Resolution of an ambiguity in dynamo theory and its consequences for back reaction studies

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ABSTRACT

An unsolved problem in turbulent dynamo theory is the “back reaction” problem: to what degree does the mean magnetic field suppress the turbulent dynamo coefficients which are needed to drive its growth? The answer will ultimately derive from a combination of numerical and analytical studies. Here we show that analytic approaches to the dynamo and back reaction problems require one to separate turbulent quantities into two components: those influenced by the mean field (which are therefore anisotropic) and those independent of the mean field (and are therefore isotropic), no matter how weak the mean field is. Upon revising the standard formalism to meet this requirement, we find that: (1) The two types of components often appear in the same equation, so that standard treatments, which do not distinguish between them, are ambiguous. (2) The usual first-order smoothing approximation that is necessary to make progress in the standard treatment is unnecessary when the distinction is made. (3) In contrast to previous suggestions, the correction to the dynamo $\alpha$ coefficient found by Pouquet et al (1976) is actually independent of the mean field, and therefore cannot be interpreted as a quenching.

Subject Headings: magnetic fields; galaxies: magnetic fields; Sun: magnetic fields; stars: magnetic fields; turbulence; accretion discs.
1. Introduction

Mean-field turbulent dynamo theory is a leading candidate to explain the origin of large-scale magnetic fields in stars and galaxies (e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983). This theory appeals to a combination of helical turbulence, differential rotation, and turbulent diffusion to exponentiate an initial seed large-scale magnetic field. Typically, the total magnetic field is broken into a large-scale or mean component and a small-scale or fluctuating component, and the rate of growth of the mean field is sought. The mean field grows on a length scale much larger than the outer scale of the turbulent velocity, with a growth time much larger than the eddy turnover time at the outer scale. Fluid helicity provides a statistical correlation of small scale loops favorable to exponential field growth. Turbulent diffusion is needed to redistribute the amplified mean field since Ohmic diffusion is negligible. By definition, the mean-field dynamo then belongs to the “slow” dynamo class (Zeldovich et al 1983). On the other hand, the small-scale field amplifies quickly without the aid of diffusion, and is thus constitutes a “fast” dynamo (Zeldovich et al 1983). Rapid growth of the small-scale field necessarily accompanies the mean-field dynamo. Its impact upon the growth of the mean field, and the impact of the mean field itself on its growth are controversial.

Dynamo theory has usually been studied in the kinematic regime, wherein the turbulent and ordered velocity fields are assumed to be given, and the seed field is assumed to grow and amplify without back reactions that affect the turbulent motions which drive its growth. Kinematic dynamo theory is incomplete, as a real dynamo should represent a solution to the nonlinear magneto-hydrodynamic (MHD) equations, which certainly embody back reaction. Although numerical simulations are necessary for our understanding of the nonlinear theory, the simplicity of the basic mean-field dynamo formalism warrants analytic investigation of the nonlinear MHD equations and the resulting back reaction to understand why or why not something like the kinematic treatment captures the evolution of the field even in the nonlinear regime.

Some aspects of how the growing magnetic field affects the turbulence have been addressed. Kulsrud & Anderson (1992) confirmed previous arguments by, e.g., Cowling (1957) & Piddington (1981) that the small-scale field builds up to approximate equipartition successively from the smallest to the largest scales in the turbulence. However, since most of the energy in a turbulent spectrum is concentrated at the largest - or outer - scale, the question of whether the outer scale reaches true equipartition is most important, since at equipartition, MHD turbulence consists largely of Alfvén waves which do not contribute to dynamo action (Moffatt 1978); this would strongly suppress dynamo growth. An equally important, and related question is: for what value of the mean field does suppression of dy-
namo action become important? The simulations of e.g. Cattaneo & Hughes (1996) suggest that suppression occurs when the energy density in the mean magnetic field is smaller by a factor of the magnetic Reynolds number than that implied by e.g., Pouquet et al. (1976).

