Optimal squeezing, pure states, and amplification of squeezing in resonance fluorescence

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It is shown that 100% squeezed output can be produced in the resonance fluorescence from a coherently driven two-level atom interacting with a squeezed vacuum. This is only possible for \(N = 1/8\) squeezed input, and is associated with a pure atomic state, \(i.e.,\) a completely polarized state. The quadrature for which optimal squeezing occurs depends on the squeezing phase \(\Phi\), the Rabi frequency \(\Omega\), and the atomic detuning \(\Delta\). Pure states are described for arbitrary \(\Phi\), not just \(\Phi = 0\) or \(\pi\) as in previous work. For small values of \(N\), there may be a greater degree of squeezing in the output field than the input — \(i.e.,\) we have squeezing amplification.

42.50.Hz, 42.50.Dv, 32.80.-t

I. INTRODUCTION

Both theoretical and experimental studies have shown that the resonance fluorescence of a driven atom can serve as a source of nonclassical light. For example, Carmichael and Walls, and Kimble and Mandel [1] predicted that the resonance fluorescence from a single two-level atom driven by a coherent laser field of low intensity would exhibit photon antibunching. The prediction has been confirmed in many laboratories [2–4]. The sub-Poissonian statistics of the fluorescent photons emitted in a short time interval by a single atom was also investigated experimentally [5].

There have been many theoretical investigations of squeezing in resonance fluorescence, both in terms of the total variances and in terms of the fluctuation spectra of the phase quadratures. Walls and Zoller, and Loudon [6] showed that the total quantum fluctuations in the phase quadratures of the resonance fluorescence of a driven two-level atom can be squeezed below the shot-noise limit. Single-mode [7], or frequency-tunable two-mode [8] squeezing with a finite bandwidth may be obtained, depending on the Rabi frequency and detuning. This internally produced squeezing results in line narrowing in the resonance fluorescence spectra [9,10]. Phase-quadrature squeezing has also been studied in the presence of an applied squeezed vacuum [10–12].

Experimental observation of squeezing in the fluorescence field has proved a great challenge, one problem being that atomic motion produces phase shifts which destroy squeezing [13]. This difficulty was surmounted in the recent experimental advances in homodyne detection schemes of the fluorescent radiation of a single trapped ion reported by Hoffges et al. [3]. Within the last couple of years, experiments carried out by Zhao et al. [14] have found some evidence of squeezing by measuring the phase-dependent fluorescence spectra of a coherently driven two-level atom with a long lifetime, stimulating the further exploration of squeezing in resonance fluorescence. Very recently, squeezing in the quadrature with phase \(\pi/4\) relative to the driving laser was observed for the first time in the resonance fluorescence of a single two-level atom [15].

Also very recently, we have found that squeezing in resonance fluorescence can be greatly enhanced in a frequency-tunable cavity [16], or in a squeezed vacuum [11,12]. The latter works mainly in the regime over which anomalous spectra such as hole-burning and dispersive profiles [17] occur, \(i.e.,\) \(\Delta = 0\) and \(\Phi = 0\), where squeezing occurs in the out-phase quadrature of the fluorescent field. In this paper we extend the study to the general case, and show that large squeezing occurs in different phase quadratures of the fluorescent field, depending upon the values of the parameters. The large squeezing is associated with an atomic pure state (a completely polarized state), and thereby with a large atomic coherence. Perfect fluorescent squeezing may only take place for the particular squeezing number \(N = 1/8\).

There is previous evidence that the value \(N = 1/8\) is special. It has been shown that large squeezing in resonance fluorescence was produced for this input squeezed field [12], and that also for this value of \(N\), the sidebands in the resonance fluorescence spectrum have the same linewidth as in the \(N = 0\) case [18].

