Nuclear Matter in Intense Magnetic Field and Weak Processes

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Abstract

We study the effect of magnetic field on the dominant neutrino emission processes in neutron stars. The processes are first calculated for the case when the magnetic field does not exceed the critical value to confine electrons to the lowest Landau state. We then consider the more important case of intense magnetic field to evaluate the direct URCA and the neutronisation processes. In order to estimate the effect we derive the composition of cold nuclear matter at high densities and in beta equilibrium, a situation appropriate for neutron stars. The hadronic interactions are incorporated through the exchange of scalar and vector mesons in the framework of relativistic mean field theory. In addition the effects of anomalous magnetic moments of nucleons are also considered.

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1 Introduction

Recently there has been a lot of interest in the study of the effects of strong magnetic fields on various astrophysical phenomena. Many of these require large magnetic fields and indeed fields with a strength of $10^{18}$ Gauss or even more, have been conceived to exist at the time of the supernova collapse inside neutron stars and in other astrophysical compact objects [1]. The matter density in neutron stars exceeds the nuclear density and the presence of high magnetic fields modifies the energy of charged particles confining them to low landau levels. This results in shifting of beta equilibrium and the composition of matter gets significantly modified [2],[3]. The intense magnetic fields also result in reduced photon opacities and enhanced cooling of neutron stars [4]. Effect of magnetic field on weak interaction rates is of great importance in neutrino emission in neutron stars and during first few seconds immediately after collapse of massive neutron stars [5], and also in light elements abundances in the early universe [6]. Specifically it has been shown [7] that intense magnetic field induces an asymmetry in neutrino emission which may be capable of imparting a large recoil momentum to the newly born neutron star. The large magnetic field would also affect the location of electron neutrino sphere through modification of neutrino scattering cross-section thus affecting the Kasenko-Segre[8] mechanism to account for pulsar picks. The dominant mode of energy loss in neutron stars is through neutrino emission. The important process leading to neutrino cooling are the so called URCA

\[ n \rightarrow p + e^- + \bar{\nu}_e \quad (1) \]

\[ p + e^- + \rightarrow n + \nu_e \quad (2) \]

modified URCA

\[ N + N \rightarrow N + N + \nu + \bar{\nu} \quad (3) \]
\[ n + n \rightarrow n + p + e^- + \bar{\nu}_e \]  
(4)

\[ n + p + e^- \rightarrow n + n + \nu_e \]  
(5)

and the pion mediated

\[ n + \pi^- \rightarrow n + e^- + \bar{\nu}_e \]  
(6)

process accompanied by along with their inverse reactions.

At low temperatures for degenerate nuclear matter, the direct URCA process can take place only near the fermi energies of participating particles and simultaneous conservation of energy and momentum require the inequality \( p_F(e) + p_F(p) \geq p_F(n) \) to be satisfied in the absence of the magnetic field. This leads to the well known threshold \([9]\) for the proton fraction \( Y_p = \frac{n_p}{n_B} \geq 11\% \) thus leading to strong suppression in nuclear matter. This condition is satisfied for \( n_b \geq 1.5n_0 \) (where \( n_0 = 0.16 fm^{-3} \), is the nuclear saturation density), in a relativistic mean field model of interacting n-p-e gas for \( B = 0 \). In this paper we calculate the energy loss due to the above process firstly for the case when the magnetic field does not exceed the critical value to confine degenerate electrons to the lowest Landau level. The interesting case arises when the magnetic field is intense enough to force electrons into the ground state such that electrons occupy the lowest Landau level with their spins aligned opposite to the magnetic field direction, the charge neutrality now pushes the protons too into the ground state. In this situation we require to use the exact wave functions obtained by solving Dirac equation in the magnetic field. We calculate the energy loss rate due to the direct URCA process which now proceeds regardless of the proton fraction. We also calculate the neutronization rate

\[ e^- + p \rightarrow n + \nu_e \]  
(7)

for a situation appropriate during collapse.
In order to make numerical estimates of the cooling rates in magnetised neutron stars, we require the composition of nuclear matter. For this we consider electrically neutral nuclear matter composed of relativistic nucleons and electrons in beta equilibrium. We take the hadronic interactions through \((\rho - \omega - \sigma)\) meson exchange in the framework of relativistic nuclear mean field theory as in [3]. In addition we also incorporate the anomalous magnetic moments of nucleons. Indeed for magnetic fields strong enough to shift the masses due to magnetic moment nearly equal to the magnitude of nucleon masses themselves, we may not be able to define the magnetic moments and we need a field theoretical calculation of self energy of particles in the external magnetic field. Such a calculation for electrons had been performed by Schwinger[10] but unfortunately is not available for nucleons for we only have phenomenological anomalous magnetic moments for these particles. When considering very intense magnetic fields we should be careful about the effect of magnetic field on strong interactions because in this situation, the nucleons and mesons interact both with with the magnetic fields through their charges and magnetic moments. However for magnetic fields not much greater than \(10^{18}\) Gauss we do not have to worry about this problem.

In Section II, we obtain the composition of interacting nuclear matter in the presence of magnetic field by taking into consideration the anomalous magnetic moments of the nucleons. In section III we calculate the energy loss due to URCA and modified URCA process for moderate magnetic fields. The direct URCA process is then studied in the presence of quantising magnetic field where the energy-momentum conservation inequality on fermion momenta of participating particles breaks down and the neutrino emission proceeds at all densities. In Section IV, we present the results and discussions.