Not only do simulations disagree on the role of the magnetic Reynolds number in the suppression formulas, but so do interpretations from well-motivated analytical approaches such as those of Zeldovich et al. (1983); Montgomery & Chen (1984); Gruzinov & Diamond (1994); Bhattacharjee & Yuan (1995); Kleeorin et al. (1995) and Field et al. (1998). The formalism of Bhattacharjee & Yuan (1995), Gruzinov & Diamond (1994) and Kleeorin et al (1995) leads to extreme suppression even in the weak mean-field limit. A second class of formalism is exemplified by Vainshtein & Kitchatinov (1983) and Montgomery & Chen (1994) who considered an expansion in the mean magnetic field, and Blackman & Chou (1997) who considered a similar perturbation theory in both the mean magnetic field and the mean velocity field. Field et al. (1998) developed a fully nonlinear approach for assessing the back reaction in the case that the gradient in the mean field is negligible. In contrast to Bhattacharjee & Yuan (1995), Gruzinov & Diamond (1994) and Kleeorin et al. (1995), but in agreement with Kraichnan (1979), Field et al.(1998) find that the magnetic Reynolds number does not strongly enter the suppression of the alpha dynamo coefficient. Why do different analytic approaches in such papers concerned with back reaction produce such different results? Here we show that the answer to this question depends on how one decomposes small-scale turbulent quantities into components which are dependent, and independent of the mean fields, respectively. We show that these components often appear together in the same equations and not distinguishing them leads to ambiguities, while properly distinguishing between them allows progress even if the first-order smoothing approximation is not satisfied.

In section 2 we review the standard derivation of the dynamo coefficients following Moffatt (1978). In Section 3 we dissect the standard treatment, point out the ambiguities, and rederive the $\alpha$ coefficient with a formalism that makes the distinction between components discussed above. In section 4 we show the effect of the revised approach on analytic back reaction studies. We conclude in section 5.

2. The standard derivation

The induction equation describing the magnetic field evolution is

$$\partial_t B = \nabla \times (V \times B) + \lambda \nabla^2 B,$$

(1)

where $\lambda$ is a constant magnetic viscosity. Here $B$ is in velocity units, obtained by dividing by $\sqrt{4\pi \rho}$; we assume incompressibility.
The equation for the mean field derived by averaging (1) is
\[ \partial_t \bar{B} = \nabla \times \langle v \times b \rangle - \bar{V} \cdot \nabla \bar{V} + \bar{B} \cdot \nabla \bar{V} + \lambda \nabla^2 \bar{B}, \] (2)
where an overbar indicates the mean field and the small-scale quantities are in lower case. The term \( \bar{B} \cdot \nabla \bar{V} \) describes the effects of differential rotation, and will not be discussed further here, while the term \( \bar{V} \cdot \nabla \bar{B} \) can be eliminated by changing the frame of reference to one moving with \( \bar{V} \); both terms will be ignored in what follows. The task for the dynamo theorist is to find the dependence of the turbulent EMF \( \langle v \times b \rangle \) on \( \bar{B} \) so that (2) can be solved. The kinematic theory assumes that the statistical properties of the small-scale velocity field are prescribed independently of the mean magnetic field; in that case it remains only to obtain the statistical behavior of the fluctuating magnetic field. Subtracting (2) from (1) then yields
\[ \partial_t b = \bar{B} \cdot \nabla v - v \cdot \nabla \bar{B} + b \cdot \nabla v - v \cdot \nabla b - \nabla \times \langle v \times b \rangle + \lambda \nabla^2 b. \] (3)

At this stage, it is customary to make the first order smoothing approximation (FOSA), namely, that the terms second order in the fluctuating quantities can be neglected. The result is then
\[ \partial_t b = \bar{B} \cdot \nabla v - v \cdot \nabla \bar{B} + \lambda \nabla^2 b, \] (4)
and for small \( \lambda \) the last term can be ignored in the kinematic theory. We use this equation to describe the statistics of \( b \) in terms of those of \( v \). In doing so we employ the Reynolds rules (Rädler (1980)), i.e. that derivatives with respect to \( x \) or \( t \) obey \( \partial_t, x \langle X_i X_j \rangle = \langle \partial_t, x \rangle (X_i X_j) \) and \( \langle \bar{X}_i x_j \rangle = 0 \) where \( X_i = \bar{X}_i + x_i \) are components of vector functions of \( x \) and \( t \). For statistical ensemble means, these hold when correlation times are small compared to the times over which mean quantities vary. For spatial means, defined by \( \langle X_i(x, t) \rangle = V^{-1} \int X_i(x + s, t) ds \), the relations hold when the average is over a large enough \( V \) that \( L \ll V^{1/3} \ll D \), where \( D \) is the size of the system and \( L \) is the outer scale of the turbulence.

