the post-shock pressure to the explosion energy and volume, but approximate the volume by \( V = \frac{4}{3} \pi a^3 \), and then substitute for \( a \) the solution of the Sedov 1D problem

\[
a = \left( \frac{E}{\rho_*} \right)^{1/5} \zeta^{2/5} t^{2/5}.
\]

The approximate post-shock pressure is then

\[
P_S = \frac{3}{25} \zeta^2 \left( \frac{E}{\rho_*} \right)^{1/5} t^{-6/5}.
\] (19)

From equations (14) and (19), the cooling time for a parcel on remnant’s minor axis, which is hit by the shock at the time \( t_s \), when the center of the remnant was at a distance \( s \) from the explosion site, is

\[
\Delta t_{\text{cool}}(s) \propto \frac{E^{3/5}}{n^{8/5}} t_s^{-9/5} \exp(5s/2h).
\] (20)

By assuming that \( \Delta t_{\text{cool}}(s) \approx \frac{5}{9} t_s \), which is an exact equality for the uniform medium only, an approximate formula for \( t_s \) can be derived. The corresponding approximate expression for \( t_{\text{cool}} \) is

\[
t_{\text{cool}} = \Delta t_{\text{cool}}(s) + t_s = \frac{14}{9} t_s = 4.60 \times 10^4 \text{yr} \left( \frac{E_{51}}{n_{-4}^{1/7}} \right) \exp \left( \frac{25}{28} \frac{s}{h} \right).
\] (21)

and approaches the cooling time for Sedov solution when \( h \to +\infty \). The dependence on \( s \) can be removed by expressing \( s \) in terms of \( a \) from equation (8) and substituting \( a \) from the Sedov 1D solution. Thus we find the iterative solution for the cooling time on the minor axis

\[
t_{\text{cool}} = 4.60 \times 10^4 \text{yr} \left( \frac{E_{51}}{n_{-4}^{1/7}} \right) \cosh^{25/14} \left( 0.16 \left( \frac{E_{51}}{n_*} \right)^{1/5} \left( \frac{a}{(\pi t_{\text{cool}})^{2/5}} \right) \right).
\] (22)

4. Accuracy of the Kompaneets Approximation

The assumptions of uniform post-shock pressure and an exponential atmosphere lead directly to an equation (8) for the displacement \( s \) of the remnant center from the explosion site, as a function of the semimajor axis, \( a \). In the hydrodynamical models performed for the particular case of SNR W44 by Shelton et al. (1998, Paper II), we noticed that this equation appeared to overestimate the displacement of the center. That implies that the post-shock pressure at the high density end is higher than at the low density end. In the following, we provide a rough justification for such a differential. Note that numerical studies of \( s(a) \) may yield a reasonably useful test case for intercomparison of 2D hydrocodes.

The upturn in pressure at the edge of the Sedov remnant can be attributed to the effective gravity in the decelerating post-shock gas. The post-shock mass velocity is \( v_2 = \frac{4}{3} v \), so the outwardly directed
effective gravity is roughly \(-\frac{3}{4} \dot{v}\). Assuming that the post-shock gas is in hydrostatic equilibrium, we get
\[
\frac{dP}{d\eta} \sim -\rho g \sim -3\rho_0 \dot{v},
\]
given \(\rho = 4\rho_0\) (\(\eta\) is the coordinate locally perpendicular to the shock front and directed outwards). The jump condition (2) however specifies the post-shock pressure, so the fractional gradient is
\[
\frac{1}{P} \frac{dP}{d\eta} \sim -\frac{4}{v^2} \dot{v}.
\]
From the same jump condition we have \(\frac{\dot{P}}{\dot{P}_S} = \frac{\dot{\rho}}{\rho_0} + 2 \frac{\dot{v}}{v}\). For the exponential atmosphere, \(\dot{\rho}_0 = \pm \rho_0 v/h\) at the dense and tenuous ends of the remnant respectively. Thus
\[
\frac{\dot{v}}{v} = \frac{1}{2} \left( \frac{\dot{P}_S}{P_S} - \frac{\dot{\rho}_0}{\rho_0} \right) = \frac{1}{2} \left( \frac{\dot{P}_S}{P_S} \mp \frac{1}{h} \frac{v}{h} \right),
\]
or finally, at the ends of the remnant,
\[
\frac{1}{P} \frac{dP}{d\eta} \sim -2 \left( \frac{\dot{P}_S}{P_S v} \mp \frac{1}{h} \right).
\]
In this equation, the leading term on the right-hand side is the positive gradient term present in the Sedov solution for a uniform medium. The second term is the correction factor we are looking for, positive on the high density end, negative on the low density end. It expresses the fact that the higher density end has a more rapid deceleration and therefore a steeper pressure gradient.

