Spinor formulation of topologically massive gravity

A. N. Aliev and Y. Nutku
TÜBİTAK - Marmara Research Center
Research Institute for Basic Sciences
Department of Physics
41470 Gebze, Turkey

In the framework of real 2-component spinors in three dimensional space-time we present a description of topologically massive gravity (TMG) in terms of differential forms with triad scalar coefficients. This is essentially a real version of the Newman-Penrose formalism in general relativity. A triad formulation of TMG was considered earlier by Hall, Morgan and Perjes, however, due to an unfortunate choice of signature some of the spinors underlying the Hall-Morgan-Perjes formalism are real, while others are pure imaginary. We obtain the basic geometrical identities as well as the TMG field equations including a cosmological constant for the appropriate signature. As an application of this formalism we discuss the Bianchi Type $VIII - IX$ exact solutions of TMG and point out that they are parallelizable manifolds. We also consider various re-identifications of these homogeneous spaces that result in black hole solutions of TMG.

1 Introduction

The interesting properties of Einstein’s theory of gravitation in three dimensions are well-recognized [1, 2, 3]. Due to the equivalence between the Riemann and Ricci tensors in three dimensions, the space-time curvature is completely determined by the energy-momentum tensor of the exterior matter sources. In the absence of sources space-time is flat, while the coupling
to particle-like sources [3] as well as the presence of a cosmological constant [4, 5] provide exact solutions with non-trivial static geometry. On the other hand the topology can be non-trivial even for vacuum [1].

There is, however, a dynamical theory of gravity in three dimensions. This is Deser, Jackiw and Templeton’s theory [6] of topologically massive gravity (TMG). The geometry of exact solutions of TMG is non-trivial and in order to investigate the properties of these exact solutions it will be useful to develop a triad formalism analogous to the Newman-Penrose formalism [7, 8] of general relativity and relate it to real 2-component spinors. Earlier Hall, Morgan and Perjes [9] (HMP) have presented a triad formalism for vacuum TMG without explicit reference to spinors. Due to an unfortunate choice of signature some of the 2-component spinors underlying HMP formalism are real while others are pure imaginary. In view of the fundamental importance of a triad formalism based on real spinors for TMG, we shall obtain the basic geometrical identities as well as the field equations of TMG with a cosmological constant for the appropriate signature and relate it to real 2-component spinors. We shall express these results in terms of differential forms with triad scalar coefficients. This is based on some unpublished work by one of us [10] for the Newman-Penrose formalism in general relativity. The formulation of TMG in terms of differential forms was considered earlier by Dereli and Tucker [11] without the use of triad scalars.

The Bianchi Type VIII and IX exact solutions of TMG [12] are obtained readily through the use of this spinor formalism and therefore they provide a simple illustration of our approach. We shall show that the Bianchi Type VIII solution can be re-identified, à la Bañados, Teitelboim and Zanelli [5], to result in a generalization of the black hole solution of TMG [13]. Finally we shall show that Bianchi Type $VIII-IIX$ exact solutions of TMG are oriented, parallelizable manifolds which provide new examples of 3-manifolds where the ideas of Cartan and Schouten are realized [14].

2 Spin frame and the triad

We shall consider a 3-dimensional Riemannian space-time described by a metric with signature $(+−−)$. In this case we can introduce real 2-component spinors in complete analogy with the spinor representation of 4-dimensional space-time [7]. For this purpose we start by introducing a
spin frame with the basis

$$\zeta^A_{(a)} = \{ \sigma^A, \iota^A \} \quad A = 1, 2 \quad a = 0, 1$$ (1)

of real 2-component spinors where, capital Latin indices refer to the components of spinors and small Latin indices enclosed by parantheses run over the two values omicron and iota respectively. These indices will be raised and lowered by the Levi-Civita symbol $\epsilon_{AB}$ from the right. Thus

$$o_A = o^B \epsilon_{BA} = -\epsilon_{AB}o^B$$ (2)

and the normalization of the frame will be determined by

$$o_A \iota^A = 1$$ (3)

with all others vanishing identically. The triad of real vectors that corresponds to such a spin frame will be given in terms of the basis spinors through the Infeld-van der Waerden connecting quantities $\sigma^i_{AB}$ which are symmetric $\sigma^i_{AB} = \sigma^i_{BA}$. We have

$$l^i \equiv \sigma^i_{AB}o^A o^B = \sigma^i_{00}$$

$$n^i \equiv \sigma^i_{AB}\iota^A \iota^B = \sigma^i_{11}$$

$$\frac{1}{\sqrt{2}} m^i \equiv \sigma^i_{AB} o^A \iota^B = \sigma^i_{01} = \sigma^i_{10}$$ (4)

where $l, n$ are null and $m$ is space-like. Corresponding to the normalization in eq.(3) they will be subject to

$$l_in^i = l_i n_i = 1, \quad m_i m^i = -1$$ (5)

with all other contractions vanishing identically. In compact notation

$$\sigma^i_{AB} = \begin{pmatrix} l^i & \frac{1}{\sqrt{2}} m^i \\ \frac{1}{\sqrt{2}} m^i & n^i \end{pmatrix}$$ (6)

and its inverse is given by

$$\sigma^i_{AB} = \begin{pmatrix} n_i & -\frac{1}{\sqrt{2}} m_i \\ -\frac{1}{\sqrt{2}} m_i & l_i \end{pmatrix}.$$ (7)
As a consequence of the definitions (6) and (7) we have the completeness relations

\[\delta^i_k = \sigma^{AB}_i \sigma^k_{AB} = l^i n_k + n^i l_k - m^i m_k,\]  

\[\sigma^i_{AB} \sigma^{AB}_i = \frac{1}{2} (\delta^A_M \delta^B_N + \delta^A_N \delta^B_M)\]

and we recall the basic spinor identity in two dimensions

\[\epsilon_M [A \epsilon_{BC}] = 0\]

where square parantheses denote skew symmetrization. One consequence of the identity (10) which will be used repeatedly is

\[\xi_A \eta_B - \xi_B \eta_A = \epsilon_{AB} \xi_M \eta^M\]

for any two 2-component spinors \(\xi\) and \(\eta\).