Using the Reynolds rules, isotropy of the turbulent velocities, and the assumption that \( \langle v(t) \times b(0) \rangle = 0 \), we obtain the classical result (Moffat 1978):
\[ \langle v \times b \rangle = \langle v(t) \rangle \times \int_0^t \partial_t \langle b dt' \rangle = \alpha \bar{B} - \beta \nabla \times \bar{B}, \] (5)
where \( \alpha = -\frac{1}{3} \tau_c \langle v \cdot \nabla \times v \rangle \) and \( \beta = \frac{1}{3} \tau_c \langle v \cdot v \rangle \) are the helicity and diffusion dynamo coefficients respectively, and \( \tau_c \) is the correlation time of the turbulent quantities. Eq. (5) is put into (2) to solve for the evolution of the mean field, often assuming \( \nabla \alpha = \nabla \beta = 0 \).

3. Including the back reaction and improving the standard formalism
There are three fundamental limitations with the derivation of section 2: (a) The kinematic approximation, according to which the velocity field is prescribed, ignoring the back reaction of both the small- and large-scale magnetic field. (b) The FOSA, according to which 1st and 2nd terms in (3) dominate the 3rd and 4th terms, even though it is known (Kulsrud and Anderson 1992) that the small-scale field $b$ builds up much faster than the large-scale field $\bar{B}$. (c) Finally, even though the same symbols $v$ and $b$ are employed throughout the derivation, they mean different things in different places. To see why, note that the scalar forms of $\alpha$ and $\beta$ follow from isotropy of the fluctuating components $b$ and $v$. However, if both are strictly isotropic, $\langle v \times b \rangle$ would vanish because it is the average of an isotropic vector. Moreover, growth of the mean field in (2) requires some violation of homogeneity, because growth requires a non-zero $\nabla \times \langle v \times b \rangle$. Thus, some way of distinguishing the small-scale quantities in the turbulent EMF from those in terms of which it is expanded using (4) is necessary.

We will first formulate the required modifications to the standard theory. Then we study their effects in attempting to derive the Reynolds number dependence of the back reaction on $\alpha$ in the limit of weak mean field. To this end, we split up the equation for the small-scale magnetic field into an equation for that component which is independent of the mean magnetic field and that component which depends on the mean magnetic field. Blackman & Chou (1997) applied this approach to first order in both the mean field and the velocity field to find a coupled vorticity-magnetic dynamo. With a similar approach, Field et al. (1998) compute rigorously the effect of the back reaction of $\bar{B}$ on the dynamo $\alpha$ to all orders in $\bar{B}$; here we consider only the first-order terms in their development.

We assume that the turbulence is weakly anisotropic and inhomogeneous. Terms linear in the slowly time-varying mean quantities contribute, but their averaged zeroth-order coefficients are taken to be isotropic and homogeneous (still allowing for reflection asymmetry). Iterating the equations using the formal solutions for the turbulent fields $b(t) = b(t = 0) + \int \partial_t b' dt'$ and $v(t) = v(t = 0) + \int \partial_t v' dt'$, and using times appropriately chosen such that the correlation $\langle v(t) \times b(0) \rangle \simeq 0$, we obtain to first order in mean quantities the turbulent EMF

$$\langle v \times b \rangle^{(1)} = \langle v^{(0)}(t) \times \int_0^t \partial_t b^{(1)}(t') dt' \rangle - \langle b^{(0)}(t) \times \int_0^t \partial_t v^{(1)}(t') dt' \rangle,$$

where the time derivatives are given by (3) and by the equation of motion, and thus to first order, (2) becomes

$$\partial_t \bar{B} = \nabla \times \langle v \times b \rangle^{(1)} + \lambda \nabla^2 \bar{B}.$$

