Spacetime Duality
and Two–dimensional Gauge Field Theory

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Abstract

In this letter we implement a recently proposed spacetime duality approach to dualize a two dimensional, Abelian, gauge field theory, which has no dual version under p–duality. Our result suggests that spacetime duality spans a new, wider, class of dual theories, which cannot be related one to another by p–duality transformations.

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Recently, a new proposal for dualizing field theories has been put forward [1]. Contrary to the approach based on internal gauge symmetries, known as p–duality (a good review of p–dualities, with an extensive reference list can be found in [2] ), the new approach exploits local spacetime symmetry as the basic tool to obtain a dual theory. In [1] such approach has been applied in two dimensions to scalar and fermion theories, as well as to their super–symmetric extensions. The resulting dual model was a (super) scalar field theory. Only in the case of (2, 2) super–symmetric models, dual theory exhibits a certain degree of complexity due to the exchange of chiral and twisted chiral super fields. These results are in agreement with the p–duality approach relating \( p \)-forms to \( D - p - 2 \) dual forms. Accordingly, for \( D = 2 \) the only allowed duality relation is between \( p = 0 \) forms, i.e. scalar fields are dual to scalar fields. Nonetheless, spacetime approach to dualities bears no obvious requirement of equivalence to p–dualities, because the two approaches follow from different symmetries.

In this letter we would like to show that the spacetime approach allows to establish new duality relations which cannot be accounted for in the p–duality approach. The massless, Abelian, vector model, in two dimensions, is a gauge theory without a dual version in the p–duality sense, but can be consistently coupled to gravity and dualized following the path integral formulation outlined in [1]. The classical action for the gauge field \( A_\mu \) coupled to the background metric \( h_{\mu\nu} \) is given by

\[
S[h, A, \overline{c}, c] = \int \! d^2x \sqrt{-h} \left[ -\frac{1}{4} h^{\mu\alpha} h^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{2} (\nabla_h h^{\mu} A_{\mu})^2 + \overline{c} \Box_h c \right]
\] (1)

where, \( \nabla_h h^\mu \) is the covariant derivative with respect to the background metric and \( \Box_h \) is the covariant D’Alembertian. We have also introduced a gauge fixing term and a pair of Fadeev–Popov ghosts \( \overline{c}, c \). Having coupled the gauge field to a background metric we can define a non–trivial generating functional as

\[
Z[h] = \int \! [DA]_h [J_{FP}]_{A,h} \exp iS[h, A]
\] (2)
In (2), the ghost action has been written in terms of the Fadeev–Popov determinant \([ J_{FP} ]_{A,h} \). The basic difficulty in functional integration is the definition of the functional measure. Quantum anomalies can appear as a result of the impossibility to define a functional measure preserving all the classical symmetries [3]. If one insists on having a covariant measure necessarily ends up producing a conformal anomaly. Therefore, it is useful to put in evidence conformal degree of freedom within the functional measure. For this purpose one first chooses a decomposition of the metric as \( h_{\mu\nu} = \hat{h}_{\mu\nu} e^\varphi \), where \( \hat{h}_{\mu\nu} \) is the traceless part of \( h_{\mu\nu} \). Then, the measure of an arbitrary field \( X \) transforms as

\[
[D X]_h = [D X]_{\hat{h}} J[\hat{h}, \varphi]
\]  

(3)

where, \( J[\hat{h}, \varphi] \) is the Jacobian corresponding to the above decomposition of the metric. The Jacobian is a local quantity and can be written in terms of the local part of the anomalous gravitational action [4] given by the Liouville action \( S^X_L[\hat{h}, \varphi] \):

\[
S^X_L[\hat{h}, \varphi] = \int d^2x \sqrt{-\hat{h}} \left[ \frac{1}{2} \varphi \Box_{\hat{h}} \varphi + R_{\hat{h}} \varphi \right]
\]  

(4)

The generating functional (2) can then be written as

\[
Z[\hat{h}] = \int[D A]_{\hat{h}} J[\hat{h}, \varphi] [J_{FP}]_{A,\hat{h}} \exp iS[\hat{h}, \varphi, A]
\]  

(5)

One may notice appearance of the gauge Fadeev–Popov determinant in the form \([ J_{FP} ]_{A,\hat{h}} \) instead of the originally introduced \([ J_{FP} ]_{A,h} \), which is due to the conformal invariance of the scalar ghost action for the gauge field.

Gravitational anomalous action can be obtained from (5) integrating out the gauge field. Classical actions for massless scalar and fermion field are conformally invariant in two dimensions, while the gauge field action is not. Classical non–invariance has to be appropriately accounted for within the path integral approach.

