The inverse redshift–space operator: reconstructing cosmological density and velocity fields

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ABSTRACT

We present the linear inverse redshift space operator which maps the galaxy density field derived from redshift surveys from redshift space to real space. Expressions are presented for observers in both the CMBR and Local Group rest frames. We show how these results can be generalised to flux–limited galaxy redshift surveys. These results allow the straightforward reconstruction of real space density and velocity fields without resort to iterative or numerically intensive inverse methods. As a corollary to the inversion of the density in the Local Group rest frame we present an expression for estimating the real space velocity dipole from redshift space, allowing one to estimate the Local Group dipole without full reconstruction of the redshift survey. We test these results on some simple models and find the reconstruction is very accurate. A new spherical harmonic representation of the redshift distortion and its inverse is developed, which simplifies the reconstruction and allows analytic calculation of the properties of the reconstructed redshift survey. We use this representation to analyse the uncertainties in the reconstruction of the density and velocity fields from redshift space, due to only a finite volume being available. Both sampling and shot-noise variance terms are derived and we discuss the limits of reconstruction analysis. We compare the reconstructed velocity field with the true velocity field and show that reconstruction in the Local Group rest frame is preferable, since this eliminates the major source of uncertainty from the dipole mode. These results can be used to transform redshift surveys to real space and may be used as part of a full likelihood analysis to extract cosmological parameters.

Key words: Cosmology: theory – large–scale structure of the Universe

1 INTRODUCTION

Galaxy redshift surveys are distorted by the intrinsic motions of galaxies along the line of sight. This distortion leads to a number of effects. On large, linear scales the radial displacement of galaxy positions is enhanced, leading to compressed structures along the line of sight. On smaller scales, near turnaround, the structure will appear completely collapsed along the line of sight, and thereafter will begin to invert itself. Finally on very small scales, after virialisation, the random motions of galaxies in clusters will stretch out the structure in redshift space.

These distortions can complicate measurements of the statistical properties of the large scale galaxy distribution, and can lead of confusion in the interpretation of redshift maps. However, the distortions also contain valuable information on the structure of cosmological velocity fields. Furthermore, in gravitational instability theory the parameter controlling the degree of distortion is a function of the mean density of the Universe (see the excellent review by Hamilton, 1997, on linear redshift distortions).

One of main problems of cosmography is how to map the large scale structure seen in galaxy catalogues from redshift space to real space. In recent years there has been much interest in reconstructing redshift surveys, with the advent of the IRAS all-sky surveys, QDOT and 1.2Jy, and in particular the new Point Source Catalogue redshift survey (hereafter PSCz; see Saunders et al 1996). While it is straightforward to write down the transformation from real space to redshift space galaxy distributions, the inverse relation has proven more elusive.

The standard approach to reconstructing real space and the velocity fields has been to apply some iterative method, to solve for the density field (Kaiser et al 1991, Taylor & Rowan-Robinson 1993, Branchini et al 1998) or reconstruct galaxy positions, again iteratively (eg, Yahil et al 1991). Peebles (1989) showed that the trajectories of galaxies could be solved numerically by a Least Action principle and Shaya
et al (1993) applied this to large scale structure. Recently Croft & Gaztanaga (1997) have shown that applying the Zel'dovich approximation to a Least Action principle leads to a combinatorics problem of finding the least path between initial and final conditions. Valentine et al (1999) have generalised this to selection function weighted galaxy surveys and applied the method to the Point Source Redshift Catalogue. Again, the process is iterative and may be time consuming. Going to a higher order reconstruction appears to be a slow and painful process.

An alternative method has been to expand the density field in spherical harmonics and solve for each multipole, either from a numerically intensive inversion method (Fisher et al 1995, Zaroubi et al 1995, Webster et al 1997), or by numerically solving the redshift space Poisson equation to find the velocity potential field (Nusser & Davis 1994). The former method has the advantage of calculating the uncertainties in the reconstruction, but assumes a Gaussian distributed density field and requires regularization in the form of a Wiener filter. The latter has the advantage of being extendable into the Zel'dovich regime, but again must be solved numerically if flux–limited.

Finally, Tegmark & Bromley (1995) proposed a Green's function solution to the linear inversion problem, allowing one to regain the real space density and velocity fields from a straightforward convolution. However their method was only valid for volume limited galaxy distributions.

With the advent of the PSCz and forthcoming completion of the AAT 2dF (Maddox et al 1998) and Sloan Digital Sky Survey (Szalay et al 1998) an accurate, but straightforward method for removing the distortion in flux–limited redshift surveys is required. Reconstruction can be seen as required for comparison of velocity field data with the inferred flow in redshift surveys, or a general method for reconstructing the distortion before subsequent analysis.

Since redshifts are taken in the rest frame of the observer, the inversion method should deal with the local dipole motion of the observer. The dipole itself is of interest, since a reconstruction of the real space dipole can be compared with the observed CMB dipole and again put constraints on the mean density of the Universe (Rowan-Robinson et al 1990, Kaiser & Lahav 1988, Rowan-Robinson et al 1999, Schmoldt et al 1999). Since the dipole is nonlocal, the whole volume of the redshift survey usually has to be reconstructed in order to estimate just the dipole.

In this paper we address the problem of linear reconstruction of galaxy redshift surveys, and put it on a firm theoretical foundation. In Section 2 we derive the inverse spherical redshift distortion operator by considering the displacement fields of galaxies in real and redshift space. This operator allows one to transform directly between the linear redshift density field and the linear real space density field, without iteration. This allows a quick and simple, yet accurate, method for removing the linear distortions seen in redshift maps. We derive versions of this inverse operator for volume and flux–limited redshift surveys and for observers in the Cosmic Microwave Background (CMB) and Local Group rest frames. As a corollary of the Local Group inverse redshift operator we derive an expression for estimating the real space observer's dipole motion directly from a redshift survey, without a full inversion. This removes the effects of the redshift distortion and corrects for the well–known “Rocket effect” (Kaiser & Lahav 1988).

We develop a spherical harmonic representation of the distortion operator and its inverse in Section 3. This allows both practical and convenient application of the operators to redshift surveys, and simplifies the calculation of the properties of the reconstructed density and velocity fields. We demonstrate the method by reconstructing both linear density fields and the real space dipole from a simple model of a redshift survey.

In Section 4 we use our spherical harmonic formalism to estimate the uncertainties in the reconstruction of the density and velocity fields with only a finite survey volume. We derive the statistical properties of the reconstructed velocity field, taking into account cosmic variance and shot noise. This allows us to place limits on the accuracy to which one can do such reconstructions.

We summarise our results and present our conclusions in Section 5.

2 MAPPING FROM REDSHIFT–SPACE TO REAL–SPACE

The problem of finding an inverse redshift–space operator depends on the specifics of the redshift survey to which it is to be applied. The redshift survey may be volume– or flux–limited. In the case of the latter, the mean observed density of galaxies is a function of distance from the observer and requires an extra correction. The redshifts of the galaxies in a survey are taken in the observer's rest frame, and so include the motion of the Earth with respect to the CMB. Traditionally this has been transformed to either the Local Group rest frame – where the Local Group represents the largest gravitationally bound structure to which we belong, and whose size is close to the linear regime – or the CMB rest frame. In the rest of this paper we shall refer to the “Local Group” rest frame in rather a loose way, referring to a hypothetical observer moving in the local linear flow field with respect to the CMB rest frame.

In the next few sections we find solutions to the inverse distortion in the CMB frame for volume–limited (Section 2.1) and flux–limited (Section 2.3) surveys, and in the Local Group frame (Section 2.4). In addition the effects of the “Local Group” motion on redshift space and its inverse operator lead us to an expression for the calculating the real space dipole from a redshift survey (Section 2.4).

We begin by considering the linear displacement of galaxies and derive the inverse redshift operator for a volume limited redshift survey.