Here another ambiguity arises if one does not properly separate the zeroth-order quantities from the higher order quantities: Note that the quantity $\langle v \times b \rangle$ can be written as $\langle v \times b \rangle = \langle v(t) \times \int \partial_t b(t') dt' \rangle = -\langle b(t) \times \int \partial_t v(t') dt' \rangle$. But not distinguishing between $v, b$ and
\( \mathbf{v}^{(0)}, \mathbf{b}^{(0)} \) in (6) leads to \( \langle \mathbf{v} \times \mathbf{b} \rangle = \langle \mathbf{v}(t) \times \int \partial_t \mathbf{b}(t') dt' \rangle = -\langle \mathbf{b}(t) \times \int \partial_t \mathbf{v}(t') dt' \rangle = \langle \mathbf{v}(t) \times \int \partial_t \mathbf{b}(t') dt' \rangle - \langle \mathbf{b}(t) \times \int \partial_t \mathbf{v}(t') \rangle = 0 \), where the second last equality follows from purposely ignoring the superscript \( (0) \) in (6), and the last equality is therefore unavoidable. This shows that separating the zeroth order fluctuating quantities from the total small-scale quantities is essential.

According to (6) the calculation of the EMF requires expressions for \( \mathbf{b}^{(1)} \) and \( \mathbf{v}^{(1)} \). The equation that determines \( \mathbf{v}^{(1)} \) is the momentum equation, given in general by

\[
\partial_t \mathbf{v} = - \nabla \cdot \nabla \mathbf{v} - \nabla p_{\text{eff}} + \nu \nabla^2 \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{G} + \mathbf{F}(x, t),
\]

where \( \mathbf{F} \) is the force that drives the turbulence, \( p_{\text{eff}} \equiv p + \frac{1}{2}B^2 \), with \( p \equiv P/\rho \), \( \nu \) is a constant viscosity, and \( \mathbf{G} \) is gravity. To put this equation in the form that we need, we first define the small-scale velocity \( \mathbf{v} \) to be \( \mathbf{V} - \bar{\mathbf{V}} \) and similarly for the other variables. Then averaging the resulting equation for \( \bar{\mathbf{V}} + \mathbf{v} \) and subtracting the result from (8), we obtain

\[
\partial_t \mathbf{v} = - \bar{\nabla} \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \bar{\mathbf{v}} - \mathbf{v} \cdot \nabla \mathbf{v} + \langle \mathbf{v} \cdot \nabla \mathbf{v} \rangle - \nabla p - \nabla \frac{1}{2}b^2 + \nabla \langle \frac{1}{2}b^2 \rangle - \nabla \langle \mathbf{b} \cdot \mathbf{B} \rangle + \mathbf{b} \cdot \nabla \mathbf{b} - \langle \mathbf{b} \cdot \nabla \mathbf{b} \rangle + \mathbf{b} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{b} + \mathbf{f}(x, t) + \nu \nabla^2 \mathbf{v},
\]

where \( p \) is now the small-scale pressure and \( \mathbf{f} \) is the small-scale (and only) part of \( \mathbf{F} \); consistent with the assumption of incompressibility, we take the small-scale gravity to be zero. We eliminate the term \( \mathbf{V} \cdot \nabla \mathbf{v} \) by a change of frame. We shall also ignore the term \( \mathbf{v} \cdot \nabla \bar{\mathbf{V}} \). In our Galaxy for example, this term is smaller than \( |\mathbf{v} \cdot \nabla \bar{\mathbf{V}}| \) (Field et al 1998). In what follows, therefore, \( \bar{\mathbf{V}} \) will not appear. Now, writing the equations (3) and (9) to zeroth and linear order in \( \mathbf{B} \), we have

\[
\partial_t \mathbf{b}^{(0)} = \mathbf{b}^{(0)} \cdot \nabla \mathbf{v}^{(0)} - \mathbf{v}^{(0)} \cdot \nabla \mathbf{b}^{(0)} - \nabla \times \langle \mathbf{v}^{(0)} \times \mathbf{b}^{(0)} \rangle + \lambda \nabla^2 \mathbf{b}^{(0)}
\]