Any classically non–invariant action can be split into an invariant piece and a classical trace contribution \( S^{\text{non–inv}}[X, \hat{h}] = S[X, \hat{h}] + S^{\text{tr}}[X, \hat{h}, \varphi] \). The invariant piece of the classical action contributes a numerical coefficient \( k_X \) to the anomalous gravitational action, while the
trace part contributes a different numerical coefficient $k_X^{tr}$. Both coefficients can be computed from the finite part of the anomalous Feynman graph a la Adler–Rosenberg [5], in diagrammatic perturbation theory, or can be extracted from the finite part of the regularized Jacobian in the Fujikawa approach [3]. In addition there is a finite and local contribution to the gravitational anomalous action which is extracted, in the process of regularization, from the divergent part of the anomalous diagram and whose numerical coefficient is an arbitrary, regularization dependent, parameter “a” [6]. The freedom in the choice of “a” allows one to shift among various regularization schemes. This coefficient is now appearing in front of the Liouville action of the functional measure and is the same both for classically invariant and for classically non–invariant theories. The evaluation of the functional integral for classically non–invariant theories proceeds as follows

$$Z[h] = \int [DX]_h \exp (iS[X,h])$$

$$= \exp \left( iaS_L[\varphi,\hat{h}] + ikX S[\hat{h}] + i\frac{k^{tr}_{X}}{X}S[\hat{h},\varphi] \right)$$

(6)

where, $S[h] = \int d^2x \sqrt{-\hbar} R(h) \Box \hbar^{-1} R(h)$ and various numerical coefficients have been put in evidence. The above result generalizes the path integral anomaly calculation to the case of classically conformally non–invariant actions. In order to compare (6) to the result of an explicit calculation, such as in [5], one has to choose a definite regularization scheme. Suppose one wants to define a diffeomorphism invariant measure. This requires $aS_L[\varphi,\hat{h}] + k_X S[\hat{h}] = k_X (S_L[\varphi,\hat{h}] + S[\hat{h}]) = k_X S[h]$ which gives $a = k_X$ within a covariant regularization scheme. One can give a general formula for evaluating numerical coefficients of the invariant piece of the anomalous gravitational action through the relation [7]

$$[\det D_q]^{-N/2^{s+1}} = \exp \left[ i \frac{N}{48\pi} (-1)^s \left( 6q^2 - 6q + 1 \right) S[q] \right]$$

(7)

where, $q$ is the Lorenz weight of the field, $D_q$ is the covariant operator acting on that field, $N$ is the number of fields and $s$ denotes statics of the field ($s = 0$ for bosons, $s = 1$ for
fermions). From (7) one sees that \( k_X = N(-1)^s(6q^2 - 6q + 1) \).

The anomalous gravitational action for a classically conformally non–invariant action turns out to be

\[
Z[h] = \exp \left( i k_{X}^{\text{non-inv}} S[h] \right) \tag{8}
\]

with \( k_{X}^{\text{non-inv}} = k_X + k_X^{tr} \). The path integral evaluation of gravitational anomalies thus reproduces results obtained in [5] by other methods. Classically conformally invariant action result is obtained simply putting \( k_X^{tr} = 0 \). With the above generalization of functional integration of matter fields in mind, one proceeds on the path of dualizing gauge theory. Following [1], we promote the background metric \( h_{\mu\nu} \) into a dynamical gravitational field \( g_{\mu\nu} \) which serves to dualize the original action (1). The corresponding mother generating functional is defined as

\[
Z[h] = \int [\mathcal{D}A]_g [\mathcal{D}g]_g \Delta[g,h] \left[ J_{FP} \right]_{A,g} \exp i S[g,A] \tag{9}
\]

\( \Delta[g,h] \) denotes a constraint condition to be chosen on the gravitational field \( g_{\mu\nu} \) in such a way to restrict it to the background field \( h_{\mu\nu} \). Then, (9) will reproduce the starting functional (2). One chooses the “ conformal gauge ” condition \( g_{\mu\nu} = e^{\phi} h_{\mu\nu} \). This gauge choice fixes the diffeomorfism transformations and, thus, introduces in (9) a gravitational Fadeev–Popov determinant which can be expressed in terms of the gravitational ghost anomalous action as \( [J_{FP}]_g = \exp ( i k_{gh} S[g] ) \) where \( k_{gh} \) is the ghost coefficient. As an example of calculation of the numerical coefficient, take a pair of gravitational ghost–antighost \((B_{\mu\nu}, C_\mu)\) corresponding to the Fadeev–Popov determinant of the conformal gauge for which \( q_B = 2, N = 2, s = 1 \). Then, one finds the known value \( k_{gh} = -26 \).