2.1 Volume limited redshift inversion

The forward problem, of mapping a density field from real space to redshift space, can be solved using the transformation

\[ s = r + (\hat{r}, v) \hat{r} \]  

* Throughout we shall use velocity units for distances. This is equivalent to setting the Hubble parameter \( H_0 = 1 \) in the formulae.
where $s$ and $r$ are the comoving redshift and real-space coordinates, and $v$ is the peculiar velocity field. In linear theory the velocity field is calculated from the density field by (Peebles 1980)

$$v(r) = \frac{f(\Omega_m)}{4\pi} \int d^3r \, \delta(r) \frac{r' - r}{|r' - r|^3}$$

(2)

where $f(\Omega_m) \approx \Omega_m^{0.8}$ is the growth rate of the density field and $\Omega_m$ is the mass density parameter of the Universe.

It is useful to introduce a displacement vector field, defined by

$$\frac{d}{dt} \equiv v$$

(3)

which denotes the displacement in real space from some initial, or Lagrangian, position to the present day. In linear theory the velocity is proportional to the displacement,

$$v = f(\Omega_m) \xi.$$  

(4)

To allow for the possibility of linear biasing of the galaxy density field we introduce the linear bias factor, $b$, defined by $\delta_b = b \delta_m$. If velocities are estimated from the galaxy distribution, rather than the true density, the constant of proportionality between velocities and displacements is $\beta$ where $\beta \equiv \Omega_m^6 b/\bar{\rho}$.

The real space and redshift space coordinates, $r$ and $s$ respectively, can be related to a set of initial or Lagrangian coordinates, $q$, by

$$r = q + \xi,$$

(5)

and

$$s = q + \xi^*.$$  

(6)

The redshift space displacement field, $\xi^*$, is related to the real space displacement, $\xi$, by (Taylor & Hamilton 1996)

$$\xi^*_i = \mathcal{P}^{-1} \xi_j,$$

(7)

where

$$\mathcal{P}^{-1} \equiv \delta^{ij} + \beta \tilde{\xi}_i \tilde{\xi}_j$$

(8)

is the redshift space projection tensor, and $\delta^{ij}$ is the Kronecker tensor. From continuity we can find the linear redshift density by taking the divergence of the redshift displacement field:

$$\delta^s = -\nabla \cdot \xi^*.$$  

(9)

Substituting equation (7) into equation (9) yields

$$\delta^s = -\nabla \cdot \xi^* = -(\nabla \xi + \beta (\tilde{\xi}_i \tilde{\xi}_j \nabla \xi_j + 2r^{-1} \tilde{\xi}_i \xi_j))$$

$$= [1 + \beta (\partial^2 + 2r^{-1} \partial_r) \nabla^{-2}] \delta,$$

(10)

where we have used the real space continuity equation

$$\delta = -\nabla \xi,$$

(11)

and $\nabla^{-2}$ is the inverse Laplacian. To linear order spatial derivatives in redshift space are equal to derivatives in real space, and the coordinates of fields are the same in both real and redshift space, i.e. $\delta(s) = \delta(r)$.

Equation (10) can be written much more succinctly if we introduce the linear, spherical redshift space operator (Hamilton 1997)

$$S = 1 + \beta (\partial^2 + 2r^{-1} \partial_r) \nabla^{-2},$$

(12)

and write the transformation from real to redshift density fields as

$$\delta^s = S \delta.$$  

(13)

Having formulated the redshift distortion in the language of displacements the inverse operator, $S^{-1}$, is now straightforward to find. We begin by inverting equation (7), relating the real and redshift displacement fields,

$$\xi_i = \mathcal{P}^{-1} \xi^*_i$$

(14)

where

$$\mathcal{P}^{-1} \equiv \delta^{ij} - \frac{\beta}{1 + \beta} \tilde{\xi}_i \tilde{\xi}_j$$

(15)

is the inverse redshift projection tensor. Here we have used the identity $\tilde{\xi} = \tilde{r}$. Taking the divergence of both sides of equation (14) will give us the inverse relationship between redshift and real space density perturbations,

$$\delta = -\nabla \xi = [1 - \frac{\beta}{1 + \beta} (\partial^2 + 2r^{-1} \partial_r) \nabla^{-2}] \delta^s.$$  

(16)

The inverse spherical redshift operator is given by

$$S^{-1} = 1 - \frac{\beta}{1 + \beta} (\partial^2 + 2r^{-1} \partial_r) \nabla^{-2}$$

(17)

To invert the redshift distortion we have used only the continuity equation in real and redshift space, and assumed potential flow in both the real and the redshift displacement fields. In Section 2.2 we shall show that if the linear redshift displacement field is curl-free then so is the linear redshift displacement field.

Having found the inverse operator we now verify that it satisfies the relation

$$SS^{-1} = S^{-1} S = 1.$$  

(18)

In order to show that equation (17) does satisfy this it is useful to first find the relation between $S$ and $\mathcal{P}$, and $S^{-1}$ and $\mathcal{P}^{-1}$. From the redshift continuity equation we find

$$\delta^s = -\nabla \cdot \xi^* = -\nabla \cdot \mathcal{P} \xi^* = -\nabla \cdot \mathcal{P} \xi^*_i \nabla^{-2} \delta.$$  

(19)

Expanding $\delta$ in the same way we find that $S$ and $\mathcal{P}$ and $S^{-1}$ and $\mathcal{P}^{-1}$ are related by

$$S \equiv (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta, S^{-1} \equiv (\nabla \mathcal{P}^{-1} \xi^*_i \nabla^{-2} \delta,$$

(20)

Multiplying these operators together yields

$$SS^{-1} = (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta) \mathcal{P}^{-1} \xi^*_i \nabla^{-2} \delta.$$  

(21)

where we have highlighted the operator $\nabla \cdot (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta)$ in the second line which is applied to the real space displacement field, $\xi$. If we apply this operator to a general potential field $A_i = \nabla \phi$ we find

$$\nabla \cdot (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta) A_j = \nabla \cdot (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta) \nabla \phi.$$

(22)

Hence for a potential vector field this operator is the identity matrix

$$\nabla \cdot (\nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta = \delta^s,$$

(23)

Substituting this into equation (21) completes the proof

$$SS^{-1} = \nabla \mathcal{P} \xi^*_i \nabla^{-2} \delta \mathcal{P}^{-1} \xi^*_i \nabla^{-2} \delta.$$
\[ \begin{align*}
\omega^s &= \beta \nabla \times \xi^s, \\
\text{and given that the real space displacement field is curl-free,}
\omega &= \nabla \times \xi = 0,
\end{align*} \]

we find
\[ \begin{align*}
\omega^s &= \beta \nabla \times \hat{r}(\hat{r}, \xi) \\
&= \beta(\hat{r} \cdot \xi)(\nabla \times \hat{r}) + \beta(\hat{r} \times \nabla)(\hat{r}, \xi) \\
&= 0.
\end{align*} \]

Hence the linear redshift displacement field is curl-free and our proof holds.

### 2.2 Is the linear redshift displacement field irrotational?

To establish the same proof for \( S^{-1} S \) we need to show that the operator \( \nabla, \nabla^{-2} \nabla \), which this time is applied to \( \xi^s \), is acting on a potential field. Introducing the redshift vorticity vector, \( \omega^s \) defined as

\[ \omega^s = \beta \nabla \times \xi^s, \]

and given that the real space displacement field is curl-free,

\[ \omega = \nabla \times \xi = 0, \]

we find
\[ \begin{align*}
\omega^s &= \beta \nabla \times \hat{r}(\hat{r}, \xi) \\
&= \beta(\hat{r} \cdot \xi)(\nabla \times \hat{r}) + \beta(\hat{r} \times \nabla)(\hat{r}, \xi) \\
&= 0.
\end{align*} \]

Hence the linear redshift displacement field is curl-free and our proof holds.