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II. PURE STATE

Our model consists of a two-level atom with ground and excited states $|0\rangle$ and $|1\rangle$, driven by a monochromatic laser field and damped by a broadband squeezed vacuum. In the frame rotating with the laser frequency $\omega_L$, the Hamiltonian (in units of $\hbar$), is

$$ H = \frac{\Delta}{2} \sigma_z + \frac{\Omega}{2} \left( e^{i\phi_L} \sigma_+ + e^{-i\phi_L} \sigma_- \right), \quad (1) $$

where $\sigma_z = (|1\rangle\langle 1| - |0\rangle\langle 0|)$ is the atomic inversion operator, $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are the atomic raising and lowering operators respectively, $\Delta = \omega_A - \omega_L$ is the detuning, $\Omega$ is the Rabi frequency and $\phi_L$ is the laser phase. The squeezed vacuum is characterized by the squeezing photon number $N$, the squeezing phase $\phi_s$, and the strength of the two-photon correlations $M$, which obeys

$$ M = \eta \sqrt{N(N+1)}, \quad (0 \leq \eta \leq 1). \quad (2) $$

The value $\eta = 1$ indicates an ideal squeezed vacuum, whilst $\eta = 0$ corresponds to no squeezing at all — a thermal field. The squeezed vacuum may be ‘turned off’ by setting $N = 0$. In the remainder of the paper, we take $\eta = 1$, and thus $M = \sqrt{N(N+1)}$.

The optical Bloch equations, modified by the squeezed vacuum, are of the form

$$ \langle \dot{\sigma}_x \rangle = -\gamma_x \langle \sigma_x \rangle - \left( \Delta + \gamma M \sin \Phi \right) \langle \sigma_y \rangle, $$

$$ \langle \dot{\sigma}_y \rangle = -\gamma_y \langle \sigma_y \rangle + \left( \Delta - \gamma M \sin \Phi \right) \langle \sigma_x \rangle - \Omega \langle \sigma_z \rangle, $$

$$ \langle \dot{\sigma}_z \rangle = -\gamma_z \langle \sigma_z \rangle + \Omega \langle \sigma_y \rangle - \gamma, \quad \quad (3) $$

with

$$ \gamma_x = \Gamma + \gamma M \cos \Phi, $$

$$ \gamma_y = \Gamma - \gamma M \cos \Phi, $$

$$ \gamma_z = \gamma_x + \gamma_y = 2\Gamma, \quad (4) $$

where $\sigma_z = (\sigma_- + \sigma_+)$ and $\sigma_y = i(\sigma_- - \sigma_+)$ are the in–phase (X) and out–phase (Y) quadrature components of the atomic polarization, respectively. (We have assumed in obtaining eq. (3) that the applied EM field is coupled to the $\sigma_z$ component of the atom.) $\Phi = 2\phi_L - \phi_s$ is the relative phase between the laser field and squeezed vacuum, and $\Gamma = \gamma (N + 1/2)$, with $\gamma$ the spontaneous decay rate of the atom into the standard vacuum modes. The modified decay rates of the X and Y components of the atomic population are $\gamma_x$, $\gamma_y$ respectively, whilst $\gamma_z$ is the decay rate of the atomic population inversion.

It has been shown that such a coherently driven two-level atom interacting with the squeezed vacuum reservoir can collapse into a steady-state which is a pure state, for the case $\Phi = 0$ or $\pi$ [17,20]. This property is associated with anomalous spectral features in the resonance fluorescence and probe absorption. The condition for the atom to be in a pure state is that the quantity $\Sigma$, where

$$ \Sigma = \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2, \quad (5) $$

takes the value one [21]. We point out here that a steady pure state can, in fact, be achieved for other values of the squeezing phase as well, the requirement being that given $\Phi$, $\Omega$ and $\Delta$ are chosen to satisfy $\Sigma = 1$. The general pure state has the form

$$ |\Psi\rangle = \frac{\sqrt{M}|0\rangle - e^{i\alpha} \sqrt{N}|1\rangle}{(M+N)^{1/2}}, \quad (6) $$

where

$$ \alpha = \arctan \left( \frac{\Gamma + \gamma M \cos \Phi}{\Delta + \gamma M \sin \Phi} \right). \quad (7) $$

