A Quasi-Exactly Solvable $N$-Body Problem with the $sl(N+1)$ Algebraic Structure

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Abstract

Starting from a one-particle quasi-exactly solvable system, which is characterized by an intrinsic $sl(2)$ algebraic structure and the energy-reflection symmetry, we construct a daughter $N$-body Hamiltonian presenting a deformation of the Calogero model. The features of this Hamiltonian are (i) it reduces to a quadratic combination of the generators of $sl(N+1)$; (ii) the interaction potential contains two-body terms and interaction with the force center at the origin; (iii) for quantized values of a certain cohomology parameter $n$ it is quasi-exactly solvable, the multiplicity of states in the algebraic sector is $(N+n)!/(N!n!)$; (iv) the energy-reflection symmetry of the parent system is preserved.
1 Motivation and results

A hidden Lie-algebraic structure of certain Hamiltonians allows one to generate quasi-exactly solvable (QES) spectral problems [1]–[4] with quantized cohomology parameters, determining a part of the spectrum which is tractable algebraically. Several $N$-body problems of this type were suggested recently [5]–[7]. The most instructive for our purposes is the model constructed in Ref. [7]. It presents an $sl(2)$-based deformation of the Calogero model [8], which was, in turn, shown to be Lie-algebraic in the fundamental work [9]. Here we report a quasi-exactly solvable $N$-body problem obtained as an $sl(N + 1)$ deformation of the Calogero model.

The Hamiltonian of the model has the form

\[
H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \left\{ -[4n + 4\alpha(N - 1) + 4\beta + 3]x_i^2 + x_i^6 + \frac{g}{4x_i^2} \right\} + g \sum_{i,j=1}^{N} \left\{ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right\},
\]

where $g$ is a positive constant, the constants $\alpha$ and $\beta$ are defined as

\[
\alpha = \frac{\sqrt{1 + 4g} + 1}{2}, \quad \beta = \frac{\sqrt{1 + g} + 1}{4},
\]

and $n$ is the quantized cohomology parameter taking integer values, $n = 0, 1, 2, \ldots$. The number $\nu$ of states in the algebraic sector is related to $n$ as follows:

\[
\nu = \frac{(N + n)!}{N!n!}.
\]

The first line in Eq. (1) presents the kinetic term plus a self-interaction while the second line contains a two-body coupling plus “mirror-reflected” terms. In Sec. 2 this Hamiltonian will be shown to be reducible to a quadratic combination of the generators of $sl(N + 1)$. In Sec. 3 we consider a specific example, while Sec. 4 is devoted to generalizations. Note that, omitting the $x^2$ and $x^6$ terms in the first line, one arrives at a particular case of the so-called $BC_n$ model. The latter was shown to be exactly-solvable, its Lie-algebraic representation is based on $sl(N + 1)$ [10]. Thus, the quasi-exactly solvable system (1) we present, can be also viewed as a deformation of the $BC_n$ model. In fact, key technical elements of the proof are essentially the same as those worked out in Refs. [7, 9, 10].

As well-known (see [11]), the original Calogero model can be obtained from the one-dimentional harmonic oscillator,

\[
H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2.
\]
To this end one observes that the eigenfunctions of the spectral problem (4) have the form
\[ \psi_n(x) = P_n(x) e^{-x^2/2}, \]  
where \( P_n(x) \) is a polynomial of degree \( n \). It can be always parametrized by the position of its zeros,

\[ P_n(x) \propto (x - x_1) \cdots (x - x_n). \]  

A generic time-dependent wave function

\[ \Psi(t, x) = \sum_n C_n \psi_n(x) e^{-iE_nt} \]  

can be written as

\[ \Psi(t, x) \propto (x - x_1(t)) \cdots (x - x_n(t)) e^{-x^2/2}, \]  
where \( x_i(t), (i = 1, \ldots, n) \) are time-dependent functions.

One can check [11] that the equations of motion for \( x_i(t) \) are those following from the Lagrangian

\[ L_{\text{daughter}} = \frac{1}{2} \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} \left\{ \sum_{i=1}^{N} x_i^2 + g \sum_{i,j=1 \atop i \neq j}^{N} \frac{1}{(x_i - x_j)^2} \right\} \]  

at \( g = 1 \). This is a particular case of the original Calogero system [8]. Since the energy eigenvalues of the harmonic oscillator are commensurate, it is quite obvious that the system (9) has infinitely many periodic (complex) classical trajectories, which is indicative of the exact solvability. Indeed, as well-known, the quantum Calogero model is exactly solvable [8, 9].