### 2.3 Flux limited redshift inversion

Since most redshift surveys are flux limited rather than volume limited it is useful to find the flux limited linear redshift operators. The density of galaxies seen in a redshift survey can be represented in real space by

\[ \rho = \phi(1 + \delta), \]

while in redshift space the galaxy density can be written as

\[ \rho^s = \phi^s(1 + \delta^s). \]

The mean observed density of galaxies in the survey is given by the selection function, \( \phi \), in real space and \( \phi^s = \phi(s) \) in redshift space. Conservation of galaxy numbers imply that

\[ \rho(r) \, d^3v = \rho^s(s) \, d^3s \]

and so the real and redshift density fields are related by

\[ \rho = \rho^s \det D_{ij}. \]

where

\[ D_{ij} \equiv \frac{\partial s_i}{\partial r_j} \]

is the distortion tensor mapping real to redshift space. Expanding the determinant of the distortion tensor to first order we find

\[ \det D_{ij} \approx 1 + \frac{\beta}{1 + \beta} \nabla \cdot \delta_i \delta_j \xi^s, \]

where we have used the relationship

\[ s_i = r_i + \frac{\beta}{1 + \beta} \hat{r} \cdot \hat{r} \xi^s \]

to express the expansion in terms of the redshift displacement vector. We can also expand the real space selection function in terms of redshift space quantities:

\[ \frac{\phi^s}{\phi} = 1 - \frac{\beta}{1 + \beta} (\hat{s} \cdot \xi^s) \partial_i \ln \phi. \]

Combining these expressions we find to first order that the spherical redshift operator for a flux limited galaxy survey is

\[ \delta = [1 - \frac{\beta}{1 + \beta} (\partial_i^2 + \alpha(s) s^{-1} \partial_i) \nabla^{-2}] \xi^s. \]

where

\[ \alpha(s) \equiv \frac{d \ln \sigma^{-1}(s)}{d \ln s} \]

is related to the local slope of the selection function. Hence the inverse redshift operator for flux limited redshift surveys is

\[ S^{-1} = 1 - \frac{\beta}{1 + \beta} (\partial_i^2 + \alpha(s) s^{-1} \partial_i) \nabla^{-2}. \]

Similarly the forward redshift operator is

\[ S = 1 + \beta(\partial_i^2 + \alpha(r) r^{-1} \partial_i) \nabla^{-2} \]

where

\[ \alpha(r) \equiv \frac{d \ln r^2 \phi(r)}{d \ln r} \]

\( \alpha \) and \( \alpha' \) are related by

\[ \alpha(r) + \alpha'(r) = 4. \]

It is clear from these relations that the effect of changing from a volume limited redshift survey to a flux-limited redshift survey can be easily incorporated by the substitution \( 2 \to \alpha \) or \( 2 \to \alpha' \) in the forward and inverse operators respectively. This simple transformation will allow us to generalise results from volume- to flux-limited.

### 2.4 Inverse Redshift Operator in the Local Group Frame

So far we have only considered the distortion operator and its inverse from the point of view of an observer at rest with respect to the CMB. However the true rest frame of an observer is influenced by local, nonlinear gravitational interactions. Since nonlinear motions induced by local galaxies cannot be treated in the linear framework it is usual to transform to the linear part of our motion, that of the Local Group of galaxies (whether our Local Group also has a substantial nonlinear component to its motion is an issue we shall side-step here). One can then either work directly in the Local Group frame, or use the known dipole, measured by the CMB dipole anisotropy and transform to the CMB frame. However, there are a number of reasons for working in the Local Group rest frame. Firstly the linear theory formalism is only valid so long as the perturbative variable \( \hat{r}_v = (v(r) - v(0))/r \to 0 \) when \( r \to 0 \). In the CMB frame the velocity field does not necessarily vanish at the origin, and an extra, unnecessary assumption is needed to satisfy this. In addition, if our relative motion does not vanish at the origin, divergences appear in the analysis of our dipole, calculated from redshift surveys. We discuss this effect in detail below. Finally, as we shall show in Section 3 reconstruction of the velocity field in the Local Group frame reduces the uncertainties due to finite survey volumes and shot-noise.

The transform from real space to redshift space in the

Local Group rest frame can be found by subtracting the Local Group displacement from the redshift displacement vector:
\[
\xi_i^{\text{LG}} = P_{ij} \xi_j - \beta \nabla \cdot (\xi \cdot n) + (P_{ij} - \delta_{ij}) \xi_j(0). \tag{42}
\]
Super- or sub-script LG denotes a variable measured in the Local Group frame, and \(\xi(0)\) is the Local Group displacement, or dipole. This is equivalent to using \(\hat{r}(v - v(0))\) instead of \(\hat{r}.v\) as the redshift coordinate. From equation (42) the transformation to redshift space is straightforward. Taking the divergence of the redshift displacement field in the Local Group frame we find
\[
\delta^{\text{LG}} = [1 + \beta(\nabla^2 + 2r^{-1}(\partial_r - \partial_{r=0}))\nabla^{-2}] \delta(0) \tag{43}
\]
where \(\partial_{r=0}\) is the radial differential operator at the origin and \(\partial_{r=0} \nabla^{-2} = -\int d^3r (\hat{s} \cdot \hat{j})/4\pi r^2\) is the radial velocity operator at the origin. It is useful to note that \(\nabla_{\xi} \delta \xi(0) = \xi(0) (\nabla \delta \delta_{\xi}(0)\) since \(\xi(0)\) is a fixed vector at the origin.

In the Local Group frame the redshift operator is then
\[
S_{\text{LG}} = 1 + \beta(\nabla^2 + 2r^{-1}(\partial_r - \partial_{r=0}))\nabla^{-2}. \tag{44}
\]
Finding the inverse is slightly trickier. A straightforward inversion of equation (42) yields
\[
\xi_i = P_{ij} \xi_j^{\text{LG}} + (\delta_{ij} - P_{ij}) \xi_j(0) \tag{45}
\]
Therefore to find the real space displacement field we need to know what the Local Group displacement is. One could take the known dipole from the CMB here, but to calculate the effect self-consistently the Local Group motion should be estimated from the redshift catalogue itself.

One might suppose that equation (42) would help us here, since it transforms real displacements to redshift displacements. But if we let \(r \to 0\) we find \(\xi^{\text{LG}}(0) \to \xi(0)\). This is just a statement that the only displacement not affected by the redshift transformation is the observer’s, since this is by definition the origin of coordinates in both systems. But while we have demonstrated that the redshift and real space Local Group displacements are equal, this does not imply that the Local Group motion estimated from redshift catalogues are the same in real and redshift space, and it is the latter quantity we need. To illustrate this, consider the dipole generated at the origin. This can be calculated from the density field by
\[
\xi(0) = \frac{1}{4\pi} \int d^3r \hat{r} \delta(r), \tag{46}
\]
Similarly, one can calculate the redshift dipole (Kaiser & Lahav 1988)
\[
\xi^*(0) = \frac{1}{4\pi} \int d^3r \hat{r} \delta^*(r)
= -\frac{1}{4\pi} \int d^3r \hat{r} \nabla \xi^*(r)
= \xi(0) \left(\frac{2}{3} \int \frac{dr}{r} + 1\right)
+ \beta \int d^3r \hat{r} (\hat{r} \cdot \nabla^2 + 2r^{-1} \partial_r) \nabla^{-2} \delta
= \frac{2}{3} \int \frac{dr}{r} \xi(0) + \beta \int d^3r \hat{r} \delta \xi(0) \xi(0), \tag{47}
\]
which is clearly different from the true dipole. If we only consider the first term, which arises from the intrinsic dipole motion of the observer,
\[
\xi^*(0) = \beta \xi(0) \frac{2}{3} \int \frac{dr}{r}, \tag{48}
\]
we find a logarithmic contribution to the dipole from the survey geometry. This is the well known “Rocket Effect” (Kaiser & Lahav 1988). This contribution is generated by the reflex action of the distorted survey on the measured dipole. The divergence of the estimated dipole arises when points in real space are mapped to the origin in redshift space. In practice we would not expect this divergence to appear if the velocity field is coherent over some scale. In this case the local matter field will be moving at the same velocity as the observer and will not be mapped to the origin. Hence the way to avoid such divergences is to smooth the velocity field on some scale. In practice we will have to smooth on large scales to allow the use of linear theory (see Section 3.6 below).