The components of the metric are given by

\[g_{ik} = \sigma^{AB}_i \sigma^k_{MN} \epsilon_{AM} \epsilon_{BN} = l^i n_k + n^i l_k - m^i m_k\]

and using eq.(7) we may introduce the co-frame

\[\sigma^i_{AB} \, dx^i = \begin{pmatrix} \frac{n}{\sqrt{2}} & -\frac{1}{\sqrt{2}} m \\ \frac{1}{2^3} m & \frac{l}{2} \end{pmatrix}\]

so that the basis 1-forms are given in terms of the triad

\[\{n, l, -m\} = e^a_i dx^i\]

where,

\[e^a_i = \{n_i, l_i, -m_i\}\]

is the inverse of the triad basis

\[e^i_a = \{l^i, n^i, m^i\}.\]

The triad components of the metric are

\[g_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\]
while the space-time interval expressed in terms of the basis 1-forms is given by

$$ds^2 = l \otimes n + n \otimes l - m \otimes m.$$  \hspace{1cm} (18)

Next we define the intrinsic derivative operators, or the derivative in the direction of the legs of the triad through

$$D = l^i \frac{\partial}{\partial x^i} \quad \Delta = n^i \frac{\partial}{\partial x^i} \quad \delta = m^i \frac{\partial}{\partial x^i}$$ \hspace{1cm} (19)

and

$$d = l \Delta + n D - m \delta$$ \hspace{1cm} (20)

is the resolution of the exterior derivative along the triad.

### 3 Ricci rotation coefficients

Starting with the triad basis we have the Ricci rotation coefficients which are the components of the Levi-Civita connection

$$\gamma_{cab} = e_{a \; i \; k} e^{i \; c \; e^k_b}$$ \hspace{1cm} \gamma_{cab} = -\gamma_{cabo} \hspace{1cm} (21)

where semicolon denotes the covariant derivative. Using definitions analogous to those in the Newman-Penrose formalism, the Ricci rotation coefficients can be named

$$\begin{align*}
\gamma_{100} &= \epsilon = l_{i; k} n^i \ell^k \\
\gamma_{101} &= -\epsilon' = l_{i; k} n^i n^k \\
\gamma_{102} &= \alpha = l_{i; k} n^i \ell^k \\
\gamma_{200} &= \kappa = l_{i; k} m^i \ell^k \\
\gamma_{201} &= \tau = l_{i; k} m^i n^k \\
\gamma_{202} &= \sigma = l_{i; k} m^i m^k \\
\gamma_{210} &= \tau' = n_{i; k} m^i \ell^k \\
\gamma_{211} &= \kappa' = n_{i; k} m^i n^k \\
\gamma_{212} &= \sigma' = n_{i; k} m^i m^k
\end{align*}$$ \hspace{1cm} (22)

where prime denotes the analogue of the prime symmetry operation of Geroch, Held and Penrose [8] which results from the interchange of $l$ and $n$ leaving $m$ fixed. This operation does not change the normalization conditions in eqs.(3), (5) and we note that $\alpha' = -\alpha$. The Ricci rotation coefficients can be grouped together in a table

5
Table 1.

One of the advantages of presenting the triad formalism of TMG in terms of differential forms lies in the great simplification it provides for the calculation of the Ricci rotation coefficients. Taking the exterior derivative of the basis 1-forms and expressing the result in terms of the basis 2-forms yields the spin coefficients through a linear algebraic system. That is, by a comparison of the results with

\[
\begin{align*}
    dl &= -\epsilon l \wedge n + (\alpha - \tau) l \wedge m - \kappa n \wedge m \\
    dn &= \epsilon' l \wedge n - \kappa' l \wedge m - (\alpha + \tau') n \wedge m \\
    dm &= (\tau' - \tau) l \wedge n - \sigma' l \wedge m - \sigma n \wedge m
\end{align*}
\]

we can obtain the Ricci rotation coefficients through the solution of a linear algebraic system.

4 Spin coefficients

Turning to the definition of spin coefficients which are the components of the spin connection we shall begin with the spin frame and use

\[
\Gamma_{(a)(b)(c)(d)} = \zeta_{(a) A; CD} \zeta_{(b)}^A \zeta_{(c)}^C \zeta_{(d)}^D,
\]

or more briefly

\[
\Gamma_{(a)(b) CD} = \zeta_{(a) A; CD} \zeta_{(b)}^A
\]

where, once again, semicolon stands for the covariant derivative. The correspondence between the covariant derivatives of the vector and spinor fields
is established by means of the Infeld-van der Waerden connecting quantities (6) and (7). Thus
\[
\Psi_{i;k} = \sigma_i^{AB} \sigma_k^{CD} \Psi_{AB;CD}
\]
\[
\Psi_{AB;CD} = \sigma_i^{AB} \sigma_k^{CD} \Psi_{i;k}
\]
as a consequence of which we can easily verify that
\[
\sigma_i^{AB;CD} = 0, \quad \sigma_i^{AB} :_{CD} = 0, \quad \epsilon_{AB;CD} = 0.
\]

The spin coefficients are symmetric both in the first and the last pairs of indices
\[
\Gamma^{(a)(b)(c)(d)} = \Gamma^{(b)(a)(c)(d)} = \Gamma^{(a)(b)(d)(c)}
\]
since the spin frame is defined solely through real spinors. The definition of spin coefficients can be given through the formula
\[
\Gamma^{(a)(b)}_{XY} = \frac{1}{2} \epsilon^{(c)(d)} \zeta^A_{(a)} \zeta^D_{(d)} \left[ \zeta_{(b)}^A \zeta_{(c)}^D \right]_{XY}
\]
which is a real version of the analogous formula in general relativity [15]. This can be verified by direct calculation
\[
\Gamma^{(a)(b)}_{XY} = \frac{1}{2} \epsilon^{(c)(d)} \left[ \zeta^A_{(a)} \zeta^D_{(d)} \zeta_{(c)}^A \zeta_{(b)}^D \right]_{XY} + \zeta^A_{(a)} \zeta^D_{(d)} \zeta_{(c)}^A \zeta_{(b)}^D
\]
\[
= \frac{1}{2} \epsilon^{(c)(d)} \delta_{(a)}^{(k)} \Gamma^{(k)}_{(b)}_{XY} = \Gamma^{(a)(b)}_{XY}.
\]
Using eq.(29) we can evaluate the spin coefficients in terms of the Ricci rotation coefficients. As an example we have
\[
\Gamma_{0000} = \frac{1}{2} \left[ o^{A}o^{B} (o^{A}o^{B})_{00} - o^{A}o^{B} (o^{A}o^{B})_{00} \right]
\]
\[
= \frac{1}{2\sqrt{2}} \left( l_{i;k} l^k m^i - m_{i;k} l^k l^i \right) = \frac{1}{\sqrt{2}} \kappa
\]
and the rest of the spin coefficients can be evaluated similarly. It is clear that due to their symmetry properties (28) only nine spin coefficients need to be evaluated. We find that the spin coefficients can be listed according to the table

7
Table 2.

\[\begin{array}{cccc}
\begin{array}{llll}
\hline
a & b & 0 & 1 \\
\hline
cd & & 0 & 1 \\
00 & \frac{1}{2} \epsilon & -\frac{1}{\sqrt{2}} \kappa & -\frac{1}{\sqrt{2}} \tau' \\
01 & \frac{1}{2\sqrt{2}} \alpha & -\frac{1}{2} \sigma & -\frac{1}{2} \sigma' \\
11 & -\frac{1}{2} \epsilon' & -\frac{1}{\sqrt{2}} \tau & -\frac{1}{\sqrt{2}} \kappa' \\
\hline
\end{array}
\end{array}\]

where \( \Gamma_{abcd} = \Gamma^q_{ac} d q_b \).