Recently it was suggested [12] to apply a similar strategy to generate a deformed Calogero model from the one-dimensional polynomial QES system with the sextic interaction. Starting from a sextic QES potential of a general form, a deformed classical Calogero Lagrangian was obtained, with the quadratic, quartic and sextic self-interaction terms, see Eq. (26) in [12]. A somewhat simplified form of this Lagrangian which will be useful for our purposes (see below) is

\[ L_{\text{daughter}} = \frac{1}{2} \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} \sum_{i=1}^{N} \left\{ -2Ax_i^2 + x_i^6 \right\} - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{N} \frac{1}{(x_i - x_j)^2}, \]  

where \( A \) is a numerical constant.

We use, as a starting point, a special case of the polynomial QES system possessing the energy-reflection (ER) symmetry [13, 14]:

\[ H = \frac{1}{2} \left[ -\frac{d^2}{dx^2} + x^6 - (8j + 3)x^2 \right]. \]  

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If the parameter $j$ is chosen to be 

$$j = \frac{N}{2},$$

the wave functions belonging to the algebraic sector have the form (see [13])

$$\psi_n = P_{2N}(x) e^{-x^4/4}$$  \hspace{1cm} (12)

where

$$P_{2N}(x) \propto (x - x_1) \cdots (x - x_N)(x - x_{N+1}) \cdots (x - x_{2N}),$$ \hspace{1cm} (13)

with the additional condition

$$x_{N+i} = -x_i, \quad i = 1, \ldots, N.$$ \hspace{1cm} (14)

Since $N = 2j$ and $j$ is half-integer, the parameter $N$ is integer, of course. We stress that the polynomial in Eq. (13) is of degree $2N$ rather than $N$.

By considering linear combinations of the type (7) we convert the parameters $x_i$ into time-dependent functions. The classical daughter Lagrangian, from which the classical equations of motion for $x_i(t)$ ensue, is that of Eq. (10).

Why we require the parent system to have the ER symmetry? In the general case neither of the trajectories $x_i(t)$ obtained in [12] is periodic. This is due to the fact that the energy eigenvalues are not commensurate. If, however, the parent system is ER symmetric, as in Eq. (11), then one can consider special combinations (7) of the type

$$\Psi(t, x) = A \psi_E(x) e^{-iEt} + B \psi_{-E}(x) e^{iEt}$$

where \{$\psi_E$, $\psi_{-E}$\} is any of the ER pairs of the system (11); there are $[(N + 1)/2]$ such pairs (here the square brackets denote the integer part).

It is obvious that this choice gives rise to a periodic (complex) trajectory for $x_i(t)$. The number of such trajectories in the daughter system following from (7) is $[(N + 1)/2]$. (Compare this with the harmonic oscillator and the daughter Calogero model where the number of periodic trajectories is infinite.) This gives us a hint that a daughter system obtained from (11) may be quasi-exactly solvable.

To build the quasi-exactly solvable system based on $sl(N + 1)$ we impose the constraints (14). More exactly, we allow for a general set of the coupling constants preserving the structure of (10). The coefficients in front of $\partial^2/\partial x_i^2$ and $x_i^6$ can be always made equal to unity by an overall rescaling of variables and the Hamiltonian. The coefficient in front of the two-body coupling term is set to $g$ where $g$ is an arbitrary positive constant (the positivity is needed for the stability of the problem). The coefficients in front of $x_i^2$ and $x_i^{-2}$ are then unambiguously fixed by the requirement of the quasi-exact solvability. In this way we arrive at the Hamiltonian (1).