and

\[
\partial_t \mathbf{b}^{(1)} = \mathbf{B} \cdot \nabla \mathbf{v}^{(0)} - \mathbf{v}^{(0)} \cdot \nabla \mathbf{B} + \mathbf{b}^{(1)} \cdot \nabla \mathbf{v}^{(0)} - \mathbf{v}^{(1)} \cdot \nabla \mathbf{b}^{(0)} + \mathbf{b}^{(1)} \cdot \nabla \mathbf{v}^{(1)} - \mathbf{v}^{(0)} \cdot \nabla \mathbf{b}^{(1)} - \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle^{(1)} + \lambda \nabla^2 \mathbf{b}^{(1)},
\]

for the small-scale magnetic field, and

\[
\partial_t \mathbf{v}^{(0)} = -\mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} + \langle \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} \rangle - \nabla p^{(0)} - \nabla \frac{1}{2}b^{(0)2} + \mathbf{b}^{(0)} \cdot \nabla \mathbf{b}^{(0)} - \langle \mathbf{b}^{(0)} \cdot \nabla \mathbf{b}^{(0)} \rangle + \nabla \langle \frac{1}{2}b^{(0)2} \rangle + \mathbf{f}(x, t)^{(0)} + \nu \nabla^2 \mathbf{v}^{(0)},
\]

and

\[
\partial_t \mathbf{v}^{(1)} = -\mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(1)} - \mathbf{v}^{(1)} \cdot \nabla \mathbf{v}^{(0)} - \nabla p^{(1)} - \nabla \langle \mathbf{b}^{(0)} \cdot \mathbf{b}^{(1)} \rangle + \nabla \langle \mathbf{b}^{(0)} \cdot \mathbf{b}^{(1)} \rangle + \langle \mathbf{v} \cdot \nabla \mathbf{v} \rangle^{(1)} - \nabla \langle \mathbf{b}^{(0)} \cdot \mathbf{B} \rangle + \mathbf{b}^{(1)} \cdot \nabla \mathbf{b}^{(0)} + \mathbf{b}^{(0)} \cdot \nabla \mathbf{b}^{(1)} - \langle \mathbf{b} \cdot \nabla \mathbf{b} \rangle^{(1)} + \mathbf{b}^{(0)} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{b}^{(0)} + \nu \nabla^2 \mathbf{v}^{(1)},
\]
for $v^{(1)}$, where we have assumed that the forcing function is the same for both the state with $\bar{B} = 0$ and that with $\bar{B} \neq 0$, so $f^{(1)} = 0$.

To further simplify we need an approximation related to, but distinct from the standard FOSA. Note that the standard FOSA, employed in section 1, would imply that $|b| = |b^{(0)} + b^{(1)} + ...| < |\bar{B}|$. Although this assumption is different from the kinematic approximation in that it says nothing about the back reaction of the mean field on the velocity flows, it does imply that the small scale field is much weaker than the mean field. In reality, this is not the case: both simulations and analytical arguments (e.g. Parker 1979) show that the small-scale field energy saturates at a value of order the small-scale kinetic energy. Fortunately, we do not have to make the FOSA here. This is because we take the zeroth-order turbulent quantities to be solutions of (10) and (12), and work with the effect of the mean field on the first order quantities. We require only that $|b^{(1)}|, |v^{(1)}| < \bar{B}$, a much weaker condition. This reduced smoothing approximation (RSA), unlike the FOSA, allows for $b^{(0)} \gg \bar{B}$.