For the Sedov case, equation (9) gives \(-P_S/P_S v = 3/R\). With the semimajor axis \(a\) in place of \(R\), we use equation (25) to estimate the relative gradients between the two ends at
\[
\frac{\left( \frac{1}{P} \frac{dP}{d\eta} \right)}{\text{Dense}} \sim \frac{1 + \frac{a}{Q h}}{1 - \frac{a}{Q h}},
\]
Upon integrating from the central pressure plateau up the slope along the major axis, the ratios of the post shock pressures should be similar to, but less extreme than, this ratio of gradients. As a rough parameterization, we can write it as
\[
\frac{P_{S,\text{Dense}}}{P_{S,\text{Tenuous}}} \sim \frac{1 + \frac{a}{Q h}}{1 - \frac{a}{Q h}},
\]
where \(Q\) is somewhat greater than 3. At the end of this section, we provide a way of estimating the value of \(Q\) from numerical studies.

Now, we can see that the Kompaneets approximation of constant post-shock pressure holds only for \(a \leq h\), i.e. at times well before blowout. Our approximation of the shock by an ellipsoid is indistinguishable from the Kompaneets solution at this stage (the \(\Delta \eta/\eta\) deviations smaller than \(10^{-3}\)).

The post-shock pressure can be expressed as a function of time only, when \(a\) in equation (27) is approximated by the 1D Sedov solution
\[
P_S \propto (1 \pm (t/t_Q)^{2/5}) t^{-6/5},
\]
where the top sign refers to the dense end, the bottom to the tenuous one, and \((t/t_Q)^{2/5}\) has replaced \(a/Q h\). Keeping in mind that we are looking for a first order correction, we used the fact that at early times
$P_S \propto t^{-6/5}$. For a strong, non-radiative shock, $v \approx \sqrt{P_S/\rho_0} \propto e^{s/2h} \sqrt{1 \pm (t/t_Q)^{2/5}} t^{-3/5}$. Taking into account that $v_L = \frac{dz_H}{dt}$, but $v_H = -\frac{dz_L}{dt}$, we can write

$$-e^{-z_H/2h} dz_H \propto \sqrt{1 + (t/t_Q)^{2/5}} t^{-3/5} dt; \quad e^{-z_L/2h} dz_L \propto \sqrt{1 - (t/t_Q)^{2/5}} t^{-3/5} dt,$$

which after integrating gives

$$\frac{e^{-z_H/2h} - 1}{1 - e^{-z_L/2h}} = \frac{1 + (t/t_Q)^{2/5}}{1 - (t/t_Q)^{2/5}} \frac{3/2 - 1}{3/2 - 1} \approx \frac{1 + \frac{1}{4}(t/t_Q)^{2/5}}{1 - \frac{1}{4}(t/t_Q)^{2/5}} \approx \frac{1 + \frac{1}{4} \frac{4}{Q_\rho}}{1 - \frac{1}{4} \frac{4}{Q_\rho}},$$

where we first expanded the numerator and denominator to the two leading terms, and then substituted $a$ from the Sedov solution. Note that for uniform post-shock pressure, the ratio (30) is equal 1, and after substituting $z_H = s - a$, $z_L = s + a$, one gets the formula (8) for the center shift $s$.