### 5 Connection 1-form

The connection 1-form is defined by

\[\Gamma^b_a = \Gamma^b_{ac} i_{cd} \sigma^i dx^i\]  \hspace{1cm} (32)

which is a traceless 2 \times 2 real matrix of 1-forms and using eq.(7) and table 2 we have

\[\begin{align*}
\Gamma^0_0 &= \frac{1}{2} l_{i;k} n^i dx^k = \frac{1}{2} (-\epsilon' l + \epsilon n - \alpha m) \\
\Gamma^0_1 &= -\frac{1}{\sqrt{2}} l_{i;k} m^i dx^k = \frac{1}{\sqrt{2}} (-\tau l - \kappa n + \sigma m) \\
\Gamma^1_0 &= -\frac{1}{\sqrt{2}} n_{i;k} m^i dx^k = \frac{1}{\sqrt{2}} (-\kappa' l - \tau' n + \sigma' m)
\end{align*}\]  \hspace{1cm} (33)

with \( \Gamma^1_1 = -\Gamma^0_0 \). Under the prime operation the entries of \( \Gamma^b_a \) flip, \( (\Gamma^0_0)' = \Gamma^0_1 \), \( (\Gamma^1_0)' = \Gamma^1_0 \) and for the rest we recall that the prime operation is an involution. The most prominent role played by the connection 1-form is in holonomy

\[P e^i \int_{-\infty}^{s} \Gamma^b_a i_{cd} \sigma^i dx^i \]  \hspace{1cm} (34)

where \( P \) denotes path-ordering.
6 Curvature 2-form

The curvature 2-form is obtained from the connection 1-form through

\[ R^b_a = d \Gamma^b_a - \Gamma^m_a \wedge \Gamma^b_m \]  

(35)

but first we must define the triad scalars for curvature. In three dimensions there is no Weyl tensor and the Riemannian curvature tensor can be decomposed in terms of the Ricci tensor \( R_{ab} = R^c_{acb} \) and the scalar of curvature \( R \).

Using the traceless Ricci tensor

\[ \Theta_{ab} = -\frac{1}{2} \left( R_{ab} - \frac{1}{3} g_{ab} R \right) \]  

(36)

the Riemann tensor is given by \([16]\)

\[ R_{abcd} = -2g_{ac} \Theta_{bd} + 2g_{ad} \Theta_{bc} - 2g_{bd} \Theta_{ac} + 2g_{bc} \Theta_{ad} - \frac{1}{6} (g_{ad} g_{bc} - g_{ac} g_{bd}) R. \]  

(37)

and we can pass to the spinor analogue of this expression

\[ R_{ABDXC} = -2 \Phi_{BDXZ} \epsilon^{AC} \epsilon^{WY} - 2 \Phi_{ACWY} \epsilon_{BD} \epsilon_{XZ} \]

\[ + 2 \Phi_{BCXY} \epsilon_{AD} \epsilon_{WZ} + 2 \Phi_{ADWZ} \epsilon_{BC} \epsilon_{XY} \]

\[ + \frac{1}{6} (\epsilon_{AC} \epsilon^{WY} \epsilon_{BD} \epsilon_{XZ} - \epsilon_{AD} \epsilon_{BC} \epsilon_{WZ} \epsilon_{XY}) R \]  

(38)

where the quantities \( \Phi_{ABDX} \) are curvature spinors. The spinor analogue of the trace-free Ricci tensor is similarly

\[ R_{ABDX} = -2 \Phi_{ABWX} + 6 \Lambda \epsilon_{AB} \epsilon_{WX} \]  

(39)

where we have introduced the definition

\[ \Lambda = \frac{1}{18} R \]  

(40)

for the curvature scalar. With these preliminaries the curvature 2-form can be written in the form

\[ R^m_n = \frac{1}{2} \zeta^M_l \zeta^{(n)}_N R_{MX}^{NX} \sigma^C_i \sigma^D_k d\xi_i \wedge d\xi_k \]  

(41)
and in terms of the basis 2-forms we get

\[ R^0_0 = \left( 2\Phi_{11} - \frac{3}{2} \Lambda \right) l \wedge n - \Phi_{12} l \wedge m + \Phi_{10} n \wedge m \]

\[ R^1_0 = -\sqrt{2} \Phi_{01} l \wedge n + \frac{1}{\sqrt{2}} \Phi_{02} l \wedge m - \sqrt{2} \Phi_{00} n \wedge m \]  \hspace{1cm} (42)

\[ R^0_1 = \sqrt{2} \Phi_{12} l \wedge n - \sqrt{2} \Phi_{22} l \wedge m + \frac{1}{\sqrt{2}} \Phi_{02} n \wedge m \]

where we have introduced the definitions

\[ \Phi_{00} := \Phi_{0000}, \quad \Phi_{01} := \sqrt{2} \Phi_{0010}, \quad \Phi_{02} := 2 \Phi_{0011}, \]

\[ \Phi_{11} := \Phi_{0101}, \quad \Phi_{12} := \sqrt{2} \Phi_{0111}, \quad \Phi_{22} := \Phi_{1111}. \]  \hspace{1cm} (43)

Taking into account the equation (39) along with the relations (6) and (7) we see that these quantities are triad scalars

\[ \Phi_{00} = -\frac{1}{2} R_{ik} l^i l^k, \quad \Phi_{01} = -\frac{1}{2} R_{ik} l^i m^k, \quad \Phi_{02} = -\frac{1}{2} R_{ik} m^i m^k, \]

\[ \Phi_{11} = -\frac{1}{2} R_{ik} l^i n^k + 3\Lambda, \quad \Phi_{12} = -\frac{1}{2} R_{ik} n^i m^k, \quad \Phi_{22} = -\frac{1}{2} R_{ik} n^i n^k. \]  \hspace{1cm} (44)

Note that \( \Phi_{ik} \) is symmetric and under the prime operation the index 1 remains unchanged, while the rest flip 0 \( \leftrightarrow \) 2.