2 Nuclear matter composition in magnetic field

We consider here the effect of strong uniform magnetic field on the composition of nuclear matter consisting of neutrons, protons and electrons in beta equilib-
rium and the neutrinos/antineutrinos are assumed to freely stream out of the system. Hadronic interactions are taken into account by considering the nucleons to interact by exchanging scalar $\sigma$ and vector $(\omega, \rho)$ mesons in the framework of relativistic nuclear mean field theory. This is taken into account in the high density limit by replacing the meson fields by their classical averages in the equations of motion based on a Lagrangian depicting the physical process. In addition we also consider the case when nucleons have anomalous magnetic moments. As discussed in the Introduction since we do not have a proper field theoretical framework for nucleons, we treat them as point particles with given phenomenological anomalous magnetic moments and restrict ourselves to fields which do not substantially change the energy of these particles. In a uniform magnetic field $B$ along $z$ axis corresponding to the choice of the gauge field $A^\mu = (0, 0, xB, 0)$, the relativistic mean field Lagrangian can be written as

$$
\mathcal{L} = \sum \bar{\psi}_i [i\gamma^\mu D^\mu - m_i + g_\sigma \sigma - g_\omega \gamma_\mu \omega^\mu - \frac{1}{2} g_\rho \sigma_{\mu\nu} F^{\mu\nu} + \kappa_i \sigma_{\mu\nu} F^{\mu\nu} \psi_i] + \frac{1}{2} [(\partial^\mu \sigma)^2 - (m_\sigma^2 \sigma^2)] - \sum_{\omega, \rho} \left[ \frac{1}{4} (\partial_{\mu} V^i_{\mu} - \partial_{\nu} V^i_{\nu})^2 - \frac{1}{2} m^2_{\omega/(V^i_{\mu})^2} \right] \tag{8}
$$

in the usual notation with $D^\mu = \partial^\mu + ieA^\mu$ and $K_i$ as the anomalous magnetic moments given by

$$
\kappa_p = \frac{e}{2m_p} \left[ \frac{g_p}{2} - 1 \right] \tag{9}
$$

$$
\kappa_n = \frac{e}{2m_n} \left[ \frac{g_n}{2} \right] \tag{10}
$$

where $g_p = 5.58$ and $g_n = -3.82$ are the Lande’s $g$-factor for protons and neutrons respectively. Writing the general solution as

$$
\psi(r) = e^{-i(Et - p_y y - p_z z)} f_s(x) \tag{11}
$$

where $f_s(x)$ is the four component Dirac solution. Replacing the meson fields in the relativistic mean field approximation by their density dependent average values $< \sigma_0 >, < \omega_0 >$ and $< \rho_0 >$ respectively, the equation of motion satisfied by the nucleons in the magnetic field become
\[-i\alpha_x \frac{\partial}{\partial x} + \alpha_y (p_y - eBx) + \alpha_z p_z + \beta (m_i + g_\sigma < \sigma_0 >) - i\kappa_i \alpha_x \alpha_y B \] \[f_{i,s}(x) \]

\[
(E^i - U_0^i) f_{i,s}(x)
\]

where

\[
U_{0}^{p,n} = g_\omega < \omega_0 > \pm \frac{q_p}{2} < \rho_0^3 >
\]

The equations are first solved for the case when momentum along the magnetic field direction is zero and then boosting along that direction till the momentum is \(p_z\). For the protons and neutrons we thus get

\[
E_{\nu,s}^{p} = \sqrt{m_i^{*2} + p_z^2 + eB(2\nu + 1 - s) + \kappa_i^2 B^2 - 2\kappa_p B s \sqrt{m_i^{*2} + eB(2\nu + 1 - s)}} - U_{0}^{p}
\]

and

\[
E_{s}^{n} = \sqrt{m_n^{*2} + p_z^2 + \kappa_n^2 B^2} - 2\kappa_n B \sqrt{p_z^2 + m_n^{*2}} - n - U_{0}^{n}
\]

respectively. In the above \(\nu\) denotes the Landau level and takes values 0,1,2,... and \(s = \pm 1\) indicates whether the spin is along or opposite to the direction of the magnetic field and \(m_i^{*} = m_i + g_\sigma < \sigma_0 >\) is the effective mass. The positive energy spinors for protons are then given by

\[
f_{p,1}(x) = C_{\nu,1} e^{-\frac{\xi^2}{4}} \begin{pmatrix} H_{\nu}(\xi) \\ 0 \\ \frac{p_z}{E_{\nu,1} + m_p} H_{\nu}(\xi) \\ \frac{\sqrt{\xi^2}}{E_{\nu,1} + m_p} H_{\nu+1}(\xi) \end{pmatrix}
\]

and

\[
f_{p,-1}(x) = C_{\nu,-1} e^{-\frac{\xi^2}{4}} \begin{pmatrix} 0 \\ H_{\nu}(\xi) \\ \frac{-2\nu\sqrt{\xi m_p}}{E_{\nu,-1} + m_p} H_{\nu-1}(\xi) \\ \frac{-p_z}{E_{\nu,-1} + m_p} H_{\nu}(\xi) \end{pmatrix}
\]
where
\[ \xi = \sqrt{eB(x + \frac{p_y}{eB})} \] (18)
\[ C_{\nu,s} = \frac{1}{\sqrt{2^{\nu+1}\sqrt{\pi}} \sqrt{E_{\nu,s}^p + m_p^*}} \] (19)
and \( H_{\nu,s} \) are the Hermite polynomials.