It is worth emphasizing that the constraint (14) is absolutely crucial, the number of degrees of freedom in the Hamiltonian (1) is twice smaller than the number of zeroes in Eq. (13), and, hence, twice smaller than the number of degrees of freedom in (10).
2 Proof of algebraization

To uncover the hidden algebraic structure of the Hamiltonian (1) we, first, perform the change of variables,

\[ x_i \rightarrow \xi_i \equiv x_i^2. \]  

(15)

In this way we arrive at

\[
\mathcal{H}_\xi = - \sum_{i=1}^{N} \left( 4\xi_i \frac{\partial^2}{\partial \xi_i^2} + 2 \frac{\partial}{\partial \xi_i} \right) + \sum_{i=1}^{N} \left\{ - [4n + 4\alpha(N - 1) + 4\beta + 3]\xi_i + \xi_i^3 + \frac{g}{4\xi_i} \right\} \\
+ 2g \sum_{i,j=1 \atop i \neq j}^{N} \xi_i + \xi_j \frac{(\xi_i - \xi_j)^2}{(\xi_i - \xi_j)^2}.
\]  

(16)

The next step of the program (for a review see [13]) is a gauge transformation (sometimes referred to as similarity transformation)

\[ \mathcal{H}_\xi \rightarrow \mathcal{H}_G = e^a \mathcal{H}_\xi e^{-a}, \]  

(17)

where the phase \( a \) appropriate to the Hamiltonian (16) is

\[ a = \ln \Psi_0, \quad \Psi_0 = \prod_{i,j=1 \atop i < j}^{N} (\xi_i - \xi_j)^{-\alpha} \prod_{i=1}^{N} \xi_i^{-\beta} \exp \left\{ \frac{1}{4} \sum_{i=1}^{N} \xi_i^2 \right\}. \]  

(18)

On physical grounds it is clear that the constants \( \alpha \) and \( \beta \) must be positive. In fact, as will be seen shortly, these constants are the positive roots of the equations

\[ \alpha^2 - \alpha = g, \quad 4\beta^2 - 2\beta = \frac{1}{4} g, \]  

(19)

see Eq. (2). The parametrization of the phase \( a \) above generalizes that exploited in the earlier proofs of the algebraization of the Calogero and \( BC_n \) models. The transformation (17) is non-unitary. The spectra of \( \mathcal{H}_\xi \) and \( \mathcal{H}_G \) are the same, however, while the eigenfunctions are related as follows:

\[ \psi_\xi = \psi_G e^{-a}. \]

Under the gauge transformation (17)

\[ \frac{\partial}{\partial \xi_i} \rightarrow e^a \frac{\partial}{\partial \xi_i} e^{-a} = \frac{\partial}{\partial \xi_i} + \sum_{j=1 \atop j \neq i}^{N} \alpha \xi_i - \xi_j + \beta \frac{1}{2} \xi_i. \]  

(20)
The corresponding transformation of \( \partial^2 / \partial \xi_i^2 \) follows from (20) with the use of the identity
\[
\sum_{i,j,k=1 \atop i \neq j \neq k}^N \frac{\xi_i}{(\xi_i - \xi_j)(\xi_i - \xi_k)} = 0. \tag{21}
\]
It is not difficult to get that
\[
H_G = -\sum_{i=1}^N \left\{ 4 \xi_i \frac{\partial^2}{\partial \xi_i^2} + (8\beta + 2) \frac{\partial}{\partial \xi_i} - 4 \xi_i^2 \frac{\partial}{\partial \xi_i} + 4n \xi_i \right\}
- 4\alpha \sum_{i,j=1 \atop i \neq j}^N \frac{1}{\xi_i - \xi_j} \left( \xi_i \frac{\partial}{\partial \xi_i} - \xi_j \frac{\partial}{\partial \xi_j} \right). \tag{22}
\]
The terms proportional to \( \xi_i^3 \), \( \xi_i^{-1} \) and \( (\xi_i - \xi_j)^{-2} \) in Eq. (16) cancel after the gauge transformation. The cancellation is ensured by the condition (19).