7 Ricci and Bianchi identities

From the definition of the curvature 2-form (35) and eqs.(33), (42) as well as the relations in eqs.(20), (23) we obtain

\[ D\epsilon' + \Delta \epsilon + 2\epsilon \epsilon' + \alpha (\tau - \tau') - \tau \tau' + \kappa \kappa' = 4 \Phi_{11} - 3\Lambda \]  \hspace{1cm} (45)

\[ \delta \kappa - D\sigma + \sigma^2 - \kappa (\tau + \tau') + \sigma \epsilon - 2\kappa \alpha = 2 \Phi_{00} \]

\[ \delta \kappa' - \Delta \sigma' + \sigma'^2 - \kappa' (\tau + \tau') + \sigma' \epsilon' + 2\kappa' \alpha = 2 \Phi_{22} \]  \hspace{1cm} (46)

\[ \Delta \sigma - \delta \tau - \sigma \sigma' + \kappa \kappa' + \epsilon \sigma + \tau^2 = \Phi_{02} \]

\[ D\sigma' - \delta \tau' - \sigma \sigma' + \kappa \kappa' + \epsilon \sigma' + \tau'^2 = \Phi_{02} \]  \hspace{1cm} (47)
\[ \Delta \alpha + \delta \epsilon' + \epsilon'(\alpha - \tau) - \sigma'(\alpha + \tau') + \kappa'(\epsilon + \sigma) = 2 \Phi_{12} \]  
(48)

\[ -D\alpha + \delta \epsilon - \epsilon(\alpha + \tau') + \sigma(\alpha - \tau') + \kappa(\epsilon' + \sigma') = 2 \Phi_{01} \]

\[ \Delta \kappa - D\tau + 2\kappa' \epsilon + \sigma(\tau - \tau') = 2 \Phi_{01} \]  
(49)

\[ D\kappa' - \Delta \tau' + 2\kappa' \epsilon - \sigma'(\tau - \tau') = 2 \Phi_{12} \]

which make up the set of Ricci identities.

The full Bianchi identities are given by

\[ dR^b_a + \Gamma^m_a \wedge R^b_m - R^b_a \wedge \Gamma^m_b = 0 \]  
(50)

and we recall that in 3 dimensions there is no difference between the full and contracted Bianchi identities. Using the same procedure that we have used to obtain the Ricci identities we can write the Bianchi identities (50) in terms of triad scalars

\[ \Delta \Phi_{01} + D\Phi_{12} - 2\delta(\Phi_{11} - \frac{3}{4} \Lambda) + 2(\tau + \tau')(\Phi_{11} - \frac{3}{4} \Lambda) + \kappa \Phi_{22} + \kappa' \Phi_{00} - (2\sigma - \epsilon) \Phi_{12} - (2\sigma' - \epsilon') \Phi_{01} + \frac{1}{2}(\tau + \tau') \Phi_{02} = 0 \]  
(51)

\[ D\Phi_{02} + 2\Delta \Phi_{00} - 2\delta \Phi_{01} + 2(\tau + \tau' + \alpha) \Phi_{01} - \sigma \Phi_{02} - 2(\sigma' - 2\epsilon') \Phi_{00} + 2\kappa \Phi_{12} - \sigma'(4\Phi_{11} - 3\Lambda) = 0 \]

\[ \Delta \Phi_{02} + 2D\Phi_{22} - 2\delta \Phi_{12} + 2(2\tau' + \tau - \alpha) \Phi_{12} - \sigma' \Phi_{02} - 2(\sigma - 2\epsilon) \Phi_{22} + 2k' \Phi_{01} - \sigma'(4\Phi_{11} - 3\Lambda) = 0 \]  
(52)

Under the prime operation the unpaired Ricci and Bianchi identities (45), (51) remain invariant while the paired equations (46), (47), (48), (49), (52) simply exchange places.

8 Cotton 2-form

The conformal properties of 3-dimensional space-time are described by the Cotton tensor [16]

\[ C^{ik} = \epsilon^{ijkl} \left( R^k_l - \frac{1}{4} \delta^k_l R \right) \]  
(53)
where $\epsilon_{ijl}$ is the completely skew 3-dimensional Levi-Civita tensor density of weight $-1$. It remains invariant under conformal transformations of the 3-dimensional metric and vanishes for conformally flat metrics. The Cotton tensor satisfies the conditions

$$C^{ik} = C^{ki}, \quad C^i_i = 0, \quad C^{ik}_{:i} = 0$$

(54)

of a symmetric, traceless, covariantly conserved tensor.

For the purpose of writing the TMG field equations in terms of differential forms we need to construct a 2-form out of the Cotton tensor. The required expression is given by

$$C^b_a = d^* R^b_a - \Gamma^m_a \wedge R^b_m - R^m_a \wedge \Gamma^b_m - \frac{1}{4} dR \wedge \Sigma^b_a$$

(55)

where the 1-form $\Sigma^b_a$ is obtained from the co-frame (13) by lowering an appropriate spinor index

$$\Sigma^b_a = \frac{1}{\sqrt{2}} \sigma^b_i dx^i$$

(56)

and therefore we need the dual of the curvature 2-form (41) which in turn requires the duals of the basis 2-forms. Starting with the definitions

$$^* dx^i = \frac{1}{2} \epsilon^{ijk} dx^j \wedge dx^k, \quad ^{**} dx^i = dx^i$$

(57)

and using the completeness relation (8) we find

$$^*(l \wedge n) = -m$$
$$^*(l \wedge m) = -l$$
$$^*(n \wedge m) = n$$

(58)

summarizing the effect of the Hodge star operator on the basis 2-forms. Then the duals of the curvature 2-forms (42) are simply

$$^* R_0^0 = \Phi_{12} l + \Phi_{10} n - 2 \left( \Phi_{11} - \frac{3}{4} \Lambda \right) m$$
$$^* R_0^1 = -\frac{1}{\sqrt{2}} \Phi_{02} l - \sqrt{2} \Phi_{00} n + \sqrt{2} \Phi_{01} m$$