One defines the constraint \( \Delta[g,h] \) as a functional integral over the Lagrange multiplier field \( \Lambda \) as

\[
\Delta[g,h] = \int [\mathcal{D}\Lambda]_g G(g) \exp \left[ -i \int d^2x \sqrt{-g} \Lambda F(h,\phi) \right] \tag{10}
\]

where \( G(g) \) is to be determined in the process of reproducing the original matter generating functional (2). The functional integral (10) is introduced in order to remove conformal
degrees of freedom. Therefore, it should be proportional to the delta function \( \delta[\phi] \). This imposes that the integrand in the exponential factor be linear in the conformal degree of freedom while the rest must depend on the background metric only as \( \sqrt{-g} F(h, \phi) = \sqrt{-h} F_h \phi \). The mother functional, can now be written, as

\[
Z[h] = \int [DA]_h [D\phi]_h [D\Lambda]_h [J_{FP}]_{A,h} [J_{FP}]_g G(g) \exp iS[\phi, A] \times \\
\exp i \left[ k_A S_L^A(g) + k_\phi S_L^\phi[g] + k_\Lambda S_L^\Lambda[g] \right] \exp \left[ -i \int d^2x \sqrt{-h} \Lambda F_h \phi \right]
\]

where, we have put in evidence the conformal degree of freedom of the functional measure through the Liouville actions with corresponding numerical coefficients. The notation bears explicit reference to different fields in order that the reader can keep track of various terms coming from different functional measures. It may be worth to remark that all the Liouville actions have the same form (4) and could be re–written as \( \left( \sum_f k_f \right) S_L[g] \), where \( \sum_f \) means summing over various fields.

Basic requirement on the mother functional is to reproduce the starting, gauge, generating functional after integrating out all the other field variables but \( A_\mu \). Let us start with the integration of the Lagrange multiplier. One obtains

\[
\int [DA]_h \exp \left[ -i \int d^2x \sqrt{-h} \Lambda F_h \phi \right] = \delta[F_h \phi] = (det F_h)^{-1} \delta[\phi]
\]

which enables us to rewrite the mother functional as

\[
Z[h] = \int [DA]_h [J_{FP}]_{A,h} \exp iS[h, A][D\phi]_h \delta[\phi] \left( det F_h \right)^{-1} \times \\
[J_{FP}]_g G(g) \exp \left[ i \left( \sum_f k_f \right) S_L[g] \right] \exp iS^{tr}[h, \phi, A]
\]

Integration over the delta function removes \( \phi \)-dependent terms giving

\[
Z[h] = \int [DA]_h [J_{FP}]_{A,h} \exp iS[h, A] \left( det F_h \right)^{-1} [J_{FP}]_h G(h)
\]

From (14) one can see that the starting functional (2) can be reproduced if

\[
( det F_h )^{-1} [J_{FP}]_h G(h) = 1
\]
The above condition determines the form $G(h)$ in terms of the Fadeev–Popov determinant for gravitational ghosts and the determinant of the operator $F_h$. This, however, does not determine the part of $G(h, \phi)$ which has been eaten by the delta function $\delta(\phi)$. We eliminate this arbitrariness by the choice $G(h, \phi) \left[ J_{FP} \right]_{h,\phi} = 1$. Inserting (15) in (11) one can write a more convenient form of the mother generating functional which serves to produce the dual theory as

$$Z[h] = \int [DA]_h [D\phi]_h [DA]_h [J_{FP}]_{A,h} \exp iS[g, A] (det F_h) \times \exp \left[ i \left( \sum_f k_f \right) S_L[g] \right] \exp \left[ -i \int d^2 x \sqrt{-h} \Lambda F_h \phi \right]$$

(16)

From this point on, we shall follow the dual route. Starting with (16) one performs a reversed order of functional integration and first integrate out the gauge field to obtain:

$$Z^{\text{dual}}[h] = (det F_h) \exp \left( ik_A^{\text{non-inv}} S_A[h] \right) \int [DA]_h [D\phi]_h \exp \left[ i \left( \sum_{f'} k_{f'} \right) S_L[g] \right] \times \exp \left[ -i \int d^2 x \sqrt{-h} \Lambda F_h \phi \right]$$

(17)

where, $\sum_{f'} = \sum_f + k_A^{\phi}$. The explicit form of the Liouville action (4) can be written in a condensed notation as $^1 S_L[h, \phi] = \int d^2 x \sqrt{-h} \left( \phi L_h \phi + M_h \phi \right)$, where $L_h$ and $M_h$ stand, respectively, for the Liouville kinetic operator and the non–minimal, linear coupling between $h_{\mu\nu}$ and $\phi$. The Gaussian integration over $\phi$ gives

$$Z^{\text{dual}}[h] = (det F_h) \exp \left( ik_A^{\text{non-inv}} S_A[h] \right) \int [DA]_h (det L_h)^{-1/2} \times \exp \left[ -i \int d^2 x \sqrt{-h} \left( \Lambda F_h + M_h \right) L_h^{-1} \left( \Lambda F_h + M_h \right) \right]$$

(18)

To simplify the above expression we introduce the new field variable

$$\hat{\phi} = L_h^{-1} \left( \Lambda F_h + M_h \right)$$

(19)

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1 We have rescaled conformal degree of freedom in such a way to absorb numerical coefficient $\sum_{f'} k_{f'}$.  

