Symmetric Instantons and Skyrme Fields

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Abstract

By explicit construction of the ADHM data, we prove the existence of a charge seven instanton with icosahedral symmetry. By computing the holonomy of this instanton we obtain a Skyrme field which approximates the minimal energy charge seven Skyrmion. We also present a one parameter family of tetrahedrally symmetric instantons whose holonomy gives a family of Skyrme fields which models a Skyrmion scattering process, where seven well-separated Skyrmions collide to form the icosahedrally symmetric Skyrmion.
1 Introduction

Skyrmions are a type of topological soliton in three-dimensional space which are of interest to physicists, in that they are candidates for a solitonic description of nuclei. Numerical simulations reveal that the minimal energy Skyrmions often have a great deal of symmetry, and in particular the charge seven Skyrmion has icosahedral symmetry [4].

One approach to the study of Skyrmions is the suggestion by Atiyah and Manton [2] that low energy Skyrmions of charge $n$ may be approximated by computing the holonomy of a charge $n$ instanton in Euclidean $\mathbb{R}^4$ along lines parallel to the Euclidean time-axis. This proposal therefore predicts the existence of a charge seven instanton with icosahedral symmetry. In this paper we verify that indeed such an instanton exists by presenting the corresponding ADHM data. We then use this data to compute the holonomy of the instanton and hence generate a Skyrme field of charge seven with icosahedral symmetry, which is an approximation to the minimal energy Skyrmion.

Manton has proposed [16] that the low energy dynamics of $n$ Skyrmions may be approximated by motion on a finite dimensional manifold of charge $n$ Skyrme fields, and one reasonable choice for such a manifold appears to be the moduli space of $n$-instantons [2]. In such an approximation a scattering of seven Skyrmions is described by a one-parameter family of 7-instantons. We present such a family of instantons, obtained by imposing tetrahedral symmetry, and compute the associated Skyrme fields. We display the baryon density isosurfaces for this scattering process, in which six Skyrmions approach along the Cartesian axes a seventh Skyrmion at the origin. The Skyrmions merge to form first a dodecahedron, then a cube, and finally the dual dodecahedron before again separating along the Cartesian axes leaving a single Skyrmion remaining at the origin. Part of this scattering process, where the dodecahedron deforms to its dual, is believed to be an important vibrational mode of the dodecahedral Skyrmion, and these are of relevance when considering the quantization of Skyrmions [3].

2 An icosahedral 7-instanton

In this section we shall present the ADHM data for an icosahedrally symmetric 7-instanton. First, we briefly recall the ADHM construction of instantons [1] and give an explanation of symmetry in a gauge theory.

The ADHM data for an $SU(2)$ $n$-instanton consists of a matrix

$$\hat{M} = \begin{pmatrix} L \\ M \end{pmatrix}$$

where $L$ is a row of $n$ quaternions and $M$ is a symmetric $n \times n$ matrix of quaternions. By the ADHM constraints we shall mean the condition

$$M^\dagger M \text{ is a real matrix,}$$

where $M^\dagger$ denotes the quaternionic conjugate transpose of $M$.

The first step in constructing the instanton from the ADHM data is to form the matrix

$$\Delta(x) = \begin{pmatrix} L \\ M - x I_n \end{pmatrix}$$
where $1_n$ denotes the $n \times n$ identity matrix and $x$ is the quaternion corresponding to a point in $\mathbb{R}^4$ via $x = x_4 + ix_1 + jx_2 + kx_3$.

The second step is then to find the $(n+1)$-component column vector $N(x)$, of unit length, which solves the equation

$$N(x)\dagger \Delta(x) = 0. \quad (2.4)$$

The final step is to compute the gauge potential $A_\mu(x)$ from $N(x)$ using the relation

$$A_\mu(x) = N(x)\dagger \partial_\mu N(x). \quad (2.5)$$

This defines a pure quaternion which can then be regarded as an element of $su(2)$ using the standard representation of the quaternions in terms of the Pauli matrices.

In order that these steps make sense it is necessary that the ADHM data satisfy an additional invertibility condition: that the columns of $\Delta(x)$ span an $n$-dimensional quaternionic space, for every $x$. Equivalently this condition can be expressed as:

$$\Delta(x)\dagger \Delta(x) \text{ is invertible for every } x. \quad (2.6)$$

In the absence of this condition ADHM data satisfying (2.2) only gives rise to a self-dual gauge field with singularities (corresponding to the points $x$ where (2.6) fails).