(59)
\[ ^* R_1^0 = \sqrt{2} \Phi_{22} l + \frac{1}{\sqrt{2}} \Phi_{02} n - \sqrt{2} \Phi_{12} m \]

and inserting into the equation (55) the quantities (20), (33), (59) and (56) we can express the Cotton 2-form in terms of the basis 2-forms. We find

\[
C_0^0 = \{ \Delta \Phi_{10} - D\Phi_{12} + 3(\tau - \tau')\Phi_{11} + \epsilon'\Phi_{01} - \epsilon\Phi_{12} - \kappa\Phi_{22} \\
+ \kappa'\Phi_{00} \} l \wedge n + \{ \delta\Phi_{12} - \Delta\Phi_{02} - 2\kappa'\Phi_{01} + (\alpha - 2\tau)\Phi_{12} \\
+ \sigma\Phi_{22} + 3\sigma'\Phi_{11} + \frac{9}{4} \Delta \Lambda \} l \wedge m + \{ \delta\Phi_{01} - D\Phi_{02} - 2\kappa\Phi_{12} \\
- (\alpha + 2\tau')\Phi_{01} + \sigma'\Phi_{00} + 3\sigma\Phi_{11} + \frac{9}{4} \Delta \Lambda \} n \wedge m \]
\]

\[
C_0^1 = \sqrt{2} \{ D\Phi_{11} - \Delta\Phi_{00} + (\tau' - 2\tau)\Phi_{11} - 2\epsilon'\Phi_{00} + \kappa\Phi_{12} \\
- \frac{3}{4} D\Lambda \} l \wedge n + \sqrt{2} \{ \Delta\Phi_{01} - \delta\Phi_{11} + \kappa'\Phi_{00} + 3\tau\Phi_{11} \\
- \sigma\Phi_{12} + (\epsilon' - \sigma')\Phi_{01} + \frac{3}{4} \delta \Lambda \} l \wedge m + \sqrt{2} \{ D\Phi_{01} - \delta\Phi_{00} \\
+ 3\kappa\Phi_{11} - (\epsilon + 2\sigma)\Phi_{01} + (\tau' + 2\alpha)\Phi_{00} \} n \wedge m \]
\]

\[
C_1^0 = \sqrt{2} \{ \Delta\Phi_{11} - D\Phi_{22} + (\tau - 2\tau')\Phi_{12} - 2\epsilon\Phi_{22} + \kappa'\Phi_{01} \\
- \frac{3}{4} \Delta \Lambda \} l \wedge n + \sqrt{2} \{ \delta\Phi_{22} - \Delta\Phi_{12} - 3\kappa'\Phi_{11} + (\epsilon' + 2\sigma')\Phi_{12} \\
- (\tau - 2\alpha)\Phi_{22} \} l \wedge m + \sqrt{2} \{ \delta\Phi_{11} - D\Phi_{12} + \sigma'\Phi_{01} - \kappa\Phi_{22} \\
+ (\sigma - \epsilon)\Phi_{12} - 3\tau'\Phi_{11} - \frac{3}{4} \delta \Lambda \} n \wedge m \]
\]

Unlike the curvature 2-form, the cotton 2-form picks up a minus sign under the prime operation in addition to the flip noted earlier \((C_0^1)' = -C_1^0\), \((C_0^0)' = -C_1^1\) which is due to its parity violating nature.

### 9 Field equations

Deser, Jackiw and Templeton’s theory [6] of TMG requires the proportionality of the Einstein and Cotton tensors in 3-dimensions. The field equations of TMG are given by

\[
G^{ik} + \frac{1}{\mu} C^{ik} = \lambda g^{ik} \]
\]
where $G^{ik}$ is the Einstein tensor, $\mu$ is the DJT constant of proportionality which can be regarded as topological mass and $\lambda$ is the cosmological constant. Using the curvature and Cotton 2-forms described in the previous sections, we can rewrite the field equations (63) in terms of 2-forms

$$R^a_b + \frac{1}{\mu} C^a_c \mu + \lambda \Sigma^m_a \wedge \Sigma^b_m = 0. \quad (64)$$

The expression of the field equations in terms of triad scalars follows from the substitution of the results in eqs.(42), (56) and (60) - (62) into eq.(64).

The full set of TMG field equations are

$$D\Phi_{12} - \Delta\Phi_{10} - 3(\tau - \tau')\Phi_{11} - \epsilon'\Phi_{01} + \epsilon\Phi_{12} + \kappa\Phi_{22} - \kappa'\Phi_{00} = \mu(\lambda + \Phi_{02}) \quad (65)$$

$$\delta\Phi_{01} - D\Phi_{02} - 2\kappa\Phi_{12} - (\alpha + 2\tau')\Phi_{01} + \sigma'\Phi_{00} + 3\sigma\Phi_{11} + \frac{3}{4} D\Lambda = -\mu\Phi_{01} \quad (66)$$

$$\delta\Phi_{12} - \Delta\Phi_{02} - 2\kappa'\Phi_{01} + (\alpha - 2\tau)\Phi_{12} + \sigma\Phi_{22} + 3\sigma'\Phi_{11} + \frac{3}{4} \Delta\Lambda = \mu\Phi_{12}$$

$$D\Phi_{11} - \Delta\Phi_{00} + (\tau' - 2\tau)\Phi_{01} - 2\epsilon'\Phi_{00} + \kappa\Phi_{12} - \frac{3}{4} D\Lambda = \mu\Phi_{01} \quad (67)$$

$$\Delta\Phi_{11} - D\Phi_{22} + (\tau - 2\tau')\Phi_{12} - 2\epsilon\Phi_{22} + \kappa'\Phi_{01} - \frac{3}{4} \Delta\Lambda = -\mu\Phi_{12}$$

$$\Delta\Phi_{01} - \delta\Phi_{11} + \kappa'\Phi_{00} + 3\tau\Phi_{11} - \sigma\Phi_{12} + (\epsilon' - \sigma')\Phi_{01} + \frac{3}{4} \delta\Lambda = -\frac{1}{2}\mu(\lambda + \Phi_{02}) \quad (68)$$

$$D\Phi_{12} - \delta\Phi_{11} + \kappa\Phi_{22} + 3\tau\Phi_{11} - \sigma'\Phi_{01} + (\epsilon - \sigma)\Phi_{12} + \frac{3}{4} \delta\Lambda = \frac{1}{2}\mu(\lambda + \Phi_{02})$$

$$D\Phi_{01} - \delta\Phi_{00} + 3\kappa\Phi_{11} - (\epsilon + 2\sigma)\Phi_{01} + (\tau + 2\alpha)\Phi_{00} = \mu\Phi_{00} \quad (69)$$

$$\Delta\Phi_{12} - \delta\Phi_{22} + 3\kappa'\Phi_{11} - (\epsilon' + 2\sigma')\Phi_{12} + (\tau - 2\alpha)\Phi_{22} = -\mu\Phi_{22}$$

Finally the DJT field equations (63) imply

$$\Lambda = -\frac{1}{3}\lambda \quad (70)$$

since the Cotton tensor is traceless.