Similarly, in the case of nonuniform post-shock pressure, we express $z_L$ and $z_H$ in terms of $a$ and $s$ and after simple algebraic manipulations we get the final formula for the center shift correction

$$\exp(s/2h) = \cosh(a/2h) - \frac{1}{4} \frac{a}{Q_h} \sinh(a/2h).$$

The first order $s$ correction is about 4 times smaller than the $P_S$ correction in equation (27). The center shift correction derived above applies to models with no cooling; for numerical models, one could plot the quantity

$$\frac{\cosh(a/2h) - \exp(s/2h)}{(a/2h) \sinh(a/2h)} = \frac{1}{2Q}$$

to evaluate the free parameter $Q$ in our method.

5. Applications to hydrodynamical models and conclusions

Assuming that the shock wave propagating in an exponentially stratified medium takes the shape of an ellipsoid, we were able to find explicit expressions for its size and expansion velocity as functions of time, and for the cooling time at both ends and on the equator. As presented in Table 1, these expressions take forms similar to the Sedov (1959) solution for a uniform ambient medium. Nevertheless, while the Sedov solution is self-similar, there are no such solutions for nonuniform media. The reason is that there are at least three independent dimensional parameters (in our case $E$, $\rho_\ast$ and $h$). A successful model of a SNR should adopt parameters that reproduce the observed quantities; for example the size, expansion velocity and progress of shell formation. If a pulsar is seen, the age can be derived as well. Our simple formulae provide quick ways to explore this 3-dimensional parameter space.

A useful approach is to generate topographic (or contour) plots on the $\rho_\ast h$ plane for assumed values for $E$ and linear size of the remnant. The contoured quantities include the time required to reach the assumed size, the post-shock pressure, the cooling and shell formation time-scales for the dense, equatorial, and tenuous directions, and the shock velocities in those directions. Although the Kompaneets model, like the Sedov model, cannot be used very reliably to estimate quantities after shell formation has occurred, we make the usual approximation that shortly after shell formation the shell velocity is about three-quarters of the shock velocity predicted for the non-radiative evolution. By this means of exploring the parameter space, a set of parameters fully reproducing the observables for SNR W44 was found and is given in Paper II.
There are other asymmetric SNRs which can be interpreted in terms of expansion into a nonuniform medium, for example a partially open H\textsc{i} shell in CTB 80 (Koo et al.1990), and an incomplete shell of CTA 1 (Pineault et al.1993). The semi-circular shape of CTB 109 is ascribed to interaction with a molecular cloud (Tatematsu et al.1990). G 84.2 -0.8 (Feldt & Green 1993) shows a striking resemblance to W44 in radio continuum, though observations of the H\textsc{i} shell are not convincing.

In the catalogue of SNRs with H\textsc{i} shell emission (Koo & Heiles 1991, Table 3) all sources show high-velocity H\textsc{i} gas on one side only: either receding or approaching. Heiles (1979), who catalogued H\textsc{i} shells in our Galaxy, noticed that for most of them, he could see only the approaching or only the receding hemisphere – a fact that he called “disturbing”. Putting aside possible observational biases, a partially formed shell appears to be more a rule than an exception. We want to point out that explanation of such asymmetric SNRs or shells does not require any abrupt density change, such as encountering the edge of a molecular cloud (Dohm-Palmer & Jones 1996). The method of estimation of the cooling time presented in §3 shows that the ratio of the cooling times on the dense and tenuous ends of the remnant is of the order of the ratio of ambient density extremes. We can see one side of the H\textsc{i} shell only, because at first there is only one side, and then by the time the other side is formed, either the part expanding into denser medium has slowed down and its emission blends with that of local gas or the tenuous end remains invisible due to the high column density contrast between low and high density ends (Silich 1992).

Our explicit analytical approximation provides an efficient tool for a quick exploration of the space of initial parameters for the SNR models. It can be used to select the initial parameters of hydro runs, which in turn can give us a detailed insight into the structure of the remnant, and can verify the above conclusions.