The neutron spinors correspond to the usual plane wave solutions of the Dirac equation with the energy given by equation (15) for spins parallel and antiparallel to the magnetic field direction. For electrons the problem was solved in QED by Schwinger to one loop in magnetic field and the energy for \( p_z = 0 \) is given[11].

\[ E_{\nu,s}^e = \sqrt{m_e^2 + (2\nu + 1 + s)eB + \frac{\alpha eB}{2\pi} S} \] (20)
This expression however breaks down for large magnetic fields for which the energy is given by

\[ E_{\nu,s}^e = \sqrt{m_e^2 + (2\nu + 1 + s)eB - \frac{\alpha m_e}{4\pi} (\log \frac{2eB}{m_e})^2 S} \] (21)
we thus see that for fields strengths of subsequent interest, term proportional to \( \alpha \) is negligible and the electron energy is simply taken to be

\[ E_{\nu,s}^e = \sqrt{m_e^2 + (2\nu + 1 + s)eB} \] (22)

The mean field values \(< \sigma_0 >, < \omega_0 > \) and \(< p_0^3 > \) are determined by minimizing the energy at fixed baryon density \( n_B = n_p + n_n \) or by maximizing the pressure at fixed baryon chemical potential \( \mu_B \). We thus get

\[ < \sigma_0 > = \frac{g_\sigma}{m_\sigma^2} (n_p^s + n_n^s) \] (23)
\[ < \omega_0 > = \frac{g_\omega}{m_\omega^2} (n_n + n_p) \] (24)
\[ < p_0^3 > = \frac{1}{2} \frac{g_\rho}{m_\rho^2} (n_p - n_n) \] (25)
where \( n_i \) and \( n_i^s \) the number and scalar number densities respectively are given by

\[
\langle n_i \rangle = \langle \psi_i^\dagger \psi_i \rangle = \frac{1}{(2\pi)^3} \int \frac{1}{e^{(E_i - \mu_i)/\beta} + 1} d^3p
\]

(26)

and

\[
\langle n_i^s \rangle = \langle \bar{\psi}_i \psi_i \rangle = \frac{1}{(2\pi)^3} \int \frac{m_i^*}{E_i} \frac{1}{e^{(E_i - \mu_i)/\beta} + 1} d^3p
\]

(27)

for each spin state. In the present case since the energy is different for different spin states, we need to calculate the densities for spin parallel and antiparallel states separately. Using the energy expressions from (14) and (13) carrying out integrals at \( T = 0 \) we get

\[
n_{n,s} = \frac{1}{2\pi^2} \left[ \frac{1}{3} (\mu_n^* - (m_n - s\kappa_n B))^2 - \frac{s}{2} \mu_n^* \kappa_n B (s - 1) \frac{m_n^*}{B} - \mu^* \right]
\]

\[- \pi \frac{m_n^* - s\kappa_n}{\mu_n^*} \sqrt{\mu_n^* - (m_n - s\kappa_n B)^2} \]

(28)

and

\[
n_{p,s} = \frac{eB}{2\pi^2} \sum_{\nu=0}^{\nu_{\text{max}}} \sqrt{\mu_p^* - m_p^2 - 2eB\nu - \kappa_p^2B^2 + 2s\kappa_p B \sqrt{m_p^2 + 2eB\nu}}
\]

(29)

where

\[
\mu_n^* = \mu_n - U_n^n
\]

(30)

\[
\mu_p^* = \mu_p - U_p^n
\]

(31)

and

\[
\nu_{\text{max}} = \text{Int} \left( \frac{\mu_p^* + s\kappa_p B}{2eB} \right)
\]

(32)

In the limit of vanishing anomalous magnetic moments, we recover the usual relations[3]. In the region of validity of defining anomalous magnetic moments for protons and neutrons and for fields of interest, we can make an expansion in powers of \( \kappa_i \) and can evaluate the densities for highly degenerate nuclear matter at finite temperatures in powers of \( \frac{\mu^*}{T} \). We get

\[
n_{p,s} = \frac{eB}{2\pi^2} \sum_{\nu=0}^{\nu_{\text{max}}} \int_{p_F} \cdots \]
\[ n_{n,s} = \frac{1}{6\pi^2} [p_F^3(n,s) + \frac{\pi^2 T^2}{2} \frac{\mu^2_{n,s}}{P_F(n,s)} + \ldots] \quad (34) \]

\[ n_e = \frac{eB}{2\pi^2} \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu,0})p_F(e)(1 - \frac{\pi^2 T^2}{6} \frac{\bar{m}_e^2}{p_F^2(e)}) \quad (35) \]

The corresponding expressions for the scalar densities are given by

\[ n_{p,\nu}^s = \frac{eB}{2\pi^2 m_p^*} \sum_{\nu} \ln\left(\frac{\mu_{p,s}^n + p_F(p,s)}{m_p^*}\right) - \frac{\pi^2 T^2}{6} \frac{\mu_{p,s}^n}{p_F^3(p,s)} + \ldots \quad (36) \]

and

\[ n_{n,s}^n = \frac{m_n^*}{4\pi^2} \ln\left(\frac{\mu_{n,s}^n + p_F(n,s)}{m_n^*}\right) + \frac{\pi^2 T^2}{3} \frac{\mu_{n,s}^n}{P_F(n,s)} + \ldots \quad (37) \]

where

\[ \bar{m}_p^2 = m_p^* + 2\nu_eB \quad (38) \]

\[ \bar{m}_e^2 = m_e^2 + 2\nu_eB \quad (39) \]

\[ \mu_{p,s}^n = \mu_{p}^* + s\kappa_pB \quad (40) \]

\[ \mu_{n,s}^n = \mu_{n}^* + s\kappa_nB \quad (41) \]

\[ p_F(p,s) = \sqrt{\mu_{p,s}^2 - \bar{m}_p^2} \]

\[ p_F(n,s) = \sqrt{\mu_{n,s}^2 - m_n^2} \]

\[ p_F(e) = \sqrt{\mu_e^2 - \bar{m}_e^2} \quad (42) \]