We see that the TMG field equations remain under the prime operation provided that the sign of the DJT coupling constant is also changed simultaneously.
10 Homogeneous Spaces

In order to illustrate the usefulness of the formalism we have presented above we shall now consider its application to an exact solution of the TMG field equations. This is a homogeneous space-time related to the Bianchi Type VIII squashed 3-pseudo-sphere [12]. We start with left-invariant 1-forms parametrized in terms of Euler angles

\begin{align}
\sigma^0 &= d\psi + \cosh \theta \, d\phi \\
\sigma^1 &= -\sin \psi \, d\theta + \cos \psi \, \sinh \theta \, d\phi \\
\sigma^2 &= \cos \psi \, d\theta + \sin \psi \, \sinh \theta \, d\phi
\end{align}

(71)

(72)

(73)

which are subject to the Maurer-Cartan equations

\[ d\sigma^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \]

(74)

with structure constants

\[ C^0_{12} = -1, \quad C^1_{20} = 1, \quad C^2_{01} = 1 \]

(75)

of Bianchi Type VIII. In terms of these left-invariant 1-forms the TMG co-frame is defined by

\[ \omega^0 = \lambda_0 \, \sigma^0, \quad \omega^1 = \lambda_1 \, \sigma^1, \quad \omega^2 = \lambda_2 \, \sigma^2, \]

(76)

where \( \lambda_0, \lambda_1, \) and \( \lambda_2 \) are constants. The exact solution of TMG results in a relationship between these scale factors \( \lambda_i \), the cosmological constant \( \lambda \) and the TMG coupling constant \( \mu \).

We define triad basis 1-forms

\[ l = \frac{1}{\sqrt{2}} (\omega^0 - \omega^1), \quad n = \frac{1}{\sqrt{2}} (\omega^0 + \omega^1), \quad m = \omega^2 \]

(77)

and taking their exterior derivative we have

\[ dl = \frac{1}{2 \lambda_0 \lambda_1 \lambda_2} \left[ (\lambda_0^2 + \lambda_1^2) \, l \wedge m - (\lambda_0^2 - \lambda_1^2) \, n \wedge m \right] \]

\[ dn = \frac{1}{2 \lambda_0 \lambda_1 \lambda_2} \left[ (\lambda_0^2 - \lambda_1^2) \, l \wedge m - (\lambda_0^2 + \lambda_1^2) \, n \wedge m \right] \]

(78)

\[ dm = \frac{\lambda_0}{\lambda_0 \lambda_1 \lambda_2} \, l \wedge n \]
Comparison of these expressions with eqs. (23) yields the Ricci rotation coefficients

\[ \begin{align*}
\epsilon &= \epsilon' = \sigma = \sigma' = 0, \\
\tau &= -\tau' = -\frac{\lambda_2}{\lambda_0\lambda_1}, \\
\kappa &= -\kappa' = \frac{1}{2\lambda_0\lambda_1\lambda_2} \left( \lambda_0^2 - \lambda_1^2 \right), \\
\alpha &= \frac{1}{2\lambda_0\lambda_1\lambda_2} \left( \lambda_0^2 + \lambda_1^2 - \lambda_2^2 \right). 
\end{align*} \] (79)

Inserting these quantities into eq. (70) we obtain

\[ 2\alpha\tau + \kappa^2 - \tau^2 = 3\lambda, \] (80)
or using eq. (79) we have the explicit form

\[ (\lambda_0 + \lambda_1 + \lambda_2)(-\lambda_0 + \lambda_1 + \lambda_2)(\lambda_0 - \lambda_1 + \lambda_2)(-\lambda_0 - \lambda_1 + \lambda_2) = 12\lambda(\lambda_0\lambda_1\lambda_2)^2. \] (81)

With the rotation coefficients (79) the TMG field equations are drastically simplified and we have only two independent equations

\[ \begin{align*}
3\kappa \Phi_{11} + (2\alpha - \tau) \Phi_{00} &= \mu \Phi_{00} \\
3\tau' \Phi_{11} + \kappa \Phi_{22} &= \frac{1}{2}\mu(\lambda + \Phi_{02})
\end{align*} \] (82) (83)

Comparison of these two equations and taking into account eq. (70) we obtain a polynomial constraint involving only the scalar factors \( \lambda_0, \lambda_1, \lambda_2 \). This can be readily factorized to yield

\[ \frac{1}{(\lambda_0\lambda_1\lambda_2)^5} \left( \lambda_1^2 - \lambda_0^2 \right) \left( \lambda_0^2 - \lambda_2^2 \right) \left( \lambda_0 + \lambda_1 + \lambda_2 \right)(-\lambda_0 + \lambda_1 + \lambda_2)(\lambda_0 - \lambda_1 + \lambda_2)(-\lambda_0 - \lambda_1 + \lambda_2) = 0. \] (84)

and the solution of the TMG field equations is now immediate. When two of these scale factors coincide, say \( \lambda_1 = \lambda_2 \), the simultaneous solution of eqs. (81), (82) and (83) has either the form

\[ \lambda_0 = \lambda_1 = \lambda_2 = \frac{1}{2\sqrt{-\lambda}}. \] (85)

16
or
\[\lambda_0 = \frac{6\mu}{\mu^2 - 27\lambda}, \quad \lambda_1 = \lambda_2 = \frac{3}{\sqrt{\mu^2 - 27\lambda}}.\] (86)

Clearly the solution in eqs. (85) gives rise to the vanishing of the Cotton tensor which is a trivial solution of TMG. The solution (86) is a generalization of the Vuorio solution [17] with a cosmological constant [13]. The general Bianchi Type \textit{VIII} solution where the scale factors are related by \(\lambda_0 \pm \lambda_1 \pm \lambda_2 = 0\) does not admit a cosmological constant. In all cases for the DJT coupling constant we get\(^1\)
\[\mu = \frac{\lambda_0^2 + \lambda_1^2 + \lambda_2^2}{\lambda_0 \lambda_1 \lambda_2}\] (87)
from eqs. (82), (83).