### 2.0.1 Symmetric instantons

It will be useful to start by considering the problem of symmetric instantons in a general context. Thus let $G \subset SO(3)$ be a subgroup of the rotation group of Euclidean 3-space, and make it act on $\mathbb{R}^4$ by rotations on $(x_1, x_2, x_3)$, leaving $x_4$ alone. This action has a very convenient description using quaternions. For this, replace $G$ by the corresponding binary group $\tilde{G} \subset SU(2)$ (the double cover of $G$ obtained from the double cover $SU(2) \to SO(3)$).

Now

$$SU(2) = Sp(1) = \{u \in \mathbb{H} : uu^* = u^*u = 1\} \quad (2.7)$$

and $x \mapsto uxu^{-1}$ clearly preserves the $x_4$ (real) component of $x$. It also acts on the imaginary part of $x$ by the rotation in $SO(3)$ corresponding to $u$ in $SU(2)$. (In fact this gives a construction of the double cover of $SO(3)$ by $SU(2)$.) From now on, whenever we speak of a subgroup of $SO(3)$ acting on $\mathbb{R}^4$, we shall mean that it acts in the way we have just described.

In order to get a grip on the problem of imposing symmetry in a gauge theory, let us introduce a rank-2 vector bundle $E$ over $\mathbb{R}^4$ and assume that our gauge potential $A_\mu$ defines a connection on $E$. If $\tilde{G}$ acts on $E$ then it makes good sense to require $A_\mu$ to be invariant (or symmetric) under this action. In this case an action of $\tilde{G}$ on $E$ consists of the data of a unitary map $\Omega(g, x) : E_x \to E_{gxg^{-1}}$ for each $g \in \tilde{G}$ and $x \in \mathbb{R}^4$, depending smoothly on $x$, and satisfying $\Omega(1, x)$ = the identity, and

$$\Omega(f, gxg^{-1})\Omega(g, x) = \Omega(fg, x) \text{ for all } f, g, \in \tilde{G}, x \in \mathbb{R}^4 \quad (2.8)$$

both sides of this equation being maps $E_x \to E_{(fg)x(fg)^{-1}}$.

It is important to note that one cannot assume that there exists a gauge in which all the $\Omega$’s are equal to the identity. What is true, however, is the following. Let $a$ be any fixed-point of the action of $\tilde{G}$, i.e. any point on the $x_4$-axis. For such a point, (2.8)
reads $\Omega(f, a) \Omega(g, a) = \Omega(fg, a)$ for all $f, g, \in \tilde{G}$, giving a representation of $\tilde{G}$ on the fibre $E_a$. This representation changes to an equivalent one under changes of gauge, so if this representation is non-trivial, there cannot be any gauge with the $\Omega$'s all equal to the identity.

Conversely, given any complex 2-dimensional representation $\rho$ of $\tilde{G}$, we may define an action of $\tilde{G}$ on a bundle over $\mathbb{R}^4$ by putting

$$\Omega(g, x) = \rho(g)$$

relative to some background gauge. Obviously this action restricts to the representation on the fibre $E_a$ on $\text{Ad}(\rho)$ occurs in at least one physically interesting example, the cubic instanton of [15].

Given a bundle $E$ with an action $\Omega$, we can state the condition for a gauge potential to be symmetric:

$$A_\mu(gxg^{-1}) = \Omega(g, x)A_\mu(x)\Omega(g, x)^{-1} - \partial_\mu\Omega(g, x)\Omega(g, x)^{-1}.$$  \hfill (2.10)

In this context, the $\Omega$'s are often referred to as ‘compensating gauge transformations’ a practice we shall occasionally follow in this paper. The term $-\partial_\mu\Omega(g, x)\Omega(g, x)^{-1}$ vanishes if the action is in the standard form (2.9). Notice that if (2.10) is satisfied then the curvature (field-strength) transforms like $F(gxg^{-1}) = \Omega(g, x)F(x)\Omega(g, x)^{-1}$ and for example the action density $-\text{tr}(F_\mu F^{\mu})$ is a $G$-invariant function on $\mathbb{R}^4$.

**Technical Remark** Since the object of physical significance, the gauge potential, takes its values in the adjoint bundle of $E$, it is most natural to impose symmetries on this bundle. Thus we suppose that $G$ acts on $\text{Ad}(E)$, covering the action of $G$ on $\mathbb{H}$ (so $g \in G$ gives a linear map $\text{Ad}(E)_x \rightarrow \text{Ad}(E)_{g x g^{-1}}$ for every $x$ in $\mathbb{H}$). Then it does not necessarily follow that such an action of $G$ on $\text{Ad}(E)$ lifts to an action of $\tilde{G}$ on $E$. Moreover this failure occurs in at least one physically interesting example, the cubic instanton of [15].