This RSA allows us to eliminate terms second order in the fluctuating quantities in (13) and (11). Note, however, that in the first $\sim t_c$ of growth, the first term in (13) could dominate the terms with the mean field. However, we are only interested in times long compared to this time, after which a steady state ensues. When we put the equation for $v^{(1)}$ into the average (6) none of the bracketed terms on the right of (11) or (13) contribute. The relevant equations for the small-scale fields then become

$$\partial_t v^{(1)} = -\nabla p^{(1)} - \nabla (b^{(0)} \cdot \bar{B}) + b^{(0)} \cdot \nabla \bar{B} + \bar{B} \cdot \nabla b^{(0)} + \nu \nabla^2 v^{(1)}, \quad (14)$$

and

$$\partial_t b^{(1)} = \bar{B} \cdot \nabla v^{(0)} - v^{(0)} \cdot \nabla \bar{B} + \lambda \nabla^2 b^{(1)}. \quad (15)$$

A remaining problem is to deal with the pressure in (14). For present purposes, we focus on the correction to the $\alpha$ dynamo coefficient, and so we consider the special case where $\nabla \bar{B}$ is negligible (Field et al. 1998). We can then take the divergence of (14). The resulting equation is $\nabla^2 (p^{(1)} + b^{(0)} \cdot \bar{B}) = 0$, whose solution is $(p^{(1)} + b^{(0)} \cdot \bar{B}) = constant$. Thus the first two terms drop out of (14).

Approximating time integrals in $\langle v \times b \rangle^{(1)}$ by factors of the correlation time $t_c$ and freely employing Reynolds rules and incompressibility, we then obtain

$$\langle v \times b \rangle^{(1)} = \alpha^{(0)} \bar{B} \quad (16)$$

where

$$\alpha^{(0)} = -\frac{1}{3} t_c [\langle v^{(0)} \cdot \nabla \times v^{(0)} \rangle - \langle b^{(0)} \cdot \nabla \times b^{(0)} \rangle]. \quad (17)$$
Upon substituting these into (2), the curls can be pulled onto the $\bar{\nabla}$ from homogeneity of the zeroth-order averages. Note that we have ignored the sub-dominant $\nu$ and $\lambda$ terms. This will not pose any problem for our discussion in the next section: the entrance of the Reynolds number into the theory that we will address is that purported to enter through the “magnetic correction” $\langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle$ in (17). Similar magnetic corrections to that in (17) have been derived before (e.g. Pouquet et al., 1976; Gruzinov & Diamond 1994) but none of the previous papers distinguish between $\alpha$ and $\alpha^{(0)}$, and we will see below that the difference is fundamental.

4. Contrast to previous work

Building on the ideas of e.g. Piddington (1981), most workers agree that the Lorentz forces from the growing magnetic field react back on the turbulent motions driving the field growth and complicating the turbulent motions. However, analytic studies and simulations disagree as to the extent to which the dynamo coefficients are suppressed by this back reaction. Some (e.g. Cattaneo & Vainshtein (1991), Vainshtein & Cattaneo (1992), Cattaneo (1994), Cattaneo & Hughes (1996), Gruzinov & Diamond (1994), Bhattacharjee & Yuan (1995), Kleeroin et al., (1995)) argue that the suppression of e.g. $\alpha$ takes the form $\alpha \sim \alpha^{(0)}/(1 + R_M \bar{B}^2/v^2)$ where $R_M$ is the magnetic Reynolds number, while others (e.g. Kraichnan 1979, Field et al., 1998) suggest $\alpha \sim \alpha^{(0)}/(1 + \bar{B}^2/v^2)$ in the fully dynamic regime. If the former formula represented the actual level of suppression, the large values of $R_M$ in nature would prevent astrophysical dynamos from working. Something else would be needed to generate large-scale fields in stars and galaxies. More analytic and numerical studies will ultimately be required to answer this question. At present, we suggest that some existing analytic arguments for extreme suppression can be challenged.

To do so, we follow the path of Gruzinov & Diamond (1994) and Bhattacharjee & Yuan (1995) but employ the formalism of section 3, rather than that of section 2, and then find that we do not arrive at the same conclusions. Consider the small-scale electric field:

$$\mathbf{e} = c^{-1}[\langle \mathbf{v} \times \mathbf{b} \rangle - \mathbf{v} \times \mathbf{b} - \mathbf{v} \times \bar{\mathbf{B}}] + \eta \mathbf{j},$$  \hspace{1cm} (18)

where $\eta = 4\pi \lambda/c^2$ is the resistivity. Dotting this with $\mathbf{b}$ and averaging gives