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REFERENCES

### Table 1.
Approximate Formulae for a SNR Expanding in a Stratified Medium

<table>
<thead>
<tr>
<th>SEDOV 1D</th>
<th>ELLIPSOIDAL 2D (EXPONENTIAL STRATIFICATION)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = \zeta^{-5/2} \left( \frac{E}{\rho_*} \right)^{-1/2} r^{5/2} )</td>
<td>( t = \zeta^{-5/2} \left( \frac{E}{\rho_*} \right)^{-1/2} (2h)^{5/2} I \left( \frac{a}{2h} \right) )</td>
</tr>
<tr>
<td>size (-) time</td>
<td>( t , [\text{yr}] = 17.32 \frac{r^{5/2}}{E^{1/2}} \frac{\rho_<em>^{1/2}}{n_</em>^{1/2}} )</td>
</tr>
<tr>
<td>( t , [\text{yr}] = 17.32 \frac{r^{5/2}}{E^{1/2}} \frac{\rho_<em>^{1/2}}{n_</em>^{1/2}} )</td>
<td>( t , [\text{yr}] = 17.32 \frac{(2h)^{5/2}}{E^{1/2}} \frac{n_<em>^{1/2}}{n_</em>^{1/2}} I \left( \frac{a}{2h} \right) )</td>
</tr>
</tbody>
</table>

where \( I(x) \equiv \frac{5}{2} \int_0^x dy \frac{\sqrt{y}}{\cosh y} \arctanh y \simeq \frac{3x^{5/2}}{3 + 2x^{5/2}} \) for \( x < 3 \)

| \( v_s = \frac{2}{5} \zeta^{5/2} \left( \frac{E}{\rho_*} \right)^{1/2} r^{-3/2} \) | \( v_s = \frac{2}{5} \zeta^{5/2} \left( \frac{E}{\rho_*} \right)^{1/2} (2h)^{-3/2} f_\pm \left( \frac{a}{2h} \right) \) |
| velocity \(-\) size | \( v_s \, [\text{km/s}] = 22600 \left( \frac{E_{51}}{n_*} \right)^{1/2} \frac{r_{pc}^{1/2}}{r_{pc}^{1/2}} \) |
| \( v_s \, [\text{km/s}] = 22600 \left( \frac{E_{51}}{n_*} \right)^{1/2} \frac{r_{pc}^{1/2}}{r_{pc}^{1/2}} \) | \( v_s \, [\text{km/s}] = 22600 \left( \frac{E_{51}}{n_*} \right)^{1/2} (2h_{pc})^{-3/2} f_\pm \left( \frac{a}{2h} \right) \)

where \( f_\pm(x) \equiv \exp(\pm x) \cosh x \sqrt{x} \arctanh x \).

| \( t_{cool} \, [\text{yr}] = 4.6 \times 10^4 \frac{E_{51}^{3/14}}{n_*^{4/7}} \) | \( t_{cool} \, [\text{yr}] = 17.32 \frac{(2h_{pc})^{5/2}}{E^{1/2}} \frac{n_*^{1/2}}{n_*^{1/2}} I(x_{\pm}^+ + A g_\pm(x_{\pm}^+)) \) |
| cooling time | where \( g_\pm(x) \equiv \frac{\cosh x \exp(\pm 5x)}{x^{3/2} (\arctanh x)^3} \), \( A = 6.2 \times 10^8 \frac{E_{51}^2}{n_*^2 (2h_{pc})} \)

and \( x_{\pm}^\pm \) is set by \( A = \frac{5}{2} \frac{x^2 \exp(\pm 5x) (\arctanh x)^4}{2 \cosh^6 x (\pm 5 e^{\mp x} + \frac{3 \cosh x}{2x} + \frac{3}{\arctanh x})} \).

Units: \( h_{pc} \, [\text{pc}] \), \( r_{pc} \, [\text{pc}] \), \( n_* \, [\text{cm}^{-3}] \), \( E_{51} \, [10^{51} \text{erg}] \)

* the top signs refer to expansion into the dense medium, the bottom into the tenuous medium; for the shock expansion velocity in the equatorial direction, the exponential factor should be dropped from \( f_\pm(x) \).