To explain how this problem can be understood, we remark that, as above, the action of $G$ on $\text{Ad}(E)$ restricts to give a representation of $G$ on $\text{Ad}(E)_a$ for any point $a$ fixed by the action. Since $\text{Ad}(E)_a$ may be identified with the Lie algebra of $SU(2)$ and hence with $\mathbb{R}^3$, in general the image of $G$ by this representation will be a subgroup of $SO(3)$ isomorphic to some quotient group $F = G/H$ of $G$. In general the double cover $\tilde{F} \subset SU(2)$ of $F$ acts on $E_a$ and this extends, as in (2.9) to an action on the whole of $E$. However it is in general not the case that $\tilde{F}$ is a quotient of the binary group $\tilde{G}$. In the case of the cubic instanton, $G$ is the rotation group of the cube, $H$ is the Klein viergruppe (consisting of the identity and the half-turns about the three coordinate axes) and $F$ is isomorphic to the dihedral group of order 6.

In the general case, it still happens that a double cover $\tilde{G}$, say, acts on $E$, but this need not be the binary group $\tilde{G}$. With $F, \tilde{F}$ as above,

$$\tilde{G} = \{(g, f) \in G \times \tilde{F} : p(g) = q(f)\}.$$  

Here $p : G \rightarrow G/H = F$ is the map to the factor group and $q : \tilde{F} \rightarrow F$ is the double-cover. The restriction to $\tilde{G}$ of the projection $G \times \tilde{F} \rightarrow G$ is then a $2 : 1$ map, as one can easily verify from the definitions.
If \( G \) is the icosahedral group, this complication does not arise owing to the fact that \( G \) is then a simple group (being isomorphic to the alternating group \( A_5 \)) and \( H \) must either be 1 or \( G \).

### 2.1 Symmetric ADHM data

Returning to the ADHM description (2.3), this will be \( \tilde{G} \)-symmetric if for every \( g \in \tilde{G} \), we have ‘compensating gauge transformations’

\[
\Delta(gxg^{-1}) = \begin{pmatrix} \rho_\infty(g) & 0 \\ 0 & U(g) \end{pmatrix} \Delta(x)U(g)^{-1}
\]

for every \( g \) in \( \tilde{G} \). The first matrix has been decomposed into blocks corresponding to (2.1) so \( \rho_\infty \) is \( 1 \times 1 \) and \( U \) is \( n \times n \). In order to preserve the shape of \( \Delta(x) \), \( U(g) \) must be the product of a real orthogonal matrix with a unit quaternion, while \( \rho_\infty(g) \) can be any unit quaternion. (Recall that the matrix \( M \) is symmetric.)

Considering the coefficient of \( x \) on each side of (2.11), we obtain

\[
U(g)xU(g)^{-1} = gxg^{-1} \mathbb{1}_n.
\]

By taking \( x = 1, i, j, k \) one deduces that \( U(g) = \rho_w(g) \cdot g \) where \( \rho_w(g) \) is real. Hence ADHM data are \( \tilde{G} \)-invariant if

\[
\rho_\infty(g)L \rho_w(g)^{-1}y^{-1} = L, \quad \rho_w(g)gM \rho_w(g)^{-1}y^{-1} = M,
\]

where \( \rho_w \) is a real \( n \)-dimensional representation of \( \tilde{G} \) and \( \rho_\infty \) is a quaternionic 1-dimensional representation of \( \tilde{G} \).

The reason for the notation \( (\rho_w, \rho_\infty) \) is that \( \Delta(x) \) can be viewed more invariantly as an \( \mathbb{H} \)-linear map \( W \rightarrow W \otimes \mathbb{H} \oplus E_\infty \) where \( W \) is an \( n \)-dimensional real vector space and \( E_\infty \) is the fibre at \( \infty \) of the bundle carrying our \( SU(2) \) gauge potential. (The space \( W \) can in turn be identified with the zero-modes of the coupled Dirac operator, but we shall not need this fact.) Here we are using the \( SU(2) \)-structure to think of \( E_\infty \) as a 1-dimensional quaternionic vector space.