$$\langle \mathbf{b} \cdot \mathbf{e} \rangle = -c^{-1}\langle \mathbf{b} \cdot (\mathbf{v} \times \bar{\mathbf{B}}) \rangle + \eta \langle \mathbf{b} \cdot \mathbf{j} \rangle,$$  \hspace{1cm} (19)

where we note that the first two terms on the right of (18) do not contribute to (19). Using the triple product on the second term in (19) and rearranging gives:

$$\langle \mathbf{b} \cdot \mathbf{j} \rangle = \eta^{-1}[-\bar{\mathbf{B}} \cdot \langle \mathbf{v} \times \mathbf{b}/c \rangle + \langle \mathbf{b} \cdot \mathbf{e} \rangle].$$  \hspace{1cm} (20)

Now the “magnetic correction” to $\alpha$ in the linear regime as computed in (17) is proportional to $c\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle^{(0)}/4\pi = \langle \mathbf{b} \cdot \mathbf{j} \rangle^{(0)}$, not to $\langle \mathbf{b} \cdot \mathbf{j} \rangle$. It is very important to distinguish between
those quantities with and without the superscript (0), as we now show. Taking the zeroth order contribution to (20) gives
\[ \langle \mathbf{b} \cdot \mathbf{j} \rangle^{(0)} = \eta^{-1} \langle \mathbf{b} \cdot \mathbf{e} \rangle^{(0)}, \tag{21} \]
but using (18) for \( \mathbf{e}^{(0)} \) in (21) shows that (21) is a trivial identity. Thus it is not possible to get information from (20). If we do not distinguish the zeroth-order quantities from first-order quantities we might conclude that the left hand side of (21) is equal to the correction to \( \alpha \) derived in (17), but that is not correct.

An almost identical discussion applies to Kleeorin et al (1995). In their Appendix A they provide a somewhat similar derivation of the back reaction to \( \alpha \), with the resulting equation
\[ \partial_t \alpha_m = C_1(\bar{\mathbf{B}} \cdot \nabla \times \mathbf{B} - \alpha \bar{\mathbf{B}}^2 / \beta) - C_2 \alpha_m, \tag{22} \]
where \( C_1 \) and \( C_2 \) are defined constants, and they write \( \alpha = \alpha_0 + \alpha_m \), with their \( \alpha_0 \propto \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle \) and \( \alpha_m \propto \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle \). The subtlety arises from their statement below their un-numbered equation between their (A4) and (A5) that “\( \alpha \) and \( \eta_T (\equiv \beta) \) are scalars when the mean field is small.” Scalars result from the assumption of isotropy, but isotropy applies only to strictly zeroth-order quantities as discussed in section 3. Thus, the ambiguity in the notation discussed above is present. We suggest that the \( \alpha \) and \( \eta_T (\equiv \beta) \) to which Kleeorin et al (1995) refer should actually be \( \alpha^{(0)} \) and \( \eta_T^{(0)} (\equiv \beta^{(0)}) \). Thus, the magnetic correction to \( \alpha \) in the weak mean field limit which they define as \( \alpha_m \), should really be \( \alpha_m^{(0)} \propto \langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle \), not \( \propto \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle \). This latter quantity is not simply related to \( \alpha_m^{(0)} \). The distinction is essential. Rather than (22), the equation for the correction would instead be
\[ \partial_t \alpha_m^{(0)} = -C_2 \alpha_m^{(0)}, \tag{23} \]
where no mean field appears. As a result, (22) implicitly contains less information than it appears to, by analogy to (20-21).

Yet another subtlety arises when one notices that the right side of (21) is proportional to the sum of a time derivative and spatial divergence of averages of zeroth order correlations. Assuming exact stationarity and isotropy of zeroth order averages then implies \( \langle \mathbf{b} \cdot \mathbf{e} \rangle^{(0)} = 0 \). The left side of (21) would then also be zero, and thus the correction in (21) would vanish for any finite \( \eta \). The current helicity would then not be a correction to \( \alpha \) but would be related directly through (20). However, the fact that the diffusivity enters (21) means that the right side is the ratio of two small quantities and requires an assessment of how accurate the assumptions of stationarity and isotropy really are.

