Quantum Gravity of a Brane-like Universe *

Aharon Davidson and David Karasik

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
(davidson@bguemail.bgu.ac.il, karasik@bguemail.bgu.ac.il)

Quantum gravity of a brane-like Universe is formulated, and its Einstein limit is approached. Regge-Teitelboim embedding of Arnowitt-Deser-Misner formalism, parameterized by the coordinates $y^A(t, x^i)$, is governed by some $\rho_{AB}(y, y', y'')$. Invoking a novel Lagrange multiplier $\lambda$, accompanying the lapse function $N$ and the shift vector $N^i$, we derive the quadratic Hamiltonian

$$\mathcal{H} = \frac{1}{2}N \left[ P_A \left( (\rho - \lambda I)^{-1} \right)^{AB} P_B + \lambda \right] + N^i y^A P_A .$$

The inclusion of matter resembles minimal coupling. Setting $P_A = -\frac{i}{\hbar} \frac{\delta}{\delta y^A}$, we derive a bifurcated Wheeler-Dewitt-like equation. Einstein gravity, associated with $\lambda$ being a certain 4-fold degenerate eigenvalue of $\rho_{AB}$, is characterized by a vanishing center-of-mass momentum $\int P_A dx = 0$. Troublesome $(\rho - \lambda I)^{-1}$ is replaced then by regular $M^{-1}$, such that $M^{-1}(\rho - \lambda I)$ defines a projection operator, modifying the Hamiltonian accordingly.

A prevailing theory is always seeded by a remarkably simple idea. Regge-Teitelboim gravity [1], a criticized rival [2] of Einstein gravity, may eventually fall into such a category. After all, who can resist the philosophy that the first principle which governs the evolution of the entire Universe is essentially the one which determines the world-manifold behavior of particles, strings and membranes. Following such a viewpoint, the Universe, to be referred to as a brane-like Universe, is viewed as a 4-dim extended object [3] floating in some (say) 10-dim flat Minkowski background. Some cosmological fingerprints [4] of such a brane-like Universe have already been revealed. Staying on practical grounds, however, Regge-Teitelboim gravity needs not be considered a target by itself. In fact, recalling its original underlying motivation, this theory attempted to establish a viable mathematical trail towards the unification of quantum mechanics with Einstein gravity. This conjecture was driven by several remarkable facts:

- Regge-Teitelboim gravity is, by construction, a continuation of string theory. Unlike in Einstein gravity, the metric tensor $g_{\mu\nu}(x)$ does not serve as a canonical field; this role has been taken over by the embedding vector $y^A(x)$.
- Although Einstein equations are traded for $[(G^{\mu\nu} - T^{\mu\nu})y^M]_{,\nu} = 0$, energy/momentum conservation is still automatic.
- Regge-Teitelboim gravity exhibits a built-in Einstein limit. In turn, every solution of Einstein equations is automatically a solution of Regge-Teitelboim equations.

It has been speculated, relying on the structural similarity to string/membrane theory, that quantum Regge-Teitelboim gravity may be a somewhat easier task to achieve than quantum Einstein gravity. The real target is then the Einstein limit of the theory, which in principle may call for additional first-class geometric constraints. The trouble is, however, that the parent Regge-Teitelboim Hamiltonian has never been derived!

In this short essay, by deriving the quadratic Hamiltonian of a gravitating brane-like Universe, we have overcome the dead-end reached by Regge-Teitelboim, thereby opening the door for the quantum Einstein gravity limit. A key role in our formalism is played by a novel non-dynamical field $\lambda$ which accompanies the standard Lagrange multipliers, the lapse function $N$ and the shift vector $N^i$. Starting from the purely gravitational case, a generic Regge-Teitelboim configuration is parameterized by $\mu^2 > 0$, recognized as the analogue of (mass)$^2$. Quite surprisingly, an Einstein configuration turns out to be characterized by $\mu^2 = 0$. In this language, Einstein gravity can be interpreted as the 'massless' limit of Regge-Teitelboin gravity.

Given the background Minkowski metric $\eta_{AB}$ and some embedding vector $y^A(t, x^i)$, the induced 4-dim line-element can be put in the Arnowitt-Deser-Misner (ADM) form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) ,$$

provided the 3-metric $h_{ij}$, the shift vector $N_i$, and the lapse function $N$ are identified with

*Honorable mentioned, Gravity Research Foundation (1998)
Notice the time-like unit vector \( n^A \equiv \frac{1}{N} (\dot{y}^A - N^i \dot{y}^i) \) orthogonal to \( y^A \).