Now we are just one step short of revealing the \( sl(N + 1) \) algebraic structure of the Hamiltonian (1). Following [9, 10], we pass from the original set of variables \( \{\xi_1, \ldots, \xi_N\} \) to a new set \( \{\tau_1, \ldots, \tau_N\} \) where \( \tau_i \)'s are the elementary symmetric polynomials,
\[
\tau_1 = \sum_{i=1}^N \xi_i, \quad \tau_2 = \sum_{i,j=1 \atop i < j}^N \xi_i \xi_j, \ldots, \quad \tau_N = \xi_1 \xi_2 \cdots \xi_N. \tag{23}
\]
It is convenient to define, additionally, \( \tau_0 = 1 \) and \( \tau_j = 0 \) if \( j < 0 \) or \( j > N \). A collection of formulae helpful in rewriting \( H_G \) in terms of \( \tau_i \)'s is given in Appendix. Applying these formulae it is not difficult to obtain
\[
H_G = -4 \sum_{k=1}^N \sum_{l=1}^N \sum_{j=1}^k (l - k + 2j - 1) \tau_{k-l} \tau_{l+j-1} \frac{\partial^2}{\partial \tau_l \partial \tau_k}
- 4\alpha \sum_{k=1}^N (N - k + 1) (N - k) \tau_{k-1} \frac{\partial}{\partial \tau_k} - (8\beta + 2) \sum_{k=1}^N (N - k + 1) \tau_{k-1} \frac{\partial}{\partial \tau_k}
+ 4 \sum_{k=1}^N [\tau_1 \tau_k - (k + 1) \tau_{k+1}] \frac{\partial}{\partial \tau_k} - 4n \tau_1. \tag{24}
\]
What remains to be done is to rewrite this expression as a quadratic combination of the generators of the \( sl(N + 1) \) algebra. The differential realizations of the generators of \( sl(N) \) are thoroughly studied in the mathematical literature. We will not dwell on details referring the reader to the textbooks [16]. The explicit parametrization appropriate for our purposes can be found e.g. in [9, 10] (a simple heuristic derivation is presented in [15]). It is built on \( N \) variables and refers to a degenerate
representation – the symmetrized product of \( n \) fundamental representations. The \( N^2 + 2N \) generators of \( sl(N+1) \) in this realization take the form \(^1\)

\[
\begin{align*}
D_i &= \frac{\partial}{\partial \tau_i}, \quad i = 1, 2, \ldots, N, \\
U_i &= \tau_i N_0, \quad i = 1, 2, \ldots, N, \\
N_0 &= n - \sum_{i=1}^{N} \tau_i \frac{\partial}{\partial \tau_i}, \\
N_{i,j} &= \tau_i \frac{\partial}{\partial \tau_j}, \quad i \neq j, \ i, j = 1, 2, \ldots, N, \\
N_i &= \tau_i \frac{\partial}{\partial \tau_i}, \quad i = 1, 2, \ldots, N,
\end{align*}
\]

where the parameter \( n \) is an integer, the number of the fundamental representations in the symmetrized product. One can explicitly verify that the algebra of the operators in (25) is that of \( sl(N+1) \). Moreover, for integer values of \( n \) the algebra of the generators (25) has a finite-dimensional representation of dimension \( \nu = (N+n)!/(N!n!) \) composed of the homogeneous products of \( \tau_i \) of degree 1, 2, \ldots, \( n \).

After some simple algebraic transformations the Hamiltonian (24) reduces to a quadratic combination of the generators, namely

\[
\mathcal{H}_G = -4 \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{j=1}^{k} (l - k + 2j - 1)(N_{k-j,l}N_{i+j-1,k} - \delta_{jl}N_{k-1,k})
\]

\[
-4\alpha \sum_{k=1}^{N} (N - k + 1)(N - k)N_{k-1,k} - (8\beta + 2) \sum_{k=1}^{N} (N - k + 1)N_{k-1,k}
\]

\[
-4 \sum_{k=1}^{N} (k + 1)N_{k+1,k} - 4U_1.
\]

Note that here we define

\[
N_{0,k} = D_k, \quad k = 1, 2, \ldots, N,
\]

for convenience. We will stick to this definition in what follows.

The representation (26) proves the quasi-exact solvability of the \( N \)-body problem (1). The number of levels in the algebraic sector is given by Eq. (3).

\(^1\)The number of the generators listed in Eq. (25) is \( N^2 + 2N + 1 \). However, one linear combination, \( N_0 + \sum_i N_i \), reduces to a constant and is, therefore, redundant. It is easy to see that \( N \) independent generators from the set \( \{N_0, N_i\} \) form the Cartan subalgebra of the \( sl(N+1) \) algebra.
Following the arguments of Ref. [14], it is easy to verify that all levels in the algebraic sector are split in pairs under the energy-reflection symmetry. The negative-energy state goes into the positive-energy state and vice versa under the transformation $x_k \rightarrow ix_k$, much in the same way as in the parent system (11).