Now the ADHM construction is natural; if we have an action of \( \tilde{G} \) on \( E \) (covering the action of \( G \) by rotations as above), such that the gauge potential is \( G \)-symmetric, then \( W \) and \( E_\infty \) automatically become representation spaces for \( \tilde{G} \). The above notation reflects the origin of the representations \( \rho_w \) and \( \rho_\infty \).

The reason why the ADHM equations become tractable after symmetry is imposed is simply that if \( W \) is a sum of not too many irreducible representations of \( \tilde{G} \), then there will not be too many parameters involved in the specification of \( L \) and \( M \) satisfying (2.13). Our next task, then, is to consider the irreducible representations of \( \tilde{G} \) and the construction of \( \tilde{G} \)-invariant maps between tensor products of certain of these representations.

### 2.2 A little representation theory

The problem of choosing the representations \( \rho_w \) and \( \rho_\infty \) and of constructing the invariant matrices (2.13) will now be considered. Let us first compare it with the analogous problem of constructing symmetric Nahm data (and hence symmetric monopoles [10, 12, 13]). The single most important difference between these problems is that in the Nahm case one
knows which representation (the analogue of $\rho_w$) is going to arise. That is because one knows that the Nahm data form the irreducible $n$-dimensional representation of $SU(2)$ at the end-points, and this $SU(2)$ really does correspond to the rotation group of $\mathbb{R}^3$. Thus in the cited work on symmetric monopoles, the approach was to understand explicitly how the standard representations of $SU(2)$ decompose under the action of $\tilde{G}$, making use of the invariants corresponding to the Klein polynomials.

In the instanton case, by contrast, we do not have such information about $W$. Since $W$ is identifiable with a space of Dirac zero-modes, one can compute the character of $W$ as a $\tilde{G}$-representation space using some equivariant index theory, but the information coming from this is not particularly useful. Instead we exploit the fact that the representation theory of finite subgroups of $SU(2)$ can be understood very explicitly. Since every irreducible representation of $\tilde{G}$ eventually appears in the standard representations of $SU(2)$, the two approaches are closely related in principle, though this may be rather cumbersome in practice.

We shall now, therefore, describe the irreducible representations of the binary icosahedral group $\tilde{G}$ following John McKay’s famous paper[18]. McKay observed that a convenient picture of the representation theory of $\tilde{G}$ is given by the extended $E_8$ Dynkin diagram:

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1 2 3 4 5 6 3' 4' 2'
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Each node stands for an irreducible representation of $\tilde{G}$. Here $1, \ldots, 6$ arise simply by restriction of the corresponding representations of $SU(2)$ to $\tilde{G}$. The other representations $2', 3', 4'$, of dimensions 2, 3, 4 respectively, will be described shortly. First we shall explain the role of the edges in the diagram: the node $\alpha$ is joined to $\beta$ if and only if $\alpha$ is a summand in the decomposition of $\beta \otimes 2$ into irreducible representations of $\tilde{G}$. In general, this would lead to a directed graph, but for subgroups of $SU(2)$, $\alpha$ occurs in $\beta \otimes 2$ iff $\beta$ occurs in $\alpha \otimes 2$. Moreover, the multiplicity is always 0 or 1. Thus we read off, for example $6 \otimes 2 = 3' \oplus 4' \oplus 5$. Since moreover as $SU(2)$-representations, $6 \otimes 2 = 5 \oplus 7$, we find that

$$7 = 3' \oplus 4'$$

as representations of $\tilde{G}$.

This decomposition corresponds to the existence of the icosahedral Klein polynomial of degree 12. Indeed the decomposition is equivalent to the existence of a non-trivial $\tilde{G}$-map $7 \rightarrow 7$, or equally to an invariant element in $7 \otimes 7$. Now the latter contains the representation $13$ which contains the above-mentioned Klein polynomial.

Notice also that $4' = 2' \otimes 2$.