The gravitational Regge-Teitelboim Lagrangian density is the standard one (the canonical fields are not). Up to a surface term, it can be written in the form

\[
\mathcal{L} = -\sqrt{h} \left[ NR^{(3)} - \frac{1}{N} (K_{ij}K^{ij} - K^2) + 2NA \right] ,
\]

where \( R^{(3)} \) denotes the 3-dim Ricci scalar constructed by means of the 3-metric \( h_{ij} \), and \( K_{ij} \equiv NK_{ij} \) is the extrinsic curvature \( K_{ij} \) factorized by the lapse function \( N \). \( K_{ij} \) is free of mixed derivative \( \dot{y}^A \)-terms, and since \( \dot{y}^A \)-terms are absent in the first place, the Lagrangian \( \mathcal{L}(y, \dot{y}, \eta, y|ij, \ldots) \) is apparently ripe for the Hamiltonian formalism.

The fact that the 3-metric \( h_{ij} \) is \( \dot{y}^A \)-independent helps us to derive the momenta \( P_A \) conjugate to \( y^A \), that is

\[
P_A \equiv \frac{\delta \mathcal{L}}{\delta \dot{y}^A} = \sqrt{h} \left[ R^{(3)} + \frac{1}{N^2} (K_{ij}K^{ij} - K^2) + 2A \right] n^A + \frac{2}{N} (K_{ij} - h^{ij}K)y^A_{\{ij\}} \right] .
\]

To simplify the algebraic structure of \( P^A \), define the \( \dot{y}^A \)-independent tensor

\[
\rho^{AB} \equiv 2\sqrt{h} \left[ (h^{na}h_{ab} - h^{ij}h_{ij})y^A_{\{ab\}ij} + \left( R^{(3)} + 2A \right) \eta^{AB} \right] ,
\]

to finally arrive at

\[
P^A = \frac{1}{2} (\eta mn) n^A + \rho^A_B n^B.
\]

One can immediately verify, in analogy with Wheeler-DeWitt theory and string theory, that the Hamiltonian \( \mathcal{H} \) vanishes

\[
\mathcal{H} = \dot{y}^A P_A - \mathcal{L} = N \left( n^A P_A - \frac{1}{N} \mathcal{L} \right) + N^i y^A P_A = 0 ,
\]

and thus can be interpreted as a sum of constraints. Invoking the powerful embedding identity \( \eta_{AB} \dot{y}^A y^B \equiv 0 \), the first constraint \( y^A P_A = 0 \) is easily extracted, reflecting the fact that \( y^A n_A = 0 \). The second constraint is hidden within \( n^A P_A - \frac{1}{N} \mathcal{L} = 0 \). A naive attempt to solve \( n^A (\rho, P) \) and substitute into \( n^2 + 1 = 0 \), falls short. The cubic equation involved rarely admits simple solutions, and even in cases it does, the resulting constraint is anything but a quadratic form in the momenta.

The way out involves the definition of a quantity \( \lambda \), such that

\[
P^A = (\rho - \lambda I)_B n^B .
\]

The price for an independent \( \lambda \) being an additional constraint \( nmn + 2\lambda = 0 \). Assuming that \( \lambda \) is not an eigenvalue of \( \rho_B \), we can solve for \( n^A (\rho, P, \lambda) \) and find

\[
n^A = \left[ (\rho - \lambda I)^{-1} \right]_B^A P^B .
\]

The leftover constraints can then be grouped into

\[
P(\rho - \lambda I)^{-2} P + 1 = 0 , \quad P(\rho - \lambda I)^{-1} P + \lambda = 0 .
\]

The first of which, owing to \( \frac{d}{d\lambda} (\rho - \lambda I)^{-1} = (\rho - \lambda I)^{-2} \), can be regarded superfluous provided we elevate \( \lambda \) to the level of a canonical non-dynamical variable. Note in passing that the special case \( \rho_B^A \sim \delta_B^A \) corresponds to
a Nambu-Goto string. Explicitly, \( \rho = 4 \Lambda \sqrt{h} I \) fixes \( \lambda = 2 \Lambda \sqrt{h} \), and gives rise to the familiar Virasoro constraint

\[
P^2 + 4 \Lambda^2 \eta_{AB} y^A y^B = 0.
\]

Altogether, the Regge-Teitelboim Hamiltonian acquires the quadratic form

\[
\mathcal{H} = \frac{1}{2} N \left[ P_A \left( (\rho - \lambda I)^{-1} \right)^{AB} P_B + \lambda \right] + N^i y^A_i P_A
\]