The conditions for the pure state (6) for a few specific cases are given below:

$$ \Phi = 0, \quad \Delta = 0, \quad \Omega = \frac{\gamma \sqrt{M}}{\sqrt{N+\Omega+\sqrt{N}}}, \quad (8) $$
\[ \Phi = \frac{\pi}{2}, \quad \Delta = \Gamma - \gamma M, \quad \Omega = \frac{\gamma \sqrt{2M}}{\sqrt{N+1} + \sqrt{N}}, \]

\[ \Phi = \pi, \quad \Delta \gg \Gamma - \gamma M, \quad \Omega = \frac{2\Delta \sqrt{M}}{\sqrt{N+1} - \sqrt{N}}. \]

Notice that for resonant excitation, a pure state is only possible if \( \Phi = 0 \). In general, the pure state (6) describes a completely polarized atom with the Bloch vector \( B \) lying on the Bloch sphere with polar angles \( \alpha \) and \( \beta \),

\[ B = \cos \alpha \sin \beta \mathbf{e}_x + \sin \alpha \sin \beta \mathbf{e}_y + \cos \beta \mathbf{e}_z \]

where

\[ \beta = \arccos \left( -\frac{M - N}{M + N} \right). \]

See Figure 1. When \( \Phi, \Omega \) and \( \Delta \) satisfy the condition (8), then \( \alpha = \pi/2 \), and the atomic Bloch vector (polarization) is in the Y–Z plane, whereas if the condition (10) holds, we have \( \alpha = 0 \) and the atom polarizes in the X–Z plane.

### III. OPTIMAL AND MAXIMAL SQUEEZING

The measurement of the quadrature squeezing spectrum requires the fluorescent radiation field to be first frequency filtered and then homodyned with a strong local oscillator [14]. The squeezing may also be detected in terms of the total normally-ordered variances of the phase quadratures in an alternative experimental scheme, where the total radiation field and the local oscillator are directly homodyned, without first frequency filtering [5,13]. In this paper we interested in the latter quantity, which can be expressed in terms of the steady state solution of the Bloch equations (3) as [6,7]

\[ S_\theta = \langle (\Delta E_\theta)^2 \rangle = 1 + \langle \sigma_z \rangle - (\langle \sigma_x \rangle \cos \theta - \langle \sigma_y \rangle \sin \theta)^2, \]

where \( E_\theta = e^{-i\theta} \xi^{(+)} + e^{i\theta} \xi^{(-)} \) is the \( \theta \)-phase quadrature of the atomic fluorescence field, measured by homodyning with a local oscillator having a controllable phase \( \theta \) relative to the driving laser. \( E_{\theta=0} \) and \( E_{\theta=\pi/2} \) are usually the in-phase (X) and out-of-phase (Y) quadratures of the fluorescent field, respectively. \( S_\theta \) is the total normally-ordered variance of the \( \theta \)-phase quadrature of the fluorescent field. The field is said to be squeezed when \( S_\theta < 0 \). The normalization we have chosen is such that maximum squeezing corresponds to \( S_\theta = -0.25 \). Eq. (13) implies that the squeezing occurs at large values of the atomic coherences, \( \langle \sigma_{x(0)} \rangle \).

It is not difficult to show that the total normally-ordered variances in the phase quadrature component of the fluorescent field reach their minimal value

\[ S_{\theta_{\text{opt}}} = 1 + \langle \sigma_z \rangle - \langle \sigma_x \rangle^2, \]

when the quadrature phase \( \theta = \theta_{\text{opt}} \), where

\[ \theta_{\text{opt}} = \arctan \left( \frac{\Gamma + \gamma M \cos \Phi}{\Delta + \gamma M \sin \Phi} \right). \]

(Not that only when \( S_{\theta_{\text{opt}}} < 0 \) is the resonance fluorescence a noise-squeezed field.) Furthermore, if the atom is in a pure state, i.e., \( \Sigma = 1 \), then \( S_{\theta_{\text{opt}}} \) reduces to

\[ S_{\theta_{\text{opt}}}^{PS} = \langle \sigma_z \rangle (1 + |\langle \sigma_z \rangle|)^2 \leq 0, \]

showing that maximum squeezing occurs when \( \langle \sigma_z \rangle = -1/2 \). Therefore, a completely polarized atom always radiates a fluorescent field with \( \theta_{\text{opt}} \)-phase quadrature squeezing. The quadrature phase \( \theta_{\text{opt}} \) is same as the longitudinal angle \( \alpha \) of the polarized atom in the Bloch sphere.