The thermodynamic potential of the system can now be written as
\[
\Omega = \frac{-g_\sigma^2 n_B^2}{2m_\sigma^2} - \frac{g_\rho^2}{8m_\rho^2} (n_p - n_n)^2 + \frac{g_\sigma^2}{2m_\sigma^2} (n_p^s + n_n^s)^2 - \frac{1}{8\pi^2} \sum_s \frac{1}{3} \mu_{n,s}^2 p_F^3(n,s) \\
- \frac{1}{2} m_n^2 \mu_{n,s} p_F(n,s) + \frac{1}{2} m_n^4 \ln\left(\frac{\mu_{n,s} + p_F(n,s)}{m_n^2}\right) + \frac{2}{3} \pi^2 T^2 \mu_{n,s} p_F(n,s) + \ldots \\
- \frac{eB}{8\pi^2} \sum_s \sum_\nu (\mu_{p,s} p_F(p,s) - \tilde{m}_p^2 \ln\left(\frac{\mu_{p,s} + p_F(p,s)}{\tilde{m}_p}\right) + \frac{\pi^2 T^2}{3} \mu_{p,s}^3 p_F(p,s) + \ldots \\
- \frac{eB}{8\pi^2} \sum_\nu (2 - \delta_{\nu,0}) [\mu_e p_F(e) - \tilde{m}_e^2 \ln\left(\frac{\mu_e + p_F(e)}{\tilde{m}_e}\right) + \frac{\pi^2 T^2}{3} \mu_e p_F(e) + \ldots]
\] 

(43)

and the other thermodynamic quantities can be determined by the usual thermodynamic relations. In the above

\[n_B = n_n + n_p\] 

(44)

\[n_i = n_{i,+} + n_{i,-}\] 

(45)

In the neutron stars the n-p-e matter is charge neutral and is in \(\beta\) equilibrium giving

\[\mu_n = \mu_p + \mu_e\] 

(46)

\[n_p = n_e\] 

(47)

Using these two equations, \(m^*\) is calculated self consistently for a given baryon density \(n_B\) and the magnetic field \(B\). The thermodynamic quantities can then be calculated. As a consequence of charge neutrality the magnetic field is strong enough to quantise the electron motion, the motion of the proton also gets quantized and the proton fraction increases. The value of the quantising magnetic field increases with baryon density and for magnetic field \(\frac{B}{B_c^e} \geq \frac{1}{2} \left(\frac{\mu_e}{m_e}\right)^2\) where \(B_c^e = m_e^2 = 4 \times 10^{13} \text{ Gauss}\), and for \(\mu_e\) corresponding to given \(n_B\), the electrons and protons would be in the lowest Landau level leading to interesting consequences.
3 Weak rates and neutrino emissivity

As discussed in the introduction direct URCA process \([1]\) the absence of magnetic field can proceed only above the threshold of 11% for the proton fraction. Since the effect of the magnetic field is to increase the proton fraction at the same time leading to a fall in the proton’s fermi momentum, the neutron to proton conversion is enhanced in the presence of magnetic field and we would see that in the presence of quantising magnetic field, there is no restriction on the proton fraction for the process to proceed. We will first consider the effect of magnetic field on the direct URCA, modified URCA and pion mediated processes for the case when the field does not exceed the critical value and will go over to the more interesting case where electrons are confined in the lowest Landau level. In the later situation, direct URCA and the neutronisation process are the important ones.

3.1 Weak magnetic field

We first consider the effect on the direct URCA process \(n \rightarrow p + e^- + \bar{\nu}_e\) and \(p + e \rightarrow n + \nu_e\), when the magnetic field is not strong enough to force the electrons in the lowest Landau level. Previous calculations\([2],[4],[5]\) show that the matrix element for the process remains essentially unaffected and the modification comes mainly from the phase space factor. In this situation nuclear energies are essentially independent of spin states and we can sum over nuclear spins. Treating the nucleons non-relativistically and electrons ultra-relativistically, the matrix element squared and summed over spins is given by

\[
\sum |M|^2 = 8G_F^2 \cos^2 \theta_c (4m_n^* m_p^*) E\nu [(1 + 3g_A^2) + (1 - g_A^2) \cos \theta_c] \tag{48}
\]

where \(g_A = 1.261\) is the axial-vector coupling constant.

The emissivity expression is given by

\[
\dot{E}\nu = \prod_i \int \frac{1}{(2\pi)^3 2E_i} d^3p_i E\nu \sum |M|^2 (2\pi)^4 \delta^4(P_f - P_i) S \tag{49}
\]
where the phase space integrals are to be calculated over all particle states. The statistical distribution function 
\[ S = f_n(1 - f_p)(1 - f_e) \]
where \( f_i \)'s are the Fermi-Dirac distributions. We can now evaluate the emissivity in the limit of extreme degeneracy, a situation appropriate in neutron star cores by replacing the electron phase space factor
\[
\int \frac{1}{(2\pi)^3} eB \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu, 0}) \int dp_z
\]
and using the standard techniques to perform the phase space integrals [11] and get

\[
\dot{E} = \frac{457\pi}{40320} G_F^2 \cos^2 \theta_c (1 + 3g_A^2) m_n^* m_p^* eBT^6 \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu, 0}) \frac{1}{\sqrt{\mu_e^2 - m_e^2 - 2\nu eB}}
\]

where

\[
\nu_{\text{max}} = \text{Int} \left( \frac{\mu_e^2 - m_e^2}{2eB} \right)
\]

In the limit of vanishing magnetic field, the sum can be replaced by an integer and we recover usual expression [9]

\[
\dot{E}(B = 0) = \frac{457\pi}{20160} G_F^2 \cos^2 \theta_c (1 + 3g_A^2) m_n^* m_p^* \mu_e T^6
\]