A second divergence appears at large radii. As \(r \to \infty\) the Rocket term diverges logarithmically, so the redshift dipole in the Local Group frame does not converge (Kaiser & Lahav 1988). This consideration shows us that the true dipole can only be estimated over a finite range of radii. In practice a reconstruction should be done for different survey depths to test if the true dipole has converged.

The true dipole can be estimated from redshift space via equation (46) and the divergence of equation (45)
\[
\xi(0) = -\frac{1}{4\pi} \int d^3r \hat{r} \nabla \xi(r)
=-\frac{1}{4\pi} \int d^3r \hat{r} \left(1 - \frac{\beta}{1 + \beta} (\nabla^2 + 2r^{-1} \partial_r) \nabla^{-2}\right) \delta_{\text{LG}}
- \frac{2\beta}{3(1 + \beta)} \xi(0) \int \frac{dr}{r}. \tag{49}
\]
Solving this we arrive at the dipole solution
\[
\xi(0) = \frac{1}{A} \int d^3r \hat{r} \left(1 - \frac{\beta}{1 + \beta} (\nabla^2 + 2r^{-1} \partial_r) \nabla^{-2}\right) \xi^*(0), \tag{50}
\]
where
\[
A = 1 + \frac{2\beta}{3(1 + \beta)} \int \frac{dr}{r}. \tag{51}
\]
The presence of a logarithmic term, \(\int dr/r\), may seem worrying but it is there to cancel the divergent term that arises from estimating the dipole in redshift space from the Local Group frame. Again the divergence at small scale can be removed by smoothing the velocity field, or estimating the dipole for a ball of matter at the origin. The redshift density field in equation (50) is acted on by the inverse operator, but no correction is made for the dipole, so this field is not a mapping to the real space density. We present the correct expression for this below. In addition, in Section 3.4 we show that the dipole is coupled only to the redshift space mass dipole. Hence we can estimate the Local Group dipole directly from redshift surveys, without reconstructing the whole survey.

The inverse redshift distortion operator in the local group frame is somewhat more complicated than in the CMB frame, and can be written as
\[
\delta = [1 - \frac{\beta}{1 + \beta} (\nabla^2 + 2r^{-1} \partial_r) \nabla^{-2}\xi^*(0) - \frac{\beta}{1 + \beta} \frac{2}{3} \delta \xi(0), \tag{52}
\]
where $\xi(0)$ is estimated from equation (50), or taken from the CMB dipole.

Writing this in the form of an operator we find that the inverse redshift operator in the Local Group frame is

$$S_{LG}^{-1} = \left( 1 + \frac{1 + \beta}{1 + \beta} \frac{2}{A_\delta} \partial_{r \sigma} \nabla^2 \right) \times \left( 1 - \frac{1 + \beta}{1 + \beta} (\partial_s^2 + 2s^{-1} \partial_s) \nabla^2 \right).$$

(53)

These operators are valid for volume-limited redshift surveys. For flux limited surveys the effects of the selection function are easily incorporated by making the substitution $2 \to \alpha(r)$ in the operator $S$ and substituting $2 \to \alpha'(s)$ in the operator $S^{-1}$, and noting that these are now functions of radius and so should appear to the right of the integral. This completes our task of finding the linear inverse redshift distortion operators.

3 SPHERICAL HARMONIC REPRESENTATION OF THE REDSHIFT OPERATORS

3.1 The redshift space operators

In the case of spherical survey geometry and radial distortions the most convenient basis is that of spherical harmonics (Heavens & Taylor 1995, Ballinger at al 1995, Fisher at al 1995, Tadros et al 1999). In this basis we can use the properties of the eigenvalues of the Laplacian to transform from an integro-differential operator to an algebraic operator. We shall assume for convenience that the survey is all sky. For a spherical harmonic function, $\psi_{\ell m}(k)$, defined by

$$\psi_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \int d^3r \psi(r) j_{\ell}(kr) Y_{\ell m}^*(\hat{r})$$

(54)

the Laplacian becomes

$$\nabla^2 \psi = \left( \partial_r^2 + \frac{2}{r} \partial_r - \frac{L^2}{r^2} \right) \psi = -k^2 \psi$$

(55)

where $L^2 \equiv \ell(\ell + 1)$ is the square of the angular momentum operator. In this basis we can rewrite the distortion operator as

$$S = 1 + \beta \left( 1 - \frac{L^2}{k^2 r^2} \right)$$

(56)

whose inverse is

$$S^{-1} = 1 - \frac{1 + \beta}{1 + \beta} \left( 1 - \frac{L^2}{k^2 r^2} \right).$$

(57)

These operators contain both wavemodes and radial coordinates and should be used in expressions such as

$$\delta(r) = \sqrt{\frac{2}{\pi}} \int dk k^2 \sum_{\ell m} \delta_{\ell m}(k) S^{-1}(k, r) j_{\ell}(kr) Y_{\ell m}^*(\hat{r}),$$

(58)

transforming real space harmonic modes into the redshift space density field. The physical interpretation of these operators is straightforward. For $S$ an isotropic term proportional to $\beta$ is added to the true field and the angular part proportional to $L^2$ is subtracted, leaving only a radial distortion. The inverse operator, $S^{-1}$, replaces the angular term, producing an isotropic distortion, and then reduces the field by a factor $1/(1 + \beta)$.

To including the effects of a selection function we add an extra term

$$S = 1 + \beta \left( 1 - \frac{L^2}{k^2 r^2} \right) + \beta \frac{\alpha - 2}{r} \partial_r \nabla^2,$$

(59)

where the integro-differential term can be replaced using the recurrence relation for spherical Bessel functions:

$$\frac{1}{r} \partial_r \nabla^2 j_\ell(kr) Y_{\ell m}(\hat{r}) = - (\ell j_{\ell-1}(kr) - (\ell + 1) j_{\ell+1}(kr)) Y_{\ell m}(\hat{r}) \frac{1}{kr},$$

(60)

causing some mixing between $\ell$-modes. Moving to the Local Group frame requires another extra term

$$S = 1 + \beta \left( 1 - \frac{L^2}{k^2 r^2} \right) + \beta \frac{\alpha}{3kr} \delta_{\ell m} \int d^3r' \delta_D(r').$$

(61)

where the last term is the dipole contribution in spherical harmonics (see section 4.5), and the integral operator acts to shift positions to the origin.

3.2 The transformation properties of harmonic modes

Another representation is the transformation of the spherical harmonic modes of the redshift survey. In this case the modes transform according to the relation

$$\delta_{\ell m}(k) = (1 + \beta) \delta_{\ell m}(k) - \frac{\ell (\ell + 1)}{2(\ell + 1)} \beta \int_0^\infty dk' \delta_{\ell m}(k') \kappa_\ell(k', k)$$

(62)

where we have used the integral relation (Watson 1966)

$$\int_0^\infty dr j_\ell(kr) j_\ell(k'r) = \frac{\pi}{2(\ell + 1)} \kappa_\ell(k', k)$$

(63)

and the radial function $\kappa_\ell(k', k)$ is defined as

$$\kappa_\ell(z, z') \equiv \frac{z'^\ell}{z^{\ell+1}}, \quad z' > z.$$  

(64)

The inverse transformation, from $\delta_{\ell m}(k)$ to $\delta_{\ell m}(k)$, is done in the same way, with the substitutions $(1 + \beta) \to (1 + \beta)^{-1}$ in the first term in equation (62), and $\beta \to -\beta/(1 + \beta)$ in the second.