The vanishing of \( \langle \mathbf{b} \cdot \mathbf{e} \rangle^{(0)} \) is reminiscent of Seehafer (1994) and Keinigs (1983), but is different because these authors do not distinguish zeroth order quantities and Keinigs (1983)
employs FOSA. Since averages of non-zeroth order fluctuating quantities are not in general isotropic, homogeneous or stationary, the full \( \langle b \cdot e \rangle \neq 0 \). When the full dynamo equation is considered, mean quantities that are functions of non-zeroth order fluctuating quantities can vary on the large scale of the system. The second term on the right of (20) can exceed the left hand side, providing the dominant balance to the first term on the right. This is not apparent from Keinigs (1983) or Seehafer (1994) because they do not distinguish between zeroth-order and higher order small-scale quantities.

5. Discussion and Conclusions

We have pointed out a fundamental ambiguity in magnetic dynamo theory, resolved it, and have shown its importance in explaining why some analytic approaches to the problem of the back reaction of the mean field in dynamo theories disagree. In particular, we showed that only part of the small-scale quantities can be thought of as isotropic, no matter how small the mean field is, and why the formalism of standard treatments is ambiguous on this point. The “magnetic correction” term shown in (17) is constructed from only that component of the small-scale field which is independent of the mean field, not from the total small-scale field. Previous studies have not separated the small-scale quantities into their zeroth and higher order components, and as a result find that the correction term in (17) appears to relate to the magnetic Reynolds number through (22) or (20). When the separation is made, the correction term cannot be interpreted as a strong suppression to the \( \alpha \) effect because then the only place for the magnetic Reynolds number to enter is through the last term in (15). This does not produce a strong suppression of \( \alpha \) (Field et al 1998). Note that the first-order smoothing approximation is not necessary in our approach.

The fact that a true dynamo cannot be kinematic need not imply that dynamo action is suppressed. An often quoted ratio, that of the total magnetic energy to that contained in the mean field (e.g. Zeldovich et al. (1983)) \( \langle b^2 \rangle = R_n \langle \tilde{B}^2 \rangle \), where \( n \) is some non-zero power (or logarithmic for a purely Kolmogorov spectrum, applicable in the completely kinematic regime), applies only when the first two terms on the right of (3) approximately balance the next three, or equivalently, when the sum of the the first three terms on the right of (10) balance the first two terms on the right of (11), since \( \partial_t \mathbf{b} = \partial_t \mathbf{b}^{(0)} + \partial_t \mathbf{b}^{(1)} \). But this state is very short lived because the small-scale field rapidly grows to near equipartition with the turbulent energy on a time scale \( t_c \), and the first three terms on the right of (10) quickly dominate. These terms lead to an MHD turbulent spectrum with most of the energy contained on the outer scale of the turbulence. In principle, the mean field can still slowly grow, while the combination of field line stretching and nonlinear damping in a turbulent cascade maintain the small-scale field at a steady energy density. A key question (Field 1995) is: what is the integrated kinetic vs. magnetic energy on the outer scale of the turbulence? We believe that this ratio is essential in assessing whether the dynamo works, and suggest
that a dynamo can work as long as there is a mismatch.

Finally, note that the growth of the small-scale field necessarily means that even galaxies at redshifts $\gtrsim 2$ would exhibit substantial fields when observed by Faraday rotation. This is because a typical turbulent energy-containing eddy in a galaxy is $\sim 100\text{pc}$ across, so that any line of sight through a disk will have significant RMS mean field component (Blackman 1998). A rough lower limit comes from assuming that the small-scale field builds up to equipartition with the turbulent motions ($\sim \text{few} \times 10^{-6}\text{G}$ in our Galaxy). Even when observed edge on, there are only $\sim 100$ such cells along the line of sight. The observed mean field would then be $\sim 10$ times smaller, or $\gtrsim 10^{-7}\text{G}$. The mere existence of apparent mean fields is therefore not a strong test for dynamo theory even for large redshift Galaxies. A better observational test would be to measure the scale and pattern of field reversals at these redshifts. The existence of small-scale fields is guaranteed to accompany any large-scale field produced by dynamo action and does not preclude its operation.
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