### 3 Example

To get a more transparent idea of the system we built, it is instructive to consider a specific example. Let us consider the two-body problem, $N = 2, \ n = 1$. The underlying algebra is $sl(3)$; at $n = 1$, the number of levels in the algebraic sector is $\nu = 3$. For simplicity, we put $g = 1$. Then

$$\alpha = \frac{\sqrt{5}+1}{2}, \ \beta = \frac{\sqrt{2}+1}{4}.\nonumber$$

The dynamical variables are $x_1$ and $x_2$ (or $\xi_1$ and $\xi_2$.) One introduces $\tau_1$ and $\tau_2$ according to Eq.(23). The algebraic representation of the (gauge transformed) Hamiltonian is

$$\mathcal{H}_G = -4\tau_1 \frac{\partial^2}{\partial \tau_1^2} - 16\tau_2 \frac{\partial^2}{\partial \tau_1 \partial \tau_2} - 4\tau_2 \tau_1 \frac{\partial^2}{\partial \tau_2^2} - 8\alpha \frac{\partial}{\partial \tau_1} - 2(8\beta + 2) \frac{\partial}{\partial \tau_1} \
-(8\beta + 2)\tau_1 \frac{\partial}{\partial \tau_2} + 4(\tau_1^2 - 2\tau_2) \frac{\partial}{\partial \tau_1} + 4\tau_2 \tau_1 \frac{\partial}{\partial \tau_2} - 4\tau_1 \
= -4N_1 D_1 - 16N_{2,1} D_2 - 4N_2 N_{1,2} - 8\alpha D_1 \
-2(8\beta + 2)D_1 - (8\beta + 2)N_{1,2} - 8N_{2,1} - 4U_1. \tag{28}$$

The algebraic sector consists of three levels. The corresponding eigenvalues and eigenfunctions are

$$E_\pm = \pm \sqrt{32(4\beta + \alpha + 1)}, \nonumber$$

$$\psi_\pm = \left(1 + \frac{\sqrt{2}(\alpha + 1 + 4\beta)}{4\beta + 2\alpha + 1}(x_1^2 + x_2^2) + \frac{2}{4\beta + 2\alpha + 1}x_1^2 x_2^2 \right) \
\times \left(x_1^2 - x_2^2\right)^\alpha x_1^2 \ x_2^2 \ \exp \left\{-\frac{x_1^4 + x_2^4}{4}\right\}, \tag{29}$$

and

$$E = 0, \nonumber$$

$$\psi = \left(1 - \frac{2}{4\beta + 1}x_1^2 x_2^2 \right) \left(x_1^2 - x_2^2\right)^\alpha x_1^2 \ x_2^2 \ \exp \left\{-\frac{x_1^4 + x_2^4}{4}\right\}. \tag{30}$$
The normalization constants in the wave functions above are omitted. We have worked out a few more examples with \( n > 1 \). The corresponding formulae, although perfectly explicit and closed, are quite cumbersome, and we do not reproduce them here.

4 Generalizations

The Hamiltonian (1) admits various generalizations. The most obvious extension is the inclusion of the quartic terms in the potential

\[
\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \left\{ \left( b^2 - a [4n + 4\alpha (N - 1) + 4\beta + 3] \right) x_i^2 + a^2 x_i^6 + 2abx_i^4 + \frac{g}{4x_i^2} \right\} 
\]

\[
+ g \sum_{i,j=1}^{N} \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2},
\]

(31)

where \( a, b \) are constants, with \( a > 0 \) and \( \alpha, \beta \) are the same as in Eq. (19). Then, after the change of variables (15),

\[
\mathcal{H}_\xi = -\sum_{i=1}^{N} \left( 4\xi_i \frac{\partial^2}{\partial \xi_i^2} + 2 \frac{\partial}{\partial \xi_i} \right) 
\]

\[
+ \sum_{i=1}^{N} \left\{ \left( b^2 - a [4n + 4\alpha (N - 1) + 4\beta + 3] \right) \xi_i + a^2 \xi_i^3 + 2ab\xi_i^2 + \frac{g}{4\xi_i} \right\} 
\]

\[
+ 2g \sum_{i,j=1}^{N} \frac{\xi_i + \xi_j}{(\xi_i - \xi_j)^2}.
\]

(32)

Next, we make the gauge transformation (17) with the phase similar to (18), namely

\[
\Psi_0 = \prod_{i,j=1}^{N} (\xi_i - \xi_j)^{-a} \prod_{i=1}^{N} \xi_i^{-\beta} \exp \left\{ \frac{a}{4} \sum_{i=1}^{N} \xi_i^2 + \frac{b}{2} \sum_{i=1}^{N} \xi_i \right\}.
\]