Modified URCA process (2) considered to be the dominant processes for neutron star cooling, had been calculated by Frinan and Maxwell [12] by treating the long range NN interactions through one pion exchange potential and the short range interactions in the frame work of Landau Fermi liquid. The matrix element squared over spins for the process

\[
n + n \rightarrow n + p + e + \bar{\nu}_e
\]

has been calculated to be

\[
\sum |M|^2 = 256 G_F^2 \cos^2 \theta_c (16 m_n^* m_p^*) g_A^2 \left( \frac{1}{m_{\pi}} \right)^4 \alpha_{\text{URCA}} \frac{E_e E_\nu}{(E_e + E_\nu)^2}
\]
where \( f \) is the \( \pi - N \) coupling constant \((f^2 \simeq 1)\) and \( \alpha_{URCA} \) has been estimated to be \( \approx 1.54 \). Using the above matrix element square, in the energy loss expression with appropriate electron phase space, \( \dot{E}_{URCA} \) is calculated to be

\[
\dot{E}_{URCA} = \frac{11513 G_F^2 \cos^2 \theta_e}{60480} g_A^2 m_n^* m_p^* \left( \frac{f}{m_\pi} \right)^4 \alpha_{URCA} T^8 \\
\sum_{\nu=0}^{\nu_{max}} \left( 2 - \delta_{\nu,0} \right) \left( \frac{1}{\sqrt{m_e^2 - m_\nu^2 - 2 \nu eB}} \right)
\]

which is the \( B \rightarrow 0 \) limit goes over to the standard result \([12]\)

\[
\dot{E}_{URCA}(B = 0) = \frac{11513 G_F^2 \cos^2 \theta_e}{30240} g_A^2 m_n^* m_p^* \left( \frac{f}{m_\pi} \right)^4 \alpha_{URCA} \mu_e T^8
\]

The presence of meson condensates in the core of the neutron stars has been considered in the literature \([13]\) if present, they could supply the required momentum as is done by the spectator nucleons in the case of modified URCA process, to beat the suppression due to energy momentum conservation for degenerate particles and would result in enhanced cooling through the process \( n + \pi^- \rightarrow n + e^- + \bar{\nu}_e \). Maxwell et al \([14]\) constructed the charged pion condensed phase through a chiral notation by the unitary operator

\[
U(\pi^e, \mu_{\pi^-}, k_c, \theta) = e^{i \int (k_c r - \mu_{\pi^-} t) r^3 d^3 r e^{i Q^2 \theta}}
\]

which generates the charged pion condensed phase with a macroscopic field of chemical potential \( \mu_{\pi^-} \), momentum \( k_c \) and chiral angle \( \theta \). The participating particles in neutrino emission are the quasi-particles which are the superposition of proton and neutron states. The matrix element squared and summed over spins for the quasi-particles \( \beta\)-decay \([3]\) was obtained in reference \([14]\) and is given by

\[
\sum |M|^2 = 8 G_F^2 \cos^2 \theta_e (4 m_n^* m_p^*) [(1 + 3 \tilde{g}_A^2) + (1 - g_A^2) \cos \theta_{ev}]
\]
The energy loss rate in the presence of magnetic field can now be easily calculated by assuming that chiral condensates are not really affected in the presence of external fields of moderate strength [15] and is given by

\[ \dot{E}_{\nu} = \frac{457}{40320} \frac{m^2_u}{k} \mu_e \frac{eB}{2} T^6 G_F^2 \cos^2 \theta_c (1 + 3 \tilde{g}_A^2) \frac{\theta^2}{4} [1 + (\tilde{g}_a k_e \mu_e / \mu_{\pi})^2] \]

(58)

where \( \mu_{\pi^-} = \mu_e \) in equilibrium. In \( B \to 0 \) limit, we get the standard result [14].

During collapse, nucleons are non-degenerate and non-relativistic except at the highest densities when neutrinos are trapped and the collapse gets underway, neutronization is the most important process for generating neutrinos. In this situation the rate for the process \( e^- + p \to n + \nu_e \) is given by

\[ \Gamma = \frac{G_F^2 \cos^2 \theta_c}{4 \pi^3} (1 + 3 \tilde{g}_A^2) n_p e B \sum_{\nu} (2 - \delta_{\nu,0}) \int_Q^\infty \frac{E_c (E_c - Q)^2}{\sqrt{E_c^2 - m^2_e - 2 \nu_e B}} f_e (1 - f_\nu) dE_e \]

(60)

where \( Q = m_n - m_p \) and the blocking factor for neutrons has been ignored.

In the initial stage electrons are extremely degenerate and relativistic \( E_e \gg m_e, \mu_e \gg kT \) and neutrinos freely escape \( f_\nu << 1 \).

\[ \Gamma = \frac{G_F^2 \cos^2 \theta_c}{4 \pi^3} (1 + 3 \tilde{g}_A^2) n_p m^5_e B / B_e \sum_{\nu} (2 - \delta_{\nu,0}) \int_{\nu_e}^\infty \frac{\varepsilon (\varepsilon - q)^2}{\sqrt{\varepsilon^2 - 1} - 2 \nu / \varepsilon} f_e d\varepsilon \]

\[ - \sum_{\nu,\nu' = 0}^{\nu_e-1} (2 - \delta_{\nu,0}) \int_{\nu_e}^q \frac{\varepsilon (\varepsilon - q)^2}{\sqrt{\varepsilon^2 - 1} - 2 \nu / \varepsilon} f_e d\varepsilon \]

\[ \simeq \frac{G_F^2 \cos^2 \theta_c}{4 \pi^3} (1 + 3 \tilde{g}_A^2) n_p m^5_e B / B_e \sum_{\nu} (2 - \delta_{\nu,0}) \int_{\nu_e}^\infty \frac{\varepsilon^3}{\varepsilon^2 - 1 - 2 \nu / \varepsilon} d\varepsilon \]

