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Abstract

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Numerical calculation of effective action

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We propose a method to calculate the effective action for scalar theories. This is an extension of the constraint effective potential method, so it is easily applied to lattice calculations. We apply this method to lattice Z_2 symmetric $\lambda\phi^4$ theory and discuss the effectiveness of this method.

1. Introduction

The effective potential $U_{eff}(\varphi)$ is useful to consider a vacuum of theory and phase transitions. The arguments of $U_{eff}(\varphi)$ do not depend on the space-time, so the calculation of $U_{eff}(\varphi)$ is easy. The work [1] is a pioneering work which is applicable to lattice study and a lot of studies have been done using this idea [2–4].

The effective action $\Gamma[\varphi]$ describes the dynamics of the theory on the vacuum.

$$\Gamma[\varphi] = \int dx (U_{eff}(\varphi) + Z(\varphi)(\partial_\mu \varphi(x))^2 + \dots)$$

In contrast to $U_{eff}(\varphi)$, $\Gamma[\varphi]$ is the space-time dependent functional, and this feature causes one of the difficulties of lattice calculations of $\Gamma[\varphi]$.

To avoid such a large number of degrees of freedom, we must reduce these degrees of freedom effectively. We propose a method to calculate effective action, named as constraint effective action (CEA) method. This method is applicable to lattice calculations, because it reduces the number of degrees of freedom of the argument of $\Gamma[\varphi]$.

2. Constraint effective action

We consider a Z_2 symmetric scalar model here. It is easy to apply the following method to other ($O(n)$ symmetric) models. We define the constraint effective action $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ as

$$e^{-S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})} \equiv \int [d\varphi] \delta^2(\varphi_p - \bar{\varphi}_p) e^{-S[\varphi]} \quad (1)$$

where $\varphi_p = \sum_x e^{ipx} \varphi_x$. This definition is a naive extension of the constraint effective potential (Ref. [1,2]).

We suggest that $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ is related to usual effective action $\Gamma[\varphi]$ ¹.

$$S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p}) \simeq \Gamma[\{\varphi_q\}] \quad (2)$$

where $\{\varphi_q\}$ denotes a configuration

$$\varphi_q = \begin{cases} \bar{\varphi}_{\pm p} & (q = \pm p) \\ 0 & (q \neq \pm p) \end{cases}$$

i.e. $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ is related to $\Gamma[\varphi]$ with a fixed configuration.

We check this relation in the following discussion. The effective action is the generating functional of one particle irreducible (1PI) graphs, so one particle reducible (1PR) graphs should not appear on RHS of (1) when one evaluates the path integral. There are two features of the Feynman rules of (1). One is that external lines correspond to $\bar{\varphi}_{\pm p}$ and carry momentum $\pm p$. The other is that the propagator $G(q)$ vanishes when $q = \pm p$. Using these rules, we see that at each order in $\bar{\varphi}_{\pm p}$ 1PR graphs do not contribute to RHS of (1).

In terms of 2 point functions, 1PR graphs can be represented by a series of 1PI graphs as

$$\begin{aligned} \text{---} \bigcirc \text{1PR} \text{---} &= \text{---} \text{shaded circle} \text{---} \text{shaded circle} \text{---} \\ &+ \text{---} \text{shaded circle} \text{---} \text{shaded circle} \text{---} \text{shaded circle} \text{---} \\ &+ \dots \end{aligned} \quad (3)$$

where shadowed circles indicate 1PI graphs. The external lines carry the momentum $\pm p$, so the momentum of the internal lines also have $\pm p$.

¹We use parentheses to denotes functions and square brackets denote functionals.

Using $G(\pm p) = 0$, we can see that (3) is equal to zero. From this discussion, $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ reproduces $\Gamma[\{\varphi_q\}]$ at the order $\varphi_{\pm p}^2$.

For 4 point 1PR graphs, we can write

$$\text{1PR} = \text{diagram 1} + \text{diagram 2} + \dots$$

We can easily see that all momenta of the internal lines have $\pm p$, and that none of these graphs contributes to S_{CEA} .

But in the case of the 6 point functions, the expansion is

$$\text{1PR} = \text{diagram} + \dots \quad (4)$$

If all the external lines of a 4 point 1PI graph on RHS have same signs, the internal line has the momentum $\pm 3p$, so it does not vanish. Because of this type of graphs, S_{CEA} cannot reproduce the effective action in $O(\bar{\varphi}_{\pm p}^6)$.

In summary, the relation (2) means that each coefficient of $\bar{\varphi}_{\pm p}$ in $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ and $\Gamma[\{\varphi_q\}]$ is the same up to $O(\bar{\varphi}_{\pm p}^4)$, but the discrepancy appears in $\bar{\varphi}_{\pm p}^6$ terms and higher terms.

3. Numerical calculation

We perform numerical simulations to examine our proposal. For this purpose, we choose Z_2 symmetric $\lambda\varphi^4$ theory in 3 dimensions.

We need a method to keep the constraint $\sum \varphi_x \exp(ipx) = \bar{\varphi}_p$. We use the Metropolis algorithm with some modification to realize it. The modification is the following. Each site is updated sequentially. This operation causes the change of the value of φ_p , so we choose two sites randomly and change the values of the two sites to compensate this update.