It is clear from eq. (14) that the squeezing is maximal, \( S_{\theta_{\text{opt}}} = -0.25 \), if the equations \( \Sigma = 1 \) and \( \langle \sigma_z \rangle = -1/2 \) are simultaneously satisfied. We find that this is possible only for \( N = 1/8 \), and then analytic solution can be found. The appropriate values of \( \Omega \) and \( \Delta \) are
\[ \Delta = \frac{t}{4}, \quad \Omega = \frac{1}{4} \sqrt{3(1 + t^2)}, \quad \alpha = \theta_0 = \arctan \left( \frac{1}{t} \right) \]  

(17)

where \( t = \tan(\Phi/2) \). These expressions are consistent with the values presented in eqs. (8) – (10). It is clear that \( \Delta \to \infty \) as \( \Phi \to \pi \), and then \( \Omega \to \sqrt{3\Delta} \).

We comment briefly why maximal squeezing is only possible for a small value of \( N \). Whilst, for arbitrary values of \( N \) and \( \Phi \) it is always possible to find values of \( \Omega \) and \( \Delta \) which produce a pure state (satisfy \( \Sigma = 1 \)), the condition \( \langle \sigma_z \rangle = -1/2 \) is much more stringent. This value of \( \langle \sigma_z \rangle \) implies that the external fields must not be so strong as to saturate the atom. This may be seen by examining the expression for \( \langle \sigma_z \rangle + 1/2 \). We require

\[ \langle \sigma_z \rangle + 1/2 \equiv \frac{1}{2} \frac{(N + 1/2 + M \cos \Phi) \Omega^2 + (2N - 1) (\Delta^2 + 1/4)}{(N + 1/2 + M \cos \Phi) \Omega^2 + (\Delta^2 + 1/4)(2N + 1)} = 0. \]  

(18)

Clearly, \( \langle \sigma_z \rangle + 1/2 > 0 \) if \( N > 1/2 \), even if \( \Omega = 0 \). The condition \( \Sigma = 1 \) requires a nonzero value of \( \Omega \) and, thus if the conditions \( \Sigma = 1 \) and \( \langle \sigma_z \rangle + 1/2 \) can be satisfied, and it is not obvious from the outset that they can, we must have \( 0 < N < \frac{1}{2} \).

This argument does not explain why solutions are only possible for the particular value \( N = 1/8 \), but as we have remarked earlier, the value \( N = 1/8 \) appears to be a special one from several points of view [12,18]. For this value of \( N \), the two-photon correlation strength, \( M \equiv \sqrt{N(N + 1)} \) is rational: \( M = 3/8 \), and the principal decay rates are respectively twice and one-half the decay rate in the absence of the squeezed vacuum: \( \gamma_x = \gamma, \gamma_y = \gamma/4 \) when \( \Phi = 0 \).

For small values of \( N \), large squeezing in the fluorescence is possible, even if it is not maximal. We may conclude from eq. (15) that when \( \Phi = 0 \) and \( \Delta = 0 \), optimal squeezing in the fluorescent field always occurs in the out-of-phase (Y) quadrature component, i.e., \( \theta_0 = \pi/2 \) [7,9–12]. When \( \Phi = \pi/2 \) and \( \Delta = 1 - \gamma \), then \( \theta_0 = \pi/4 \), and optimal squeezing takes place in the \( \pi/4 \) phase quadrature [15]. When \( \Phi = \pi \) and \( \Delta \gg 1 - \gamma \), then \( \theta_0 = 0 \), and optimal squeezing is always in the in-phase (X) quadrature [8,16].

### A. In-phase quadrature squeezing

We here present a detailed study of the fluorescence squeezing in the case of \( \Phi = \pi \) and \( \Delta \neq 0 \), which has previously received little attention. As we know from the above discussion, the squeezing is exhibited in the in-phase (X) quadrature component.