(61)
where $\varepsilon = \frac{E}{m_e}$, $q = \frac{Q}{m_e}$ and $\nu_{\text{max}} = \text{Int}\left(\frac{q^2}{2B}\right)$ which reduces to

$$\Gamma(B = 0) = \frac{G_F^2}{2\pi^3} \cos^2 \theta_c \frac{(1 + 3g_A^2)n_p m_e^5}{5}$$

in the limit of $B \to 0$. Later on when neutrinos are trapped $\mu_e >> kT, \mu\nu >> kT$, but as long as $\mu_e >> \mu\nu, f_\nu = 0$ and we get

$$\Gamma = \frac{G_F^2}{4\pi^3} \cos^2 \theta_c (1 + 3g_A^2)n_p m_e^5 \int_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu,0}) \int_{\varepsilon = 0}^{\infty} \frac{\varepsilon (\varepsilon - q)^2}{\sqrt{\varepsilon^2 - 1} - 2\nu \frac{B}{B_c}} f_e d\varepsilon$$

$$- \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu,0}) \int_{\varepsilon = 0}^{\infty} \frac{\varepsilon (\varepsilon - q)^2}{\sqrt{\varepsilon^2 - 1} - 2\nu \frac{B}{B_c}} f_\nu d\varepsilon$$

$$\simeq \frac{G_F^2}{4\pi^3} \cos^2 \theta_c (1 + 3g_A^2)n_p m_e^5 \frac{B}{B_c} \sum_{\nu}(2 - \delta_{\nu,0})$$

$$\int_{\varepsilon = 0}^{\frac{\mu_e}{m_e}} \varepsilon^3 \sqrt{\varepsilon^2 - 1} - 2\nu \frac{B}{B_c} d\varepsilon$$

which reduces to

$$\Gamma(B = 0) = \frac{G_F^2}{2\pi^3} \cos^2 \theta_c \frac{(1 + 3g_A^2)n_p m_e^5 - \mu_\nu^5}{5}$$

in the $B \to 0$ limit.

4 Quantising magnetic field

In the case of super strong magnetic fields such that $2eB > p_F(e)^2$ all electrons occupy the Landau ground state at $T=0$ which corresponds to $\nu = 0$ state with electron spins pointing in the direction opposite to the magnetic field. As has been discussed above, charge neutrality now forces the degenerate non-relativistic protons also to occupy the lowest Landau level with proton spins pointing in the direction of the field. In this situation we can no longer consider the matrix elements to be unchanged and they should be evaluated using the exact solutions of Dirac equation. Further because nucleons have anomalous magnetic moment, matrix elements need to be evaluated for specific spin states separately.
The electron wave function in $\nu = 0$ state has energy $E_e = \sqrt{m_e^2 + p_{ez}^2}$ with a wave function

$$\psi_e(r) = \left(\frac{eB}{\pi}\right)^{1/4} \frac{1}{\sqrt{L_yL_z}} e^{-i(E_e t - p_{ey} y - p_{ez} z)} e^{-\xi^2/2} U_{e,-1}(E_e)$$

(65)

and the positive energy spinor in $\nu = 0$ state is given by

$$U_{e,-1} = \frac{1}{\sqrt{E_e + m_e}} \begin{pmatrix} 0 \\ E_e + m_e \\ 0 \\ -p_{ez} \end{pmatrix}$$

(66)

Protons are treated non-relativistically, from equation (14) we define

$$\bar{m}_p = m_p^* - \kappa_p B$$

(67)

and the energy in $\nu = 0$ state is

$$E_p \approx \bar{m}_p + \frac{P_z^2}{2\bar{m}_p} + U_{0}^p$$

(68)

The proton wave function is given by

$$\Psi_p(r) = \left(\frac{eB}{\pi}\right)^{1/4} \frac{1}{\sqrt{L_yL_z}} e^{-i(E_p t - p_{ey} y - P_z z)} e^{-\xi^2/2} U_{p,+1}(E_p)$$

(69)

where

$$\xi = \sqrt{eB(x - \frac{P_y}{eB})}$$

(70)

and $U_{p,+1}$ is the non-relativistic spin up operator

$$U_{p,+1} = \frac{1}{\sqrt{2\bar{m}_p}} \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}$$

(71)

For neutrons we have

$$\Psi_{n,s}(r) = \frac{1}{\sqrt{L_xL_yL_z}} e^{-iP_{n,s} \cdot r} U_{n,s}(E_{n,s})$$

(72)

$$U_{n,s} = \frac{1}{\sqrt{2m_n}} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}$$

(73)
and

\[ E_{n,s} = m_n^* + \vec{p}_n^2 - \kappa_n B s + U_0^n \]  

(74)

in the non-relativistic limit. The neutrino wave function is given by

\( \Psi_\nu(r) = \frac{1}{\sqrt{L_x L_y L_z}} e^{-i p_\nu \cdot r} U_{\nu,s}(E_\nu) \)  

(75)

here \( U_{\nu,s} \) is the usual free particle spinor, \( \chi_s \) is the spin spinor and the wave function have been normalised in a volume \( V = L_x L_y L_z \) and we have used the normalization \( \sum U_\alpha U_\beta = 2m \delta_{\alpha,\beta} \) We will now calculate the matrix element for the direct URCA process (1) for different polarisation states of neutron. The neutrino polarisation will be summed over

\[ M_s = \frac{G_F \cos \theta_c}{\sqrt{2}} \int \bar{\Psi}_{N,s}(r) \gamma_\mu (1 - g A \gamma_5) \Psi_{p,+1} \gamma^\mu (1 - \gamma_5) \Psi_{e,-1}(r) d^4 r \]  

(76)

Substituting for the wave functions and carrying out the integrals, we obtain

\[ M_s = K_s \bar{U}_{n,s} \gamma_\mu (1 - g A \gamma_5) U_{p,+1} \gamma^\mu (1 - \gamma_5) U_{e,-1} \]  

(77)

where

\[ K_s = \frac{2 \pi^3 G_F \cos \theta_c}{\sqrt{2}(L_y L_z)^2 L_x} \delta(E_n - E_p - E_e - E_\nu) \delta(p_n - p_z - p_{e,z} - p_{\nu,z}) \delta(p_y - p_y - p_{e,y} - p_{\nu,y}) e^{-Q^2} \]  