11 Black holes in TMG

The Bianchi Type \textit{VIII} homogeneous space-time can be re-identified to yield a black hole solution of TMG in the same way that AdS has been re-identified by Bañados, Teitelboim and Zanelli to yield the black hole solution of the 3-dimensional Einstein field equations with a cosmological constant [5]. This process was further clarified by Horowitz and Welch [18] and we shall use their approach here. Earlier [13], such a re-identification was carried out for \(\lambda_1 = \lambda_2\) which results in a stationary 2-parameter black hole solution. But now we shall show that the general Bianchi Type \textit{VIII} solution can also be re-identified to yield a non-stationary, non-axisymmetric 3-parameter solution that has many properties akin to those of a black hole.

For the general case of Bianchi Type \textit{VIII} solution we shall use introduce a parameter \(p\)
\[\lambda_1 = \frac{3 + p^2}{\mu(1 - p)}, \quad \lambda_2 = \frac{3 + p^2}{\mu(1 + p)}\] (88)
and use the root
\[\lambda_0 = \lambda_1 + \lambda_2 = \frac{2(3 + p^2)}{\mu(1 - p^2)}\] (89)

\(^1\)We take this opportunity to correct an extra factor of 2 that crept into the denominator of this expression in [12].
of eq.(81). Then the metric is given by
\[ ds^2 = \mu^2 \left( \frac{1}{3} - \frac{p^2}{3 + p^2} \right)^2 \left\{ 4 (d\psi + \cosh \theta d\phi)^2 - (1 + p^2 - 2p \cos 2\psi) d\theta^2 
+ 4p \sin 2\psi \sin \theta \ d\theta \ d\phi - (1 + p^2 + 2p \cos 2\psi) \sin^2 \theta \ d\phi^2 \right\} \]
and introducing a new variable
\[ \sinh \left( \frac{\theta}{2} \right) = \frac{\mu}{2} \frac{1 - p^2}{3 + p^2} R \]
we find that the Bianchi VIII metric (90) becomes
\[ ds^2 = \frac{4}{\mu^2} \left( \frac{3 + p^2}{1 - p^2} \right)^2 \left\{ d\psi + \left[ 1 + \frac{1}{4} \left( \frac{1}{3 + p^2} \right)^2 \mu^2 R^2 \right] d\phi \right\}^2 
- \frac{(1 + p^2 - 2p \cos 2\psi)}{1 + \frac{1}{4} \left( \frac{1}{3 + p^2} \right)^2 \mu^2 R^2} dR^2 + 4p \sin 2\psi R \ dR \ d\phi 
- (1 + p^2 + 2p \cos 2\psi) \left[ 1 + \frac{1}{4} \left( \frac{1}{3 + p^2} \right)^2 \mu^2 R^2 \right] R^2 d\phi^2 \]
which is a limiting form of the black hole metric. We can dress up this metric with two parameters through the transformations \[18]\]
\[ \left( \frac{\mu}{2} \right)^2 \left( \frac{1}{3 + p^2} \right)^2 R^2 = \frac{r^2 - r_+}{r_+ - r_-} \]
\[ \psi = F(t, \varphi) \]
\[ \phi = \gamma \varphi \]
where \( r_+ \), \( r_- \) and \( \gamma \) are constants. The proper choice of \( F \) that will enable us to interpret \( \varphi \) as an angle is determined from the solution of
\[ \left( \frac{\partial F}{\partial \varphi} - \frac{M}{6} \sqrt{\frac{6}{6aJ - M}} \right)^2 = \frac{3}{8} \frac{a^2 J^2}{6aJ - M} (1 + p^2 + 2p \cos 2F) \]
where
\[ a = \frac{1 - p^2}{3 + p^2}, \quad M = \frac{3}{2} a^2 \mu^2 \left( r_+^2 + r_-^2 \right), \quad J = a \mu^2 r_+ r_- \] (95)
and we shall find it convenient to define
\[ f(t, \phi) = \frac{2}{a \mu} \sqrt{\frac{6}{6aJ - M}} \frac{\partial F}{\partial t}. \] (96)

Note that \( F \) is determined from eq. (94) up to an arbitrary additive function of \( t \) alone which can be fixed by physical requirements. Then the metric (92) takes the form
\[
ds^2 = \frac{1}{6} (6aJ - M) f^2 dt^2 + 2f \left( a \mu r^2 - \frac{J}{2\mu} \sqrt{B} \right) dt d\phi
- \frac{1 + p^2 - 2p \cos 2F}{4a^2 \mu^2 r^2 - \frac{1}{6} M + \frac{J^2}{4\mu^2 r^2}} dr^2 + 4p \sqrt{\frac{6}{6aJ - M}} \sin 2F \ r \ dr d\phi
- \frac{1}{6aJ - M} \left[ (6aJ - M \sqrt{B}) \sqrt{B} - \frac{3}{2} a^2 \mu^2 r^2 (4 - B) \right] r^2 d\phi^2
\] (97)
with the definition
\[ B = 1 + p^2 + 2p \cos 2F. \] (98)

It is seen that the constants \( r_+ \) and \( r_- \) introduced above are given by
\[ r_\pm = \frac{M}{3a^2 \mu^2} \left[ 1 \pm \left( 1 - 9 \frac{a^2 J^2}{M^2} \right)^{\frac{1}{2}} \right] \] (99)
describe the location of the outer and inner event horizons.

If we further specialize to \( p = 0 \) which is equivalent to \( \lambda_1 = \lambda_2 \), the metric becomes stationary [13].

## 12 Tele-parallelism

The solution of TMG that we have discussed above has a very interesting property that was not noted before. It describes a parallelizable manifold where we can introduce a global frame so that the curvature can be made to
vanish at the expense of introducing torsion. The notion of a parallelizable manifold is due to Cartan and Schouten [14] and the simplest example they chose as an illustration of their ideas was $S^3$ as a parallelizable manifold. We shall now show that the vacuum Bianchi Type $VIII \sim IX$ exact solutions of TMG provide new examples of a parallelizable 3-dimensional manifolds. They are therefore of interest from a purely mathematical point of view as well.