(11)

with \( N, N^i \), and notably \( \lambda \) serving as Lagrange multipliers. \( (\rho - \lambda I)^{-1} \) plays a role analogous to the Wheeler-DeWitt metric on superspace. Here, however, superspace has been traded for the embedding spacetime itself, and \( (\rho - \lambda I)^{-1} \) needs not be confused with the metric \( \eta_{AB} \). Once matter is included, the momenta \( P_A \) conjugate to \( y^A \) receives an extra contribution

\[
\Delta P_A = \frac{\delta L_{\text{matter}}}{\delta \dot{y}^A} = \frac{1}{2} N \pi^A \sqrt{h} T_{AB} \delta y^B \eta_{ij} y^j_i + \lambda \eta_{AB} \delta y^B \Delta y^A = \frac{1}{2} N \pi^A \sqrt{h} T_{AB} \delta y^B \eta_{ij} y^j_i + \lambda \eta_{AB} \delta y^B \Delta y^A.
\]

(13)

To be more specific, consider the case where \( \Phi(x) \) stands for a scalar field. The corresponding energy/momentum projections are

\[
T_{nn} = \left( T^{\mu \nu} y^A_{\mu} y^B_{\nu} \right) n_A n_B, \quad T_{ni} = \left( T^{\mu \nu} y^A_{\mu} y^B_{\nu} \right) n_A y^B_i.
\]

(12)

In a more general case, e.g. for a gauge field \( A^i \), the door is open for non-gravitational constraints to enter the Hamiltonian.

At the quantum level, we set \( P_A \equiv -i \delta \delta y^A \). Up to order ambiguities, the wave functional \( \Psi \) of an empty brane-like Universe [5–7] is subject to three Virasoro-type constraints: The momentum constraint equation

\[
y^A_i \frac{\delta \Psi}{\delta y^A_i} = 0,
\]

(15)

is accompanied by the bifurcated Wheeler-Dewitt-like equation

\[
\frac{\delta}{\delta y^A} ( (\rho - \lambda I)^{-1} )^{AB} \frac{\delta}{\delta y^B} \Psi = \lambda \Psi
\]

\[
\frac{\delta}{\delta y^A} ( (\rho - \lambda I)^{-2} )^{AB} \frac{\delta}{\delta y^B} \Psi = \Psi
\]

(16)

Upon the inclusion of matter, the ordinary functional derivatives are replaced by covariant functional derivatives (and \( \rho \) gets modified) according to the above prescription.

The Einstein limit of Regge-Teitelboim gravity has two faces:

- First, using the purely geometric relation

\[
2G_{nn} = R^{(3)} + \frac{1}{N^2} (K_{ij} K^{ij} - K^2),
\]

we infer that

\[
\rho_{AB} - \lambda \eta_{AB} = 2 \sqrt{h} \left[ (h^{ia} h^{jb} - h^{ij} h^{ab}) y_{A|a} y_{B|b} + (G_{nn} - T_{nn}) \eta_{AB} \right].
\]

(18)

Appealing now to the embedding identity \( \eta_{AB} y^A_{ij} y^B_{ki} = 0 \), one concludes that Einstein equation \( G_{nn} = T_{nn} \) can be satisfied if and only if
\begin{equation}
(\rho_{AB} - \lambda \eta_{AB})y^B_{\alpha} = 0.
\end{equation}

We have learned that the Einstein case is characterized by \( \lambda \) being a 4-fold degenerate eigenvalue of \( \rho_{AB} \). In turn, \( (\rho - \lambda I)^{-1} \) does not make sense, and we face the unpleasant consequence that not all components of \( n^A \) are expressible in terms of momenta. This is, however, a curable situation. The residual \( n \)'s are treated as non-dynamical variables, and the troublesome \( (\rho - \lambda I)^{-1} \) is replaced by some regular \( M^{-1} \), such that \( M^{-1}(\rho - \lambda I) \) defines the proper projection operator.

\bullet Second, using the dynamical relation

\begin{equation}
P_A = \sqrt{\hbar} \left[ (G_{nn} - T_{nn}) n^A - (G_{ni} - T_{ni}) h^{ij} y^A_j + \left( y^A_{ij} n_B y^B_{ji} \delta^{ik} h^{jl} - h^{ij} h^{kl} \right) \right],
\end{equation}

one observes that if Einstein equations \( G_{ni} = T_{ni} \) and \( G_{nn} = T_{nn} \) are both satisfied, \( P^A \) makes a total derivative. On the other hand, reflecting the Poincare invariance of the embedding spacetime, we know that the center-of-mass momentum \( \mu^A \equiv \int d^3x P^A \) is a Noether conserved vector. And since the Arnowitt-Deser-Misner formalism exclusively involves compact 3-spaces, \( \mu^A \) must vanish if Einstein equations are to be respected. Whereas a generic Regge-Teitelboim configuration exhibits a non-vanishing Casimir \( \mu^2 = \eta_{AB} \mu^A \mu^B \), easily recognized as the analogue of \((\text{mass})^2\). Einstein configurations come with \( \mu^2 = 0 \). In this language, Einstein gravity can be interpreted as the 'massless' limit of Regge-Teitelboim gravity.