(78)

and

\[ Q^2 = \frac{p_y^2 + p_{e,y}^2}{2 e B} - \frac{(p_{e,y} + P_y) + i(p_{n,z} - p_{e,z})^2}{4 e B} \]  

(79)

now calculate the matrix element squared and summed over neutrino states to get

\[ \sum |M_+|^2 = |K_+|^2 4(4m_n^* \vec{m}_p)(1 + g_A^2)(E_e + p_{e,z})(E_\nu + p_{\nu,z}) \]  

(80)

and

\[ \sum |M_-|^2 = |K_-|^2 16(4m_n^* \vec{m}_p)g_A^2 (E_e + p_{e,z})(E_\nu - p_{\nu,z}) \]  

(81)

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The neutrino emissivity is now given by

\[
\dot{E} = \frac{2G_f^2 \cos^2 \theta_c}{V^2 L_y} \frac{v_{\nu}}{2} \frac{L_y}{\pi} \int_{\infty}^{\infty} \frac{L_z}{2} d\nu \int_{-\infty}^{\infty} \frac{L_y}{2} d\nu \int_{-\infty}^{\infty} \frac{L_x}{2} d\nu \int_{\infty}^{\infty} \frac{L_y}{2} d\nu \int_{\infty}^{\infty} \frac{L_z}{2} d\nu \]

\[
\int_{-\infty}^{\infty} \frac{L_z}{2} d\nu \int_{-\infty}^{\infty} \frac{L_y}{2} d\nu \int_{-\infty}^{\infty} \frac{L_x}{2} d\nu \int_{\infty}^{\infty} \frac{L_y}{2} d\nu \int_{\infty}^{\infty} \frac{L_z}{2} d\nu
\]

\[
E_\nu (2\pi)^3 [(1 + g_A)^2 (E_n + p_{\nu, z}) \delta (E_n - E_p - E_e - E_{\nu}) f_{n,+}
\]

\[
+ 4g_A^2 (E_\nu - p_{\nu, z}) \delta (E_{n,-} - E_p - E_e - E_{\nu}) f_{n,-} (E_n + p_{e,z})(1 - f_e)(1 - f_p)
\]

\[
\delta (p_{n,z} - p_z - p_{e,z} - p_{\nu,z}) \delta (p_{n,y} - p_y - p_{e,y} - p_{\nu,y})
\]

\[
ed^{-\frac{1}{2}} E_\nu (p_{\nu, z} + p_{e, z})^2 + (p_{\nu, y} + p_{e, y})^2
\]

(82)

Integrals over \( dp_y \) and \( dp_{ey} \) can be performed by using the y-component momentum delta function. The integrals over \( dp_{ez} \) and \( dp_z \) are converted into integrals over \( dE_e \) and \( dE_p \) respectively and making use of the fact that for strongly degenerate matter, particles at the top of their respective fermi seas alone contribute. The integral over neutron solid angle is performed by using the z-component momentum conserving delta function and neglecting neutrino momentum. The integral over angle can be performed explicitly and we obtain

\[
\dot{E} = \frac{2G_f^2 \cos^2 \theta_c}{(2\pi)^5} \frac{v_{\nu}}{p_F(p)} \int E_n^3 dE_e dE_p dE_n dE_{\nu} [(1 + g_A)^2
\]

\[
\delta (E_{n,+} - E_p - E_e - E_{\nu}) f_{n,+} + 4g_A^2 \delta (E_{n,-} - E_p - E_e - E_{\nu}) f_{n,-}
\]

\[
[\frac{1}{\pi^2} (p_{\nu, z}^2 - 4p_{\nu, e}^2) \theta (p_{\nu, z}^2 - 4p_{\nu, e}^2)
\]

\[
ed^{-\frac{1}{2}} p_F(p) [1 - f_p] (1 - f_e)
\]

(83)

The two \( \theta \) functions correspond to the fact that the z-component momentum conservation \( p_{n,z} = p_z + p_{e,z} = p_F(p) \pm p_F(e) \) depending on whether both electrons and protons in their Landau ground states move in the same direction or in opposite directions. Further since charge neutrality implies \( p_F(p) = p_F(e), p_{n,z} = 2p_F(e) \) or zero. The energy integrals can now be performed by the standard techniques for degenerate matter and we get
\[ \dot{E} = \frac{457\pi G^2 F^2}{40320 G^2 F^2} \cos^2 \theta \sin B \sin \theta \left[ \frac{(1 + gA)^2}{2} \right] \]

where \( p_f(n, +) \) and \( p_f(n, -) \) are the neutron fermi momenta for the neutron spins along and opposite to the magnetic field direction respectively and are given by (sec. eq.(60))

\[ p_f^2(n, \pm) = \mu_n - m_n - U_0 n^2 \pm \frac{\kappa_n B}{2m_n} \]

The energy loss rate due to \( p + e^- \rightarrow n + \nu_e \) gives the same contribution under equilibrium and the total emissivity is thus a factor of two greater than (69) We thus see as advertised that in the presence of quantising magnetic field the inequality \( p_F(e) + p_F(p) \geq p_F(n) \) is no longer required to be satisfied for the process to proceed.Regardless of the value of proton fraction determined by \( p_F(p) = p_F(e) \), we get non zero energy loss rate.The rate however, would show an increase at the threshold of the proton fraction \( Y_p \geq 11\% \).