To check our proposal, we compare a renormalized 2 point function in momentum space $G_R(p)$ obtained from normal Monte Carlo (MC) method

Figure 1. $G_R^{-1}(p) - G^{-1}(p)$ as a function of p^2 . The results are given in a 10^3 lattice and the parameters are $\kappa = 0.15$ and $\lambda = 1.0$. Circles correspond to the results of normal MC and triangles are the CEA results.

and from our method. The $G_R(p)$ is obtained by differentiating $\Gamma[\{\varphi_q\}]$ in $\bar{\varphi}_p$ and $\bar{\varphi}_{-p}$.

$$\begin{aligned} \left. \frac{\partial^2 \Gamma[\{\varphi_q\}]}{\partial \bar{\varphi}_p \partial \bar{\varphi}_{-p}} \right|_{\bar{\varphi}_{\pm p}=0} &= \frac{1}{V} G_R^{-1}(p) \\ &= \frac{1}{V} \frac{1}{Z} (p^2 + m_R^2) \end{aligned}$$

where V is the volume of the system, m_R^2 is the renormalized mass and Z is the wave function renormalization constant.

On the other side, we apply the same operation to S_{CEA} ,

$$\begin{aligned} \left. \frac{\partial^2 S_{CEA}}{\partial \bar{\varphi}_p \partial \bar{\varphi}_{-p}} \right|_{\bar{\varphi}_{\pm p}=0} &= \\ &= \frac{1}{V} G(p)^{-1} + \frac{\lambda}{2} \frac{1}{V} \langle \varphi_x^2 \rangle \\ &+ \left(\frac{\lambda}{3!} \right)^2 \left(\frac{1}{V} \right)^2 [\langle [\varphi^3]_p [\varphi^3]_{-p} \rangle - \langle [\varphi^3]_p \rangle \langle [\varphi^3]_{-p} \rangle] \end{aligned}$$

where $[\varphi^3]_p = \sum_x e^{ipx} \varphi_x^3$. $G(p)$ is the free propagator and $\langle \dots \rangle$ means VEV of the constraint system.

We show a result in Fig. 1. In this figure we plot $G_R^{-1}(p) - G^{-1}(p)$ as a function of p^2 . We observe that the two quantities are in good agreement in the small p^2 region, especially an intercept is the same. This indicates that our proposal

Figure 2. $G_R(p) - G(p)$ as a function of p^2 in different volume in $\kappa = 0.15$ and $\lambda = 1.0$. Diamonds for the 10^3 lattice and triangles for the 14^3 .

works well for m_R . From the results of normal MC, we observe that Z is close to one in these parameters region. It is also seen that the slope of CEA method's results is almost flat. This indicates that CEA method also work well for Z .

In Fig. 1, the CEA simulations are done in 4×10^3 configuration at each momentum, and normal MC simulations are done in 100 times larger numbers of configurations. Nevertheless, the statistical errors of normal MC results are larger than CEA one. This is one of the superior points of CEA method. We guess that this property would be due to smaller phase space in the Monte Carlo integration. This is a good feature of CEA method and it is useful for precise measurements.

We also measure this quantity in different volumes to see the finite size effects (Fig. 2). The finite size effects are seen. We observe that the difference between two volume results is smaller as p^2 increases.

4. Conclusion and discussion

We proposed a method to calculate effective action. This method reproduces the effective action $\Gamma[\varphi]$ up to $O(\varphi^4)$. We checked this method by comparing $O(\varphi_{\pm p}^2)$ terms in $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ and

$\Gamma[\{\varphi_q\}]$ using the lattice simulations.

We observed good agreements between MC and CEA results, and the statistical errors of CEA method are small compared with normal MC results. We can calculate $O(\bar{\varphi}_{\pm p}^4)$ terms using CEA method in principle. This is for a further study.

We see in Sect. 2 $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ reproduces $\Gamma[\{\varphi_q\}]$ up to $O(\bar{\varphi}_{\pm p}^4)$. The discrepancy between $S_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ and $\Gamma[\{\varphi_q\}]$ at $O(\varphi_{\pm p}^6)$ is due to internal $3p$ propagator in (4). To remove this graphs, we impose an additional constraint to (1).

$$e^{-S'_{CEA}} = \int [d\varphi] \delta^2(\varphi_p - \bar{\varphi}_p) \delta^2(\varphi_{3p}) e^{-S[\varphi]}$$

This additional constraint causes $G(3p) = 0$ in the Feynman rules of the theory and the external lines with momentum $\pm 3p$ in 1PR graphs vanish because of $\varphi_{3p} = 0$, so no 1PR graphs contribute to 6 point 1PR graphs.

We can extend this argument by imposing multiple constraints to (1).

$$e^{-S''_{CEA}} = \int [d\varphi] \delta^2(\varphi_p - \bar{\varphi}_p) \prod_{n>1} \delta^2(\varphi_{(2n-1)p}) e^{-S[\varphi]}$$

We think that $S''_{CEA}(\bar{\varphi}_p, \bar{\varphi}_{-p})$ would reproduce $\Gamma[\{\varphi_q\}]$ at all order of $\bar{\varphi}_p$, but we have not proved it yet.

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