Similarly the redshift transformation in the Local Group frame is

$$\delta_{\ell m}^{LG}(k) = (1 + \beta) \delta_{\ell m}(k) - \frac{\ell (\ell + 1)}{2f + 1} \beta \int_0^\infty dk' \delta_{\ell m}(k') \kappa_\ell(k', k)$$

$$+ \frac{2}{3} \delta_{\ell m} \int_0^\infty dk' \delta_{\ell m}(k') (k'/k^2)$$

(65)

where the radial transform of the dipole term is completed by use of the integral (Watson 1966)

$$\int_0^\infty r dr j_\ell(kr) = \frac{\sqrt{\pi} \Gamma(\ell + 2/2)}{k^{\ell+1} \Gamma(\ell + 1/2)}$$

(66)

In Figure 1 we show set of Gaussian density fields, $\delta_{\ell m}(r)$ in real and redshift space, using the transformation equation (62). The harmonic modes of a constrained Gaussian field can be generated by the relation
presented here provide a simpler expression. Expanding \( \delta \) in harmonics, and using the relation

\[
\delta_{\ell m}(k) = \sqrt{P(k)} e^{i \theta_{\ell m}(k)}
\]

where \( P(k) \) is the linear power spectrum derived by Peacock & Dodds (1994) and \( \theta_{\ell m}(k) \) is randomly distributed for each mode. The waves have \( \ell = 1 \) to 6 and \( m = 0 \). Since the angular modes are independent and the distortion is \( m \)-independent this choice of azimuthal mode does not affect this demonstration. The real space density wave was recovered using equation (58). It is clear in this test of linear reconstruction that modes can be recovered with great accuracy.

### 3.3 The redshift power spectrum

Having calculated the harmonic modes of the density field in redshift space, we can calculate their correlations. This has been calculated before by Heavens & Taylor (1995) for a discrete set of spherical harmonics and by Zaroubi & Hoffmann (1996) for a set of Fourier harmonics. However the methods presented here provide a simpler expression.

Taking the two-point expectation value of the harmonic modes we find

\[
\langle \delta_{\ell m}(k) \delta_{\ell' m'}(k') \rangle = (1 + \beta)^2 P(k) k^{-2} \delta_D(k - k') + \beta^2 \ell^2 (\ell + 1)^2 \int_0^\infty dk_1 P(k_1) k_1^{-2} \kappa_1(k_1) \kappa_1(k_1, k') - \beta(1 + \beta) (\ell (\ell + 1) / 2F + 1) (P(k) k^{-2} \kappa_1(k', k) + P(k') k'^{-2} \kappa_1(k, k')) \tag{68}
\]

and zero for different \( \ell \)'s or \( m \)'s. As well as redshift space distortions creating correlations between different modes, the second term shows that the shape of the redshift power spectrum depends on the overall shape of the real power spectrum through a convolution.

### 3.4 The cosmological dipole

In Section 2.4 we derived an expression for the real space Local Group dipole. Again this takes a simple form in spherical harmonics. Expanding \( \delta \) in harmonics, and using the relation

\[
\hat{r}_i = \sqrt{\frac{4\pi}{3}} Y_{11}(\hat{r})
\]

and the orthogonality of spherical harmonics, we can reduce equation (50) to

\[
\xi_i(0) = \frac{1}{\sqrt{6\pi^2 A(1 + \beta)}} \int_0^\infty dk \int_0^\infty dr S_m^{-1}(k, r) \delta_i(k) j_1(kr) \tag{70}
\]

The radial integral can be performed to give

\[
\xi_i(0) = \frac{1}{\sqrt{6\pi^2 A(1 + \beta)}} \int_0^\infty dk \delta_i(k) \times [j_0(t) + 2/3 \beta (j_0(t) + j_1(t))/t - C(t)] \frac{\kappa_0}{k R} \tag{71}
\]

where \( R \) and \( r_0 \) are the upper and lower radial limits of the survey and \( C(t) = -\int dt \cos(t)/t \) is the cosine integral.

Alternatively one can express the dipole in terms of the redshift space density field without reference to the harmonic expansion. Integrating equation (70) and using the definition of the density dipole we find the real space dipole reduces to

\[
\xi_i(0) = \frac{1}{A(1 + \beta)} \int_0^\infty \frac{d^3 s}{4\pi} \delta^s(s) \hat{s} \left( \frac{1}{s^2} + \frac{\beta}{3} \left( \frac{1}{s^2} + \frac{1}{R^2} - \frac{2}{r_0 A} \right) \right) \tag{72}
\]

Equation (71) or (72) can also be used to estimate the real space dipole contribution in shells. In the limit of no distortions \( (\beta \rightarrow 0) \) equation (72) reduces to the real space dipole, given by equation (46), and equation (71) reduces to the harmonic equation for the dipole (equation (109), Section 4.5).

Hence we see that the real space velocity dipole can be reconstructed from the redshift space density field, without a full reconstruction. Added complications arise if there is incomplete sky coverage, since modes will be mixed, and this mixing must be included in a dipole estimate.

Finally, we can add on the effects of a selection function. Making the substitutions \( 2 \rightarrow a' \) in equation (50) and working through, we find the selection function correction term to equation (72) is

\[
\xi_i^s(0) = \frac{1}{12\pi A(1 + \beta)} \int d^3 s \delta^s(s) \hat{s} \int_0^R \frac{d^3 \phi(s')}{s^2} \partial_s \ln \phi(s'). \tag{73}
\]
they originate from. The small distances between galaxies in clusters can also lead to divergences in the calculated velocity field when the distance between galaxies becomes small. Hence smoothing would seem to be a good thing.

However smoothing in redshift space is not the same as smoothing in real space, since the coordinate system changes between the two. An isotropic smoothing kernel in one space will not be isotropic in the other. Anisotropic smoothing kernels, such as in adaptive, or Lagrangian smoothing which uses the local moment of interia as a kernel, will preserve the shape of structure and can be mapped back to real space smoothed with the correct kernel.

The spherical harmonic decomposition uses sharp $k$- and ($\ell, m$)-space filtering. This has the advantage that both redshift and real space will have the same smoothing, although since the harmonics are related by a convolution over $k$-space we need to sample higher $k$-modes in the inverted space. This will be limited by nonlinearities in the higher $k$ range. The sharpness of the $k$-space filter can be removed by introducing extra filters.

4 PROPERTIES OF RECONSTRUCTED REDSHIFT SURVEYS

Having found the inverse redshift space operator, we can now transform redshift surveys from redshift space to real space, and in the process predict the linear cosmological velocity field and the Local Group dipole. In doing so it is useful to have some idea of what uncertainties arise in the process. In this Section we calculate some of the properties of the reconstructed density and velocity fields. Throughout we shall assume that the survey is spherically symmetric, which is nearly the case for the IRAS Point Source Redshift Survey (Saunders et al 1996).

It is useful to disentangle the various uncertainties in analysing redshift surveys. In the following sections we treat independently the effects of: a biased and uncertain input distortion parameter, $\beta$ (Section 4.1); a finite survey volume on the reconstructed density field (Section 4.2); the properties of the velocity field in the CMB frame for finite survey volumes (Section 4.3); a finite survey on the dipole (Section 4.4); the properties of the velocity field in the Local Group frame (Section 4.5); shot-noise (Section 4.6). Apart from Sections 4.1 and 4.2 we shall ignore the effects of the redshift distortion. While this over-simplifies the case, it is useful to understand each of the effects in isolation.