Figure 2 shows \( S_{\theta=0} = S_X \), the in-phase quadrature of the fluorescent field, in a 3D plot against the Rabi frequency \( \Omega \) and the squeezed phase \( \Phi \), for \( N = 0.1, \Delta = 10\gamma T \) and \( \gamma = 1 \). Clearly, the greater squeezing occurs for large phases. When \( \Omega \approx 16\gamma \) and \( \Phi = \pi \), then \( S_X \approx 0.25 \) displaying the optimal (100%) degree of squeezing. Noting that the degree of squeezing in the squeezed vacuum input for \( N = 0.1 \) is 46%, we see that the squeezing in the resonance fluorescence output is substantially enhanced.

We plot \( S_X \) against \( \Omega \) and \( \Delta \) in Fig. 3 where \( N = 0.125, \Phi = \pi \), and \( \gamma = 1 \). This figure clearly shows that for this value of \( N \), near maximal squeezing occurs over wide values of the Rabi frequency \( \Omega \) and detuning \( \Delta \).

Figure 4 presents the total normally-ordered variance of the in-phase quadrature of the fluorescent field, \( S_X \), and the magnitude of the atomic Bloch vector, \( \Sigma = \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \), for the parameters: \( \Phi = \pi, \Delta = 12.5\gamma, \gamma = 1 \) and different squeezed photon numbers (a) \( N = 0.05 \), (b) \( N = 0.125 \) and (c) \( N = 0.5 \). The solid and dashed lines represent \( S_X \) and \( \Sigma \) respectively. When \( \Sigma = 1 \), the atom is in a pure state. It is obvious that large squeezing in the resonance fluorescence of the two-level atom occurs for pure atomic states [11,19]. When \( N = 0.125 \), maximal squeezing (\( S_X = 0.25 \)) is achieved at the Rabi frequency \( \Omega = 21.65\gamma \). The large squeezing is due to the large atomic coherence in the pure state.

When eq. (10) is satisfied, the atom is in the pure state (6) with \( \alpha = 0 \). The corresponding total, normally-ordered variance \( S_X \) of the in-phase quadrature of the fluorescent field is of the form

\[ S_X^{PS} = \frac{N - M}{N + M + 1/2}. \]  

(19)

When \( N = 1/8, M = 3/8 \). Then, from eq. (19) we have \( S_X^{PS} = -0.25 \) (100% squeezing). The corresponding value of the Rabi frequency is \( \Omega = \sqrt{3\Delta} \).

We plot \( S_X^{PS} \), indicated by the solid line, against \( N \) in Fig. 5, which demonstrates that large squeezing occurs for small photon numbers. For comparison, we also present the normally-ordered variance \( S_X^{PS} \) of the in-phase quadrature in the squeezed vacuum field, represented by the dashed line in this figure. It is clear that the squeezing of the output field (fluorescence) is greatly enhanced over the region \( 0 < N \leq 0.562 \), compared with the squeezing of the input (squeezed vacuum) field. Hence, the atom may be applied as a nonlinear optical element to amplify squeezing.
B. $\pi/4$-phase quadrature squeezing

Fluorescent field squeezing can also occur in other phase quadratures with the phase between 0 (in-phase) and $\pi/2$ (out-of-phase)—for example, $\theta = \pi/4$. The variance, $S_{\theta=\pi/4}$, of such a phase quadrature is shown in Fig. 6, where $\gamma = 1$, $\Phi = \pi/2$, $N = 0.125$. It is obvious that fluorescence squeezing, $S_{\pi/4} < 0$, occurs over a range of $\Omega$, $\Delta$, and the optimal (100%) degree of squeezing takes place around $\Omega = 0.612\gamma$ and $\Delta = 0.25\gamma$.

As with in-phase quadrature squeezing, the optimal squeezing in the $\pi/4$-phase quadrature is also associated with a pure state (a highly polarized atomic state). We plot $S_{\theta=\pi/4}$ and $\Sigma$ together in Fig. 7 for the parameters: $\gamma = 1$, $\Phi = \pi/2$, $\Delta = 0.25$ and different squeezed photon numbers (a): $N = 0.05$, (b): $N = 0.125$ and (c): $N = 0.5$. This figure clearly shows again that a completely polarized atom can emit a fluorescent field with large squeezing [11,19]. When $N = 0.125$, maximal squeezing ($S_X = -0.25$) is obtained at the Rabi frequency $\Omega = 0.612\gamma$.