As a result, the Hamiltonian takes the form of a quadratic combination of the generators of \( sl(N+1) \),

\[
\mathcal{H}_G = -4 \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{j=1}^{k} (l - k + 2j - 1) (\mathcal{N}_{k-j,l} \mathcal{N}_{l+j-1,k} - \delta_{j1} \mathcal{N}_{k-1,k})
\]

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\[-4a \sum_{k=1}^{N} (N - k + 1)(N - k)N_{k-1,k} - (8\beta + 2) \sum_{k=1}^{N} (N - k + 1)N_{k-1,k} - 4aU_1\]

\[-4a \sum_{k=1}^{N} (k + 1)N_{k+1,k} + 4b \sum_{k=1}^{N} kN_k + 2baN(N - 1) + (4b\beta + b)N. \quad (33)\]

This representation makes the quasi-exact solvability of (31) obvious. Needless to say that with the quartic terms added the energy-reflection symmetry is lost, generally speaking.

Another generalization of the model (1) is supersymmetrization along the lines of [10]. The sufficient condition for a straightforward supersymmetrization is the following representation of the potential:

\[V(x_i) = \sum_{i=1}^{N} \left( \frac{\partial W}{\partial x_i} \right)^2. \quad (34)\]

One can check that the potential in Eq. (1) does have the required form provided \(n = 0\). In this case the superpotential \(W\) is

\[W(x_i) = -\frac{1}{4} \sum_{i=1}^{N} x_i^4 + \alpha \sum_{i,j=1}^{N} \ln(x_i^2 - x_j^2) + \beta \sum_{i=1}^{N} \ln x_i^2. \quad (35)\]

Adding the fermion terms following the standard prescription, we get a Hamiltonian with the minimal supersymmetry. It describes a system of \(N\) particles with spin, with a certain two-body interaction (including the spin interaction). Since we were forced to put \(n = 0\), only the ground state wave function is known algebraically. Whether one can built algebraically the wave functions of the excited states remains to be seen. Another open question is whether one can built non-minimal supersymmetric models at \(n \neq 0\).

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Appendix : Transition $\xi_i \rightarrow \tau_j$

The variable $\tau_i$ (the elementary symmetrical polynomials) are defined in Eq. (23). The transition $\xi_i \rightarrow \tau_j$ can be carried out by virtue of the following basic identities:

$$
\tau_{k-1} - \xi_i \frac{\partial \tau_{k-1}}{\partial \xi_i} = \frac{\partial \tau_k}{\partial \xi_i},
$$

(A.1)

$$
\sum_{i=1}^{N} \xi_i \frac{\partial \tau_k}{\partial \xi_i} = k \tau_k, \quad k = 1, 2, \ldots, N,
$$

(A.2)

$$
\sum_{i,j=1, i \neq j}^{N} \frac{\partial^2 \tau_k}{\partial \xi_i \partial \xi_j} \xi_i \xi_j = k(k - 1) \tau_k.
$$

(A.3)

Exploiting these identities, after some straightforward but tedious algebraic manipulations, one gets

$$
\sum_{i=1}^{N} \frac{\partial}{\partial \xi_i} = \sum_{k=1}^{N} \left[ N \tau_{k-1} - (k - 1) \tau_{k-1} \right] \frac{\partial}{\partial \tau_k},
$$

(A.4)

$$
\sum_{i=1}^{N} \xi_i^2 \frac{\partial}{\partial \xi_i} = \sum_{k=1}^{N} \left[ \tau_1 \tau_k - (k + 1) \tau_{k+1} \right] \frac{\partial}{\partial \tau_k},
$$

(A.5)

$$
\sum_{i=1}^{N} \xi_i \frac{\partial^2}{\partial \xi_i^2} = \sum_{j=1}^{k} (l - k + 2j - 1) \tau_{k-j} \tau_{l+j-1},
$$

(A.6)

$$
\sum_{i,j=1, i \neq j}^{N} \frac{1}{\xi_i - \xi_j} \left( \frac{\xi_i \partial}{\partial \xi_i} - \frac{\xi_j \partial}{\partial \xi_j} \right) = \sum_{k=1}^{N} (N - k + 1)(N - k) \tau_k \frac{\partial}{\partial \tau_k}.
$$

(A.7)
References