During collapse when the neutronization process is important, the density is of the order of \( 10^{12} \text{gm/cc} \) and the nucleons are non-degenerate and non-relativistic. The strong interaction effects can be ignored. The nucleon expressions can be simplified and we have from equations (68) and (74)

\[ E_p \approx m_p - \kappa_p B + \frac{p_p^2}{2m_p} \]

\[ E_{n,s} \approx m_n - \kappa_n B + \frac{p_n^2}{2m_n} \]

defining \( Q_s = E_{n,s} - E_p \approx Q - (\kappa_n s - \kappa_p)B \), the rate for the neutronization process can be similarly evaluated in the limit \( E_e \sim \mu_e >> Q_s \) and we obtain
\[ \Gamma \approx \frac{2\sqrt{2}G_F^2 \cos^2 \theta}{3\pi^3} (m_p)^3 n_p \int dE e^{\frac{2}{E}} \left( 1 + gA \right)^2 e^{-\frac{E}{E_c}} \left\{ \left( 1 + gA \right) e^{-\frac{E}{E_c}} + \frac{Q}{E_c} e^{-\frac{E}{E_c}} \right\} f_e(E_c) \] (88)

5 Results and Discussions

We now present our results. For the case of non-interacting, cold, degenerate matter, containing nucleons and electrons with neutrinos freely streaming out, we choose \( n_B \) to lie in the range of \((\sim 0.5 \text{ to } 6) n_0\), a situation appropriate for the neutron star core and show the composition of matter as a function of density in figure 1 for different values of the magnetic field. For \( B = 0 \), the proton fraction remains less than the threshold value \( \left( \frac{n_p}{n_B} < 11\% \right) \) for direct URCA process to take place, even for \( n_B \) as large as \( 10 n_0 \). With the increase in magnetic field, the proton fraction rises steadily and crosses the threshold for \( n_B < 6 n_0 \) for \( B = 10^7(\text{MeV})^2 \) say. A further increase in B results in the presence of more protons than neutrons for example, for \( B = 5 \times 10^7(\text{MeV})^2 \) there are more protons than neutrons up to \( n_B \leq 3 n_0 \) and at large densities, the proton fraction again decreases but remains large. Any further increase in B may result in a prominently proton matter star. The effect of including anomalous magnetic moment of nucleons is to increase the proton fraction at all densities as can be seen from figure 1 and the proton fraction rises above the neutron fraction. Effect of including interactions (fig.2) is to raise the proton fraction and the threshold for direct URCA process comes down to \( n_B \geq 1.5 n_0 \). Magnetic field has the effect of generally raising the proton fraction and for the highest magnetic field \( B \approx 5 \times 10^7 \text{MeV}^2 \) considered here, there are far more protons than neutrons in the star. The effect of magnetic field on neutrino emissivity is shown in table 1 where the ratio of the emissivity \( R = \frac{\xi(B)}{\xi(0)} \) for the direct URCA process is shown as a function of density. These results are for the weak magnetic field which is our case of degenerate matter could even be as large as \( 10^4 \text{MeV}^2 \). The inter-
esting case of magnetic field that is capable of totally polarising the electrons and protons is shown in figure 3 where we have plotted the neutrino emissivity in units of $\mathcal{E}_0 = \frac{457\pi}{20160} G_F^2 \cos^2 \theta_c T^6 m_p^3$. As has been discussed above, in this case the threshold for direct URCA process is evaded and the emissivity is enhanced by up to two orders of magnitude.

\[ B = 10^2 \quad B = 10^3 \quad B = 10^4 \]

\begin{tabular}{|c|c|c|c|}
\hline
$n_B$ & $R$ & $R$ & $R$ \\
\hline
0.23 & 0.94 & 0.94 & 0.65 \\
0.36 & 0.96 & 0.89 & 0.62 \\
0.54 & 0.98 & 1.23 & 1.10 \\
0.71 & 0.99 & 1.29 & 0.89 \\
0.98 & 0.97 & 1.03 & 1.06 \\
\hline
\end{tabular}

Table 1. Ratio $R = \frac{\mathcal{E}_\nu(B)}{\mathcal{E}_\nu(0)}$ as a function of density for different values of magnetic field ($\text{in MeV}^2$)

Similar situation obtains for neutronization rate during collapse when the nucleons are non-degenerate and density $\sim 10^{12} \text{gm/cc}$. In this case even the magnetic field of moderate strength ($B > 10^2 \text{MeV}^2$) is able to polarise the electrons completely, but unlike the cold matter, the charge neutrality here does not force the protons to be totally polarised. However for $B \sim 10^4 \text{MeV}^2$ and above, the protons too are completely polarised. In figure 4 we have plotted the reaction rates $\Gamma$ in units of $\Gamma_0 = 10^6 G_F^2 \cos^2 \theta_c m_p^3 \pi^3$ for $B = 0$ and $B = 10^4 \text{MeV}^2$ for non-interacting non-degenerate hot nuclear matter at $T = 5\text{MeV}$ as a function of density and find the rates to be greatly enhanced in the presence of strong magnetic field. Strong magnetic field thus changes the composition of nuclear matter in a very substantial way and enhances the cooling rates as well as the neutronization rates and this would have serious implications for neutron and pulsar dynamics.
References


Figure 1: Effect of magnetic field on the composition of non-interacting cold nuclear matter. The baryon density, neutron density and proton density are given in units of $fm^{-3}$. The effect of including anomalous magnetic moment of nucleons on the composition is shown for $B = 5 \times 10^5$ by the dashed-dotted curve. The magnetic field is in units of $MeV^2$. 
Figure 2: Effect of magnetic field on the composition and effective proton mass as a function of baryon density for cold interacting nuclear matter for different values of the magnetic field. The densities are in units of $fm^{-3}$, proton effective mass in Mev and magnetic field in units of $MeV^2$. The effect of including anomalous magnetic moment of nucleons on the composition and on proton mass is shown for $B = 10^5$ by the dotted curve.
Figure 3: Neutrino emissivity in units of $E_0$ (see text) due to direct URCA process as a function of baryon density for $B = 0$ and $B = 10^5 MeV^2$ when the electrons and protons are completely polarised. 27
Figure 4: Neutronization rate in units of $\Gamma_0$(see text) as a function of baryon density for $B = 0$ and $B = 10^4 MeV^2$ for hot non-interacting non-degenerate nuclear matter at $T = 5 MeV$. The baryon density is given in gm/cc.