We have shown that when $\Phi = \pi/2$, $\Delta = \Gamma - \gamma M$ and $\Omega = \gamma\sqrt{2M}/\left(\sqrt{N+1} + \sqrt{N}\right)$, the atom develops into a stationary pure state, given by (6) with $\alpha = \pi/4$. This highly polarized atom radiates a fluorescent field with $\pi/4$-phase quadrature squeezing. The expression for the squeezing is same as eq. (19), but for the $\pi/4$-phase quadrature. Perfect squeezing is obtained for $N = 0.125$, $\Delta = \gamma/4$ and $\Omega = \sqrt{6}\gamma/4$.

C. Out-of-phase quadrature squeezing

The resonance fluorescence exhibits out-of-phase (Y) quadrature squeezing when $\Phi = 0$ and $\Delta = 0$ [10–12]. Furthermore, if $\Omega$ obeys eq. (8), the atom collapses into the pure state (6) with $\alpha = \pi/2$. Consequently, the optimal Y-quadrature squeezing, $S_Y^{PS}$, is given by eq. (19), as well. When the squeezed photon number $N = 0.125$ and the Rabi frequency $\Omega = \sqrt{3}\gamma/4$, the maximal degree of the Y-quadrature squeezing in the resonance fluorescence is achieved.

IV. SUMMARY

We have shown that the total normally-ordered variance of the phase quadrature of the fluorescent field emitted from a coherently driven two-level atom interacting with a squeezed vacuum can be greatly (or completely) squeezed. The squeezing in the fluorescent field is greatly increased for small values of $N$ compared with the degree of squeezing of the input squeezed vacuum field. Therefore, squeezing may be amplified through resonance fluorescence. The squeezing in the output may indeed be the maximum obtainable, when $N = 1/8$. Analytic expressions for the conditions for maximum squeezing are obtained in this case. Depending upon the values of the squeezed phase $\Phi$ and the detuning $\Delta$, squeezing can occur in the in-phase, out-of-phase, or any other phase quadrature of the resonance fluorescence. Large squeezing is always associated with the atom evolving into a pure state (a highly polarized atomic state).

From the experimental point of view, a cavity configuration may be the best candidate to investigate the atom/squeezed-vacuum interactions [22]. As shown in Refs. [22,23], many squeezing-induced effects in free space can carry over to the cavity situations in the bad cavity limit, where the atom evolves in accordance with formally the same equations as those in free space. We expect that large squeezing in resonance fluorescence will still take place in the cavity configuration.

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FIG. 1. The polar angles $\alpha$ and $\beta$ defining the position of the Bloch vector $\mathbf{B}$ on the Bloch sphere.

FIG. 2. The total normally-ordered variance of the in-phase quadrature, $S_X$, against the Rabi frequency $\Omega$ and the squeezed phase $\Phi/\pi$, for $\gamma = 1$, $\Delta = 10$, $N = 0.1$.

FIG. 3. Same as FIG. 2, but against $\Omega$ and $\Delta$, with $\gamma = 1$, $\Phi = \pi$, $N = 0.125$.

FIG. 4. $S_X$ and $\Sigma$ as functions of $\Omega$, for $\gamma = 1$, $\Phi = \pi$, $\Delta = 12.5$ and (a): $N = 0.05$, (b): $N = 0.125$ and (c): $N = 0.5$. The solid and dashed lines represent respectively $S_X$ and $\Sigma$.

FIG. 5. $S_X^{PS}$ and $S_X^{SV}$ as functions of $N$, represented by the solid and dashed lines respectively.

FIG. 6. Same as FIG. 2, but the variance of the $\pi/4$-phase quadrature, $S_{\theta=\pi/4}$, with $\gamma = 1$, $\Phi = \pi/2$, $N = 0.125$.

FIG. 7. $S_{\theta=\pi/4}$ and $\Sigma$ as functions of $\Omega$, for $\gamma = 1$, $\Phi = \pi/2$, $\Delta = 0.25$ and (a):$N = 0.05$, (b): $N = 0.125$ and (c): $N = 0.5$. The solid and dashed lines represent respectively $S_{\pi/4}$ and $\Sigma$. 