4.1 The wrong $\beta$.

Using the incorrect distortion parameter, $\beta$, will lead to a systematic offset in the recovered fields. If the true distortion is $\hat{\beta}$, and we use a different value $\beta^\prime$ in the reconstruction, a residual term is left in the reconstructed density field. This can be characterised by finding the product $S^\prime - 1 S$, where $S^\prime - 1$ is the inverse operator with the incorrect distortion parameter. This residual is

$$S^\prime - 1 S = 1 + \frac{\beta - \beta^\prime}{1 + \beta^\prime}(\partial_r^2 + 2r^{-1} \partial_r)\nabla^{-2},$$

(74)

for a constant offset distortion. Another possibility is that there is a large uncertainty on the distortion parameter (as

3.6 Filtering redshift surveys

A number of methods for reconstructing the density and velocity fields do not use a smoothed galaxy distribution, but rather treat the galaxies in the catalog as test particles, inverse weighted by the selection function. The question then arises, should one smooth or not? Clearly information is lost whenever one smooths. However, there are a number of reasons why one can smooth in reconstruction. Firstly we are applying linear theory to the density field, so the scales we are considering must be large. Secondly, as well as nonlinear features in real space, there are nonlinear caustics and “fingers of god” in redshift space that we must remove before applying the present formalism. An alternative to smoothing “fingers of god” is to collapse them back onto the cluster

3.5 Simulated reconstruction of the dipole motion

Figure 2 shows the amplitude of the dipole reconstructed from a simple random Gaussian model of the density field. The true dipole was calculated from equation (71) with $\beta = 0$ (black line). Transforming the spherical harmonic modes to redshift space in the CMB frame was then done via equation (62), with $\beta = 1$, and the dipole re-calculated in redshift space (grey line). Finally the true dipole (dark grey line) was recovered via equation (71). It is clear from figure 2 that the true dipole can also be recovered to good accuracy from equation (71).

Figure 2. True CMB redshift and recovered dipole from a random Gaussian realisation of a density field, calculated in the CMB rest frame. Only the amplitude of the dipole is plotted. The light grey line is the redshift space dipole. The darker lines are the true and reconstructed dipoles. The fields have been Gaussian smoothed on a scale of 5$h^{-1}$Mpc.

where the factor of 2 in $A$ is also transformed to $A^\prime$.
Figure 3. Cosmic variance in reconstructed density field due to external structure for a volume-limited redshift survey with radius $R = 100\, h^{-1}\text{Mpc}$. The upper line is the total uncertainty from all modes. The line decreasing with radius is the $\ell = 1$ dipole contribution. This diverges at small radius. The flat light grey line is the $\ell = 2$ quadrupole contribution. The increasing line is the total contribution from all modes $\ell \geq 3$.

<table>
<thead>
<tr>
<th>$r$ (h$^{-1}\text{Mpc}$)</th>
<th>$\Delta r_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>40</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>60</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>80</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

4.2 Incompleteness in reconstructed density fields

The reconstruction of the cosmological density field is always incomplete, since the distortion is nonlocal and we are trying to reconstruct with only a finite volume. Hence uncertainties arise in our estimates of the velocity and shear fields which cause the distortion. We can quantify this uncertainty in a model-dependent fashion by calculating the effects of a finite survey and finding the rms contribution from external structure not included in the survey volume.

The effect of a finite survey volume is to alter the true inverse redshift operator, $S^{-1}$, to an effective one, $S_{\text{eff}}^{-1}$, using only the information within the surveyed region. Using the definition of the inverse redshift operator (equation 20) we see that this is equivalent to the operator

$$S_{\text{eff}}^{-1} \equiv (\nabla, P_{ij}^{-1} \nabla_j)\nabla^{-2}_R$$

where $\nabla^{-2}_R$ is the inverse Laplacian defined over a finite volume. It is useful to split the inverse Laplacian defined over all space, $\nabla^{-2}$, into a term defined within the survey volume, $\nabla^{-2}_{< R}$, and a term defined outside the survey volume, $\nabla^{-2}_{> R}$:

$$\nabla^{-2} = \nabla^{-2}_{< R} + \nabla^{-2}_{> R}.$$  

(77)

By doing so we can find the difference between the reconstructed density field and the true real space density field in terms of $\nabla^{-2}_{> R}$. Since our effective inverse Laplacian is equivalent to $\nabla^{-2}_{< R}$, we can write

$$\nabla^{-2}_{< R} = \nabla^{-2} - \nabla^{-2}_{> R}.$$  

(78)

Let us define

$$\Delta \equiv [S_{\text{eff}}^{-1} S - I] \delta$$

as the residual field after reconstruction. We find that this field can be expressed as

$$\Delta = -[S^{-1} \nabla^2 \nabla^{-2}_{> R}] \delta.$$  

(80)

Expanding the density field in spherical harmonics and applying these operators we find

$$\Delta(r) = \frac{2}{\pi} \sum_{\ell m} \langle \delta_{\ell m}(k') \rangle (\ell + 1) F(\ell, k, r) Y_{\ell m}(\hat{r}).$$

(81)

where

$$F(\ell, k, r) = \frac{\ell}{(kr)^{\ell + 1}}.$$  

(82)

For $\ell = 0$ we have the trivial solution that the residual field is zero, since no reconstruction is necessary. We also note that the monopole mode is zero from Newtons’ First Theorem. The $\ell = 1$ dipole term increases with decreasing radius and diverges at the origin, suggesting that this should be removed in reconstructions by working in the Local Group frame (we shall return to this point when reconstruction velocity fields), while the $\ell = 2$ quadrupole residual is independent of radius.

The variance of these terms can be found by squaring and ensemble averaging using the result that the ensemble average of the $\delta_{\ell m}(k)$’s is (Heavens & Taylor 1997)

$$\langle \delta_{\ell m}(k) \delta_{\ell' m'}(k') \rangle = P(k) k^{-2} \delta_{\ell \ell'} \delta_{m m'};$$

(83)

where $\delta_{\ell m}(x)$ is the Dirac delta function and $\delta_{m m'}$ is the Kronecker delta. The cosmological variance introduced by incompleteness is given by

$$\langle \Delta^2(r) \rangle = \left( \frac{\beta}{1 + \beta} \right)^2 \int_0^{\infty} \frac{k^2 dk}{2\pi} \frac{P(k)}{2\ell + 1} \left[ F(\ell, k, r) \right]^2.$$

(84)
the dipole increases at the origin, while the quadrupole is uniform across the reconstructed survey. The higher modes, \(\ell \geq 3\) contribute to a term increasing with radius, but which does not diverge at the survey boundary. Interestingly the dipole, quadrupole and the contribution from the rest of the multipoles are all equal at about half the survey radius, independent of the survey radius. Thus a nice way to characterise the uncertainty is in terms of the quadrupole contribution, which is independent of radii, and a third of minimum uncertainty.

### 4.3 Incompleteness in the velocity field

The incompleteness that arises when reconstructing the density field is not the only worry when calculating cosmic fields from a finite region. In the rest of this Section we consider the effect of a finite volume on estimates of the velocity field in the CMB and Local Group frames. To simplify the analysis we shall assume no distortion, and only consider the uncertainty due to a finite survey volume. We shall also consider the effects of shot noise on the reconstructed velocity field. We begin by expanding the velocity field in spherical harmonics.

The Newtonian Green’s function in equation (2) can be expanded in spherical harmonics

\[
\frac{1}{|r’ - r|} = 4\pi \sum_{\ell m} \frac{1}{2\ell + 1} \kappa_\ell(r, r’) Y_{\ell m}(\hat{r}) Y_{\ell m}(\hat{r}’) \tag{85}
\]

and is related to the velocity field by \(\nabla (1/r) = -\hat{r}/r^2\). The radial gradient \(\hat{r}\) is given by equation (64)

The gradient operator can be conveniently decomposed into radial and transverse terms;

\[
\nabla \kappa Y_{\ell m}(\hat{r}) = [\partial_r \kappa_i Y_{\ell m}(\hat{r}) - i\sqrt{\ell(\ell + 1)}\kappa r^{-1} Y_{\ell m}(\hat{r}) \tag{86}
\]

where we have defined the vector spherical harmonics

\[
Y_{\ell m}^L = i Y_{\ell m}, \quad Y_{\ell m}^M = \frac{1}{\sqrt{\ell + 1}}(\hat{r} \times L) Y_{\ell m}. \tag{87}
\]

and \(L \equiv -i\hat{r} \times \nabla\) is the classical angular momentum operator. With these definitions we find that the radial velocity component is given by

\[
v_r(r) = \sqrt{\frac{2}{\pi}} \sum_{\ell m} \int \, dk k^2 \delta_{\ell m}(k) U_{\ell m}(k, r) \tag{88}
\]

where

\[
U_{\ell m}(k, r) = \frac{1}{2\ell + 1} \int_R \, dr’ r'^2 [\partial_r \kappa_\ell(r, r’)] j_\ell(k r’) Y_{\ell m}(\hat{r}). \tag{89}
\]