In order to facilitate comparison with Cartan and Schouten’s treatment of $S^3$ as much as possible, we shall now use a triad with 2 space-like legs and a third leg which can be either time-like or space-like depending on a parameter $\epsilon = \mp 1$. Thus we write the metric in the form

$$ds^2 = \epsilon (\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2$$

where the co-frame is given by eqs.(76) and depending on $\epsilon$ the $\sigma^i$ are now the left-invariant 1-forms of either Bianchi Type $VIII$, or $IX$. We shall henceforth write $S^3$ in quotation marks as our treatment will also cover the case of Lorentz signature. Left-invariant 1-forms $\sigma^i$ satisfy the Maurer-Cartan equations (74) with structure constants

$$C^0_{12} = \epsilon, \quad C^1_{20} = 1, \quad C^2_{01} = 1.$$ (101)

that have a dependence on $\epsilon$ to match the signature of space-time. Note that for Bianchi Type $IX$ we shall require the satisfaction of the Euclidean TMG field equations. Then starting with the frame (76) we find the connection 1-forms

$$\omega^1_0 = \frac{\epsilon}{2\lambda_0 \lambda_1 \lambda_2} (\lambda_0^2 + \lambda_1^2 - \lambda_2^2) \omega^2$$
$$\omega^2_0 = \frac{1}{2\lambda_0 \lambda_1 \lambda_2} (\lambda_0^2 - \lambda_1^2 + \lambda_2^2) \omega^1$$
$$\omega^1_2 = \frac{1}{2\lambda_0 \lambda_1 \lambda_2} (\lambda_0^2 + \lambda_1^2 - \lambda_2^2) \omega^0$$

from Cartan’s equations of structure.

The crucial relation which makes the Bianchi Type $VIII - IX$ solutions of TMG a parallelizable manifold is the relation

$$\epsilon_0 \lambda_0 + \epsilon_1 \lambda_1 + \epsilon_2 \lambda_2 = 0 \quad \epsilon_i^2 = 1, \quad i = 0, 1, 2$$ (103)

that is required for the vanishing of the curvature scalar. This is also the condition for the embeddability of the 3-manifold into 4-dimensional (anti-de
Sitter universe, or the (pseudo)-sphere [19]. Now using eq.(103) the connection 1-forms (102) reduce to

\[
\begin{align*}
\omega^0_1 &= -\epsilon \epsilon_0 \epsilon_1 \sigma^2, \\
\omega^2_0 &= -\epsilon_0 \epsilon_1 \sigma^1, \\
\omega^1_2 &= -\epsilon_0 \epsilon_1 \sigma^0,
\end{align*}
\]

which up to a harmless looking, but nevertheless very important factor of 2 are the same as the corresponding results for \(S^3\). Similarly the curvature 2-forms are simplified drastically

\[
\begin{align*}
\theta^1_2 &= 2 \epsilon_1 \epsilon_2 \sigma^2 \land \sigma^1, \\
\theta^2_0 &= 2 \epsilon_0 \epsilon_2 \sigma^0 \land \sigma^2, \\
\theta^1_0 &= 2 \epsilon_0 \epsilon_1 \sigma^0 \land \sigma^1
\end{align*}
\]

which are identical to their respective expressions for \(S^3\) up to a factor of 8.

Following Cartan and Schouten we now introduce the contorsion 1-forms \(K^i_k\) so that the full connection is given by

\[
\tilde{\omega}^i_k = \omega^i_k + K^i_k
\]

where we make the usual Ansatz

\[
K^i_k = \nu \omega^i_k
\]

with \(\nu\) a constant. The condition for the vanishing of the full curvature

\[
\tilde{\theta}^i_k = d \tilde{\omega}^i_k + \tilde{\omega}^i_m \land \tilde{\omega}^m_k = 0
\]

has the trivial solution \(\nu = -1\) but there is also the non-trivial solution

\[
\nu = -2
\]

which we shall hence-forth adopt. For \(S^3\) we had +1 for the non-trivial value of \(\nu\). The torsion 2-form is given by

\[
T^i = K^i_k \land \omega^k
\]
and with \( \nu = -2 \) we get

\[
\begin{align*}
T^0 &= -2 \epsilon \lambda_0 \sigma^1 \wedge \sigma^2, \\
T^1 &= -2 \lambda_1 \sigma^2 \wedge \sigma^0, \\
T^2 &= -2 \lambda_2 \sigma^0 \wedge \sigma^1
\end{align*}
\] (111)

which differ from the torsion 2-forms for “S^{3}” by a factor of 2 after allowing for the identity of the scale factors. It is straightforward to verify that the covariant exterior derivative of the torsion 2-form with respect to the full connection \( \tilde{D} \) vanishes

\[
\tilde{D} T^i = 0
\]

which, in view of eq.(108), is the first Bianchi identity.

It remains to verify that the Cartan-Schouten equations are satisfied for the Bianchi Type VIII and IX solutions. We recall that for “S^{3}” these equations are given by

\[
T_i \wedge^* T_k = g_{ik} \ast 1
\]

where \( \ast 1 \) denotes the volume 3-form and

\[
\omega^i \wedge T_i = -4 \epsilon l^2 K_{ij} \wedge K_{jk} \wedge K_{ki}
\]

respectively.

For the first Cartan-Schouten equation the check that \( T_i \wedge^* T_k = 0 \) for \( i \neq k \) is immediate from eqs.(111). For the verification of the diagonal components we have from eqs.(111)

\[
\begin{align*}
T_0 &= 2 \lambda_0 \sigma^1 \wedge \sigma^2 \implies \ast T_0 = 2 \epsilon \lambda_1 \lambda_2 \sigma^0, \\
T_1 &= 2 \lambda_1 \sigma^2 \wedge \sigma^0 \implies \ast T_1 = 2 \lambda_0 \lambda_2 \sigma^1
\end{align*}
\]

see [20] and similarly for the last component. Thus we have

\[
\frac{1}{4} T_i \wedge^* T_k = g_{ik} \ast 1
\]

in place of eq.(113). Finally a straight-forward calculation shows that

\[
\omega^i \wedge T_i = 2 \mu \ast 1 = \frac{\epsilon}{24} \mu \lambda_0 \lambda_1 \lambda_2 K_{ij} \wedge K_{jk} \wedge K_{ki}
\]

which up to a constant of proportionality agrees with eq.(114).
The starting point in the work of Chern and Simons [21] is that all oriented 3-manifolds are parallelizable. The gravitational Chern-Simons term is the Cotton tensor which vanishes for $S^3$, the classic example of a parallelizable manifold. We have shown that the Bianchi Type VIII − IX exact solutions of TMG are also orientable and parallelizable as they satisfy the conditions (108) for zero-curvature and the Cartan-Schouten equations up to simple modifications of the factors of proportionality. These are new examples of 3-dimensional parallelizable manifolds.

13 Acknowledgements

One of us (Y.N.) thanks T. Dereli for an interesting discussion on ref. [14]. This work was in part supported by Turkish Academy of Sciences.

References

   Ann. Phys. NY 140 372