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Accordingly, the functional measure for the Lagrange multiplier field transforms as

\[ [D\hat{\phi}]_h = (\det L_h)^{-1} (\det F_h) [D\Lambda]_h \]  

(20)

and the dual generating functional turns out to be of the form

\[ Z_{\text{dual}}[h] = (\det L_h)^{1/2} \exp \left( ik_{A}^{\text{non-inv}} S_A[h] \right) \int [D\hat{\phi}]_h \exp \left[ -i \int d^2 x \sqrt{-h} \hat{\phi} L_h \hat{\phi} \right] \]  

(21)

From (21) one can see the cancellation of \((\det F_h)\) which shows that the actual form of \(F_h\) is not really important. One could have as well started with the simple \(\delta(\phi)\) corresponding to the choice \(F_h = 1\). The only reason we can think in favor of choosing a covariant constraint \(\delta(R(g) - R(h)) \equiv \delta(\Box \phi)\), exploited in [1], is purely aesthetical, and because it also mimics covariant Lorenz gauge fixing in \(p\)-duality approach. The coefficient \(k_{A}^{\text{non-inv}}\) has been calculated in [5] and results to vanish, \(k_{A}^{\text{non-inv}} = 0\). Therefore, (21) reduces to

\[ Z_{\text{dual}}[h] = (\det L_h)^{1/2} \int [D\hat{\phi}]_h \exp \left[ -i \int d^2 x \sqrt{-h} \hat{\phi} L_h \hat{\phi} \right] \]  

(22)

The dual action (21) turns out to be of the same form for all fields with numerical coefficients of the conformally invariant piece of the classical action given by (7), while the trace coefficients (if any) have to be calculated case by case. The scalar (fermion) anomalous action cancels against \((\det L_h)\). The final result is a scalar–to–scalar duality, exactly like in the \(p\)–duality approach, or fermion bosonization [1]. In the case of the gauge field, there is no cancellation, and we are left with the determinant of the Liouville kinetic operator \(L_h\) together with the scalar field \(\hat{\phi}\). One can re–write the remaining determinant as a functional integral of another scalar field which, however, has to be ghost–like due to the positive power of the determinant exponential factor \(+1/2\). So, we obtain the action dual to the gauge field action as

\[ Z_{\text{dual}}[h] = \int [D\hat{\phi}]_h [D\xi]_h \exp \left[ -i \int d^2 x \sqrt{-h} \left( \hat{\phi} L_h \hat{\phi} + \xi L_h \xi \right) \right] \]  

(23)

We have shown that dual action for the gauge field in two dimensions exist within the spacetime approach to dualities. The result (23) cannot be obtained in the \(p\)-duality approach, and suggests the existence of a new, larger, class of dual theories. With hindsight,
a non–trivial dual action for a two dimensional vector gauge field is not a surprise and can be explained as follows. In the widest sense, dual theories should be such to preserve the number of degrees of freedom in passing from one to another. The explanation of our result is then clear. One starts with the gauge field carrying no physical degrees of freedom in two dimensions, and ends up with the dual theory written in terms of one physical scalar field and one scalar ghost which, together, still carry no physical degrees of freedom. The price one has to pay is that the dual theory contains more than one field, which is not high price at all. The possibility to have a wider class of dual theories in the above sense is by no means inherent to two dimensions. The procedure described in this letter has no obvious dependence on the dimensionality of the physical spacetime. It should be, therefore, applicable to higher than two dimensions provided the Liouville action remains local and Gaussian. The locality of the Liouville action is expected on the grounds that it is a synonym for a regularization counter–term contribution, as explained earlier, which is always local. In particular, the four dimensional analogue of the two–dimensional gauge field is the third rank anti–symmetric tensor field with no dynamical degrees of freedom. It is expected, therefore, to be dualized in terms of scalar and ghost–like fields in the above sense. Moreover, there is no a priori reason for higher dimensional cancellation of the various determinants in (21), even for scalar fields. It is reasonable to expect that dual theories will in general belong to the wider class of dualities described in this paper. We hope to return to this point in future.
REFERENCES