The transverse velocity components are given by

\[
v_i(r) = \sqrt{\frac{2}{\pi}} \sum_{\ell m} \int \, dk k^2 \delta_{\ell m}(k) W_{\ell m}(k, r) \tag{90}
\]

where the kernel is a transverse vector field;

\[
W_{\ell m}(k, r) = -i\sqrt{\frac{\ell(\ell + 1)}{(2\ell + 1)r}} \int_R \, dr’ r'^2 \kappa_\ell(r, r’) j_\ell(k r’) Y_{\ell m}^M(\hat{r}). \tag{91}
\]

Using equation (83) for the expectation value of two harmonic modes we find the variance of the velocity field in terms of the power spectrum is

\[
\langle v^2_r(r) \rangle = \int_0^\infty \frac{k^2 \, dk}{2\pi^2} P(k) |U_{\ell m}(k, r)|^2, \tag{92}
\]

where the effective window function is

\[
|U_{\ell m}(k, r)|^2 = \frac{\ell^2}{(2\ell + 1)} \left[ \int_R \, dr’ r'^2 [\partial_r \kappa_\ell(r, r’)] j_\ell(k r’) \right]^2 \tag{93}
\]

and

\[
\langle v^2_i(r) \rangle = \frac{1}{r^2} \int_0^\infty \frac{k^2 \, dk}{2\pi^2} P(k) |W_{\ell m}(k, r)|^2, \tag{94}
\]

with the window function

\[
|W_{\ell m}(k, r)|^2 = \frac{\ell(\ell + 1)}{(2\ell + 1)} \left[ \int_R \, dr’ r'^2 \kappa_\ell(r, r’) j_\ell(k r’) \right]^2 \tag{95}
\]

The range of these radial integrals in the window functions determines the source of the uncertainties. One of the main uncertainties in reconstruction is the effect of the density field external to the survey volume. In the next Section we study the effects of structure beyond the survey. These results can also be used to calculate the expected variance in the velocity field from shells or bulk motions.

#### 4.3.1 Incompleteness due to external structures

The range of the radial integral in these equations depends on the region of space under consideration. If we are determining the sampling variance due to structures external to the redshift survey, then the range is \(r > R\), where \(R\) is the radius of the survey (here we shall only consider sharp edges to the survey. A more complete analysis will include weighting and the effects of an angular mask). In this case \(r \leq r’\) and so \(\kappa_\ell = r/r’\ell + 1\). Hence the radial integral in equations (93) and (95) is

\[
\int_R \, dr’ r'^2 \kappa_\ell(r, r’) j_\ell(k r’) = k^2 (kr) \left( j_{\ell + 1}(k R) \over (k R) - j_\ell(k R) \right) \tag{96}
\]

The radial derivative with respect to \(r\), in equation (93) is trivial.

The total sampling variance due to external structure is then

\[
\langle v^2_r(r) \rangle = \int_0^\infty \frac{dk}{2\pi^2} P(k) \sum_{\ell = 1}^\infty \ell (r/R)^{2(\ell - 1)} j_{\ell - 1}^2(k R) \tag{97}
\]

The summation is only non-zero from \(\ell = 1\), since external structure cannot affect the monopole mode.

Figure 4 shows the radial, transverse and total rms velocities, \(v_{\text{rms}} = \sqrt{\langle v^2 \rangle}\) for a survey with \(R = 100h^{-1}\) Mpc in the CMB and Local Group frames (see Section 4.5) frame. The upper set of three lines correspond to the radial, total and transverse rms uncertainties in the velocity field and its components in the CMB rest frame. Interestingly, for a spherical survey the uncertainties obey the relation \(\langle v^2_r(r) \rangle < \langle v^2_i(r) \rangle < \langle v^2_{\text{rms}}(r) \rangle\). At the origin the uncertainty is about 100 kms\(^{-1}\), which is the dipole uncertainty for a survey of radius \(R = 100h^{-1}\) Mpc (see Section 4.4). The main effect is a rise in uncertainty from the center of the survey to the edges. However, even on the survey boundary the error is finite.
**VELOCITY FIELDS: 1-D COSMIC VARIANCE**

![Cosmic variance in reconstructed velocity field](image)

**Figure 4.** Cosmic variance in reconstructed velocity field due to structure external to a redshift survey with radius $R = 100h^{-1}$Mpc. The upper set of three lines are the radial, total and transverse rms uncertainty in the velocity field and its components in the CMB rest frame. The lower set of lines are for the same survey, but calculated in the Local Group rest frame.

The radial and transverse velocity cross-correlation functions are

$$\langle v_r(r)v_r(r') \rangle = \int_0^\infty \frac{dk}{2\pi^2} P(k) \sum_\ell \frac{\ell^2}{(2\ell + 1)} \left( \frac{rr'}{R^2} \right)^{\ell-1} j_{\ell-1}(kR) P_\ell(\mu)$$

and

$$\langle v_t(r)v_t(r') \rangle = \frac{1}{r^2} \int_0^\infty \frac{dk}{2\pi^2} P(k) \sum_\ell \frac{\ell(\ell + 1)}{(2\ell + 1)} \left( \frac{rr'}{R^2} \right)^{\ell-1} j_{\ell-1}(kR) P_\ell(\mu)$$

where $P_\ell(\mu)$ is the Legendre function and $\mu = \hat{r} \cdot \hat{r}'$. No cross terms arise in the case of an all-sky survey, as we have chosen orthogonal projections.

### 4.3.2 Sampling variance from interior structure

Over the range $r < R$ the radial integrals in equations (93) and (95) can be again calculated:

$$\int_0^R dr' r'^2 \kappa_\ell(r,r') \tilde{j}_\ell(kr') = \frac{(2\ell + 1)}{k^2} j_\ell(kr) - \frac{(kr)^\ell}{k^2} \frac{j_{\ell-1}(kr)}{j_{\ell}(kr)} \left( \frac{R}{k} \right)^{\ell-1}$$

(100)

Differentiating with respect to $r$ yields

$$\left( \frac{2\ell + 1}{k} \right) j'_\ell(kr) - \frac{\ell}{k} \left( \frac{r}{R} \right)^{\ell-1} j_{\ell-1}(kr)$$

(101)

In the limit that $R \to \infty$ we recover the expression for the true peculiar velocity field (Regős & Szalay 1989).

**VELOCITY FIELDS: 1-D DIPOLE**

![Uncertainty in the reconstructed dipole](image)

**Figure 5.** Uncertainty in the reconstructed dipole due to cosmic variance from external structure (dark line) and shot noise due to the discrete sampling of galaxies in the survey (light line; see Section 4.6). The upper curve is the total rms uncertainty due to cosmic variance and shot noise added in quadrature. The power spectrum is described in the text and the redshift survey selection function is modelled on the PSCz (Saunders et al 1996).

$$v_r(r) = \sqrt{\frac{2}{\pi} \sum_\ell \int dk k^2 \delta_{\ell m}(k) k^{-1} j'_\ell(kr) Y_\ell(m(r))}$$

(102)

where we a dash on the spherical Bessel function denotes differentiation with respect to the argument. The autocorrelation function of the radial velocities is

$$\langle v_r(r)v_r(r') \rangle = \int_0^\infty \frac{dk}{2\pi^2} P(k) \sum_\ell (2\ell + 1) j'_\ell(kr) j'_\ell(kr') P_\ell(\mu)$$

(103)

and their variance is

$$\langle v_r^2(r) \rangle = \int_0^\infty \frac{dk}{2\pi^2} P(k) \sum_\ell (2\ell + 1) j'_2(kr).$$

(104)

### 4.4 Incompleteness in the cosmological dipole

As $r \to 0$ only the $\ell = 1$ (dipole) term survives and we recover the cosmic variance on the dipole

$$\langle v_r^2(0) \rangle = \int_0^\infty \frac{dk}{2\pi^2} P(k) \tilde{j}_0^2(kR).$$

(105)

In Figure 5 we show how the dipole uncertainty changes as a function of survey radius, $R$, smoothed on a scale of $5h^{-1}$Mpc. We have again assumed the linear power spectrum suggested by Peacock & Dodds (1994). At zero radius the uncertainty tends towards the 1-d rms velocity. This fairly well matches the observed local group velocity if one assumes that our motion is a fair estimate of the 3-d rms velocity. In that case $v_{rms, 1d} = v_{rms, 3d}/\sqrt{3} \approx 350$ km s$^{-1}$. As the survey radius increases the dipole uncertainty de-
creases. However, in a flux limited redshift survey the uncertainty from shot noise increases with radii (see Section 4.6), and eventually dominates over the sampling variance. For the PSCz we find that the uncertainties are equal at about $R = 150 h^{-1}$ Mpc, thereafter becoming shot-noise dominated (see Section 4.6).

For a power-law spectrum, $P(k) = A k^n$, the integral can be evaluated and we find

$$\langle v^2(0) \rangle = \frac{A}{2n+1} R^{1-n} \Gamma(n-1) \cos n \pi /2, \quad (106)$$

where the spectral slope is in the range $-1 < n < 1$.

The equations for the variance and correlations can be also be simplified for power-law spectra. In this case the Bessel integral obeys the scaling relations

$$\int_0^\infty dk k^2 j_2^2(kR) = \int_0^\infty \frac{dk}{2} j_0^2(kR) \Gamma((5-n)/2) / \Gamma((3-n)/2)$$

Hence the cosmic variance can be evaluated in terms of the dipole uncertainty;

$$\langle v^2(r) \rangle = 3 \langle v^2(0) \rangle \sum_{\ell=1}^\infty \frac{\ell^2}{2 \ell + 1} \left( \frac{r}{R} \right)^{2(\ell-1)}$$

$$\times \frac{\Gamma((\ell + n - 3)/2) \Gamma((5-n)/2)}{\Gamma((\ell + 3 - n)/2) \Gamma((n-1)/2)}. \quad (108)$$

The transverse and total variances can be expressed similarly. Hence we find that the uncertainty on the velocities in the Local Group rest frame is small compared with the magnitude of the velocity field.

4.5 Transforming to the Local Group rest frame

As we shall see it is more accurate to calculate velocities in the observer’s Local Group rest frame, rather than the CMB frame used so far. Projecting the local dipole along arbitrary radial direction we find that

$$v_i(0) = v(0) \hat{r} = \frac{1}{6n^2} \int_0^\infty dk k^2 \delta_{10}(k) k^{-1} j_0(kR). \quad (109)$$

Similarly the transverse dipole projected perpendicular to an arbitrary radial vector is

$$v_{t_i}(0) = \frac{1}{6n^2} \int_0^\infty dk k^2 (\delta_{1i}(k) - \delta_{10}(k) \hat{n}) k^{-1} j_0(kR). \quad (110)$$

Similar results for the dipole have been found by Webster et al 1997.

With these equations it is straightforward to show that subtracting the dipole term is equivalent to ignoring the monopole and dipole terms in the harmonic summation over $\ell$. This is because the local group velocity and other velocities are only connected by a dipole term. In Figure 4 we plot the uncertainty in the velocity field in the Local Group rest frame. With the major source of uncertainty, the dipole mode, removed we find that the relative uncertainty in the velocity field is small compared with the magnitude of the flow fields we expect.

4.6 Shot–noise contribution to the reconstructed redshift survey

The shot–noise contribution to the density field can also be calculated from equation (2), taking into account that the autocorrelation function for Poisson noise is

$$\langle \delta(r) \delta(r') \rangle = \frac{1}{\phi(r)} \delta_D(r-r'), \quad (111)$$

where $\phi(r)$ is the survey selection function. We find that the shot–noise variance is (Taylor & Rowan-Robinson 1993)

$$\langle \sigma^2_{sn}(r) \rangle = \int_0^R \int_{r'}^1 d\rho G(r-r')^2$$

However, this diverges as $r$ approaches $r'$, as infinite variance is produced by infinite close discrete particles. To avoid this problem in the reconstruction one smooths the density field. Applying a Gaussian filter here we find that the smoothed shot noise variance is

$$\langle \sigma^2_{sn}(r) \rangle = \int_0^R d\rho G(r-r') \int_{r'}^1 d\mu G(r-r')^2$$

where

$$G(r) = \frac{\sqrt{2}}{\sqrt{\pi r}} e^{-r^2/2R_s^2} \quad (114)$$

is the Gaussian smoothed Greens function.

The shot noise uncertainty on the velocity can also be decomposed into radial and tangential components along the line of sight. The radial term is

$$\langle \sigma^2_{vr}(r) \rangle = \int_0^R d\rho G(r-r') \int_{r'}^1 d\mu |G(r-r')|^2, \quad (115)$$

and the tangential component can be found from the relation

$$\langle \sigma^2_{vt}(r) \rangle = \langle \sigma^2_{vt}(r) \rangle - \langle \sigma^2_{sn}(r) \rangle.$$

Figure 5 shows the shot noise contribution to the dipole for the example of the PSCs. We see that the shot–noise contribution rises just as the sampling variance contribution is falling (Figure 4).

Figure 3 shows the radial, transverse and total rms velocity dispersion due to shot noise in the CMB and Local Group rest frames. The overall effect of moving to the Local Group rest frame is marginal, and only significant for velocities at small radii. Hence moving to the Local Group rest frame does not affect the shot–noise contribution to the uncertainties.

5 CONCLUSIONS

In this paper we have derived the inverse redshift space operator. By applying this operator to a redshift survey with the correct distortion parameter, found either from some other method, eg a study of the anisotropy of clustering in redshift space, or by comparing the reconstructed velocity field with the true one, a real space map of the density field
can be recovered. From this one can reconstruct to linear order the real space potential and velocity fields. We have derived inverse operators for redshift surveys for observers in the CMB or Local Group rest frames, and we have derived operators which include the effects of a selection function.

As a corollary to the calculation of an inverse operator in the Local Group rest frame we have also shown how to calculate the observer’s dipole directly from a redshift survey, without having to fully reconstruct the density field, again in either the CMB or Local group rest frames. Simple tests on an ensemble of Gaussian random density fields have shown that both the reconstruction of the density fields, and the observers dipole are accurate.

To simplify calculations in redshift space we have developed the formalism using a spherical harmonic representation for the fields and the distortion operators. This approach allows one to easily estimate the reconstructed field from a spherical harmonic decomposition of the redshift space density field. We have also found expressions that relate the real space dipole to the redshift space density dipole.

The spherical harmonic representation is also used to estimate the effects of a finite survey volume on the reconstruction. We find that the uncertainty on the dipole mode diverges at the origin, and suggest that this can be removed by working in the Local Group rest frame. The quadrupole uncertainty is a constant across the survey volume, while the sum of the remaining multipoles increases with radius.

The velocity field can also be inferred from a reconstructed redshift survey and we have calculated the effects of a finite survey volume and shot noise on the reconstruction of the velocity field, for the special case of no distortion. We have calculated the sampling variance expected for the velocity field reconstructed from a redshift survey and find that the velocity field relative to the observers motion can be calculated far more accurately than absolute velocities. Hence one should work in the Local Group rest frame when comparing velocities. Calculation of the shot-noise contribution shows that the shot-noise is fairly insensitive to the rest frame.

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**REFERENCES**