In view of this last observation, it remains to describe $2'$ and $3'$. In the character table of $\tilde{G}$, these two representations look exactly like $2$ and $3$, but with the sign of $\sqrt{5}$ changed. As was pointed out to the first author by Jørgen Tornehave, this extends to the representations, in the following sense. Identify $\tilde{G}$ with a finite subset of points of $\mathbb{H}$ such that all the coordinates of these points lie in $\mathbb{Q}(\sqrt{5})$. In other words, for every $g \in \tilde{G}$, the coefficients of $i, j, k, (as well as the real part)$ are each of the form $a + b\sqrt{5}$, where $a$ and $b$ are rationals. (We shall show how to do this in a moment.) Consider the ‘conjugation’ on $\mathbb{Q}(\sqrt{5})$, $\alpha \mapsto \alpha'$ which takes $a + b\sqrt{5}$ to $a - b\sqrt{5}$. Then $2'$ is the representation $x \mapsto g'x$, where $g'$ is the conjugation of $g$.
**3'** is the representation $x \mapsto g'x(g')^{-1}$ (on pure imaginary $x$) and **4'** is the representation $x \mapsto g'xg^{-1}$. This recipe defines a representation because of the property $\alpha'\beta' = (\alpha\beta)'$ for any $\alpha, \beta \in \mathbb{Q}(\sqrt{5})$ and it is clear that the character of the 'primed' representation is obtained by changing the sign of $\sqrt{5}$, just as required. It so happens that only the primed versions of **2** and of **3** are new representations. Remark that despite the notation, **4'** is not the primed version of **4**; we hope that no confusion will result from this notation.

In this discussion note that **3'** and **4'** are real representations: they act on $\text{Im}(\mathbb{H})$ or $\mathbb{H}$, viewed as 3 and 4-dimensional real vector spaces. By contrast, **2'** is essentially complex; it is not the complexification of any real representation. We point out further that **4'** has the following alternative description. A 5-dimensional representation of $\tilde{G}$ arises through the identification of $G$ with the alternating group $A_5$ and letting this act by permutations of the coordinates in $\mathbb{R}^5$. This is the sum of a trivial representation and an irreducible 4-dimensional representation on $V \subset \mathbb{R}^5$, $V = \{ y \in \mathbb{R}^5 : y_1 + y_2 + y_3 + y_4 + y_5 = 0 \}$. This representation on $V$ is isomorphic to **4'**.

Coxeter, in [9], gives a description of the binary icosahedral group (his notation is $(5, 3, 2)$) which has the property mentioned above, namely that all coordinates lie in $\mathbb{Q}(\sqrt{5})$. For this, the icosahedron is oriented in such a way that the coordinate axes in $\mathbb{R}^3$ pass through edge-midpoints. In particular, the half-turns about the coordinate axes lie in $G$. In terms of quaternions, these half-turns correspond to $i, j, k$. Then $\tilde{G}$ is generated by these together with one other element of order two such as $U_2 = -(i + \tau j - \tau^{-1}k)/2$. Setting

$$U_1 = j, \quad U_2 = -(i + \tau j - \tau^{-1}k)/2, \quad U_3 = i$$

(2.14)

one gets a set of generators of $\tilde{G}$. In terms of these, the five-fold rotation is given by $A = U_1U_2$, the three-fold rotation by $B = U_2U_3$ and the two-fold rotation by $C = U_3U_1$ and

$$A^5 = B^3 = C^2 = ABC$$

and this equals $-1$ in $\tilde{G}$ but 1 when projected to $G$. ([9], p.78, eqn (7.54) and p.69 eqn (6.65)). Here $\tau = (\sqrt{5} + 1)/2$ is the golden ratio.

We can now write explicitly the action of the generators $U_1, U_2, U_3$ in the irreducible representations of $\tilde{G}$. We shall confine ourselves to the cases needed in this paper. Where necessary, we shall denote by $\rho_\alpha$ the action of $\tilde{G}$ in the representation corresponding to $\alpha$. The following identities involving $\tau$ will be used without comment in what follows:

$$\tau^{-1} = (\sqrt{5} - 1)/2, \quad \tau' = -\tau^{-1}, \quad \tau - \tau^{-1} = 1, \quad \tau + \tau^{-1} = \sqrt{5}, \quad \tau^2 + \tau^{-2} = 3.$$  

- Thinking of **2** and **2'** as quaternionic 1-dimensional representations, we have  
  $$\rho_2(i) = \rho_2'(i) = i, \quad \rho_2(j) = \rho_2'(j) = j, \quad \rho_2(k) = \rho_2'(k) = k;$$  

  and  
  $$\rho_2(U_2) = -\frac{1}{2}(i + \tau j - \tau^{-1}k), \quad \rho_2'(U_2) = -\frac{1}{2}(i - \tau^{-1}j + \tau k).$$  

- In terms of quaternions, **3** is obtained by letting $\tilde{G}$ act by conjugation on the imaginary quaternions $\text{Im}(\mathbb{H})$. Identifying $\text{Im}(\mathbb{H})$ with $\mathbb{R}^3$ via the coordinates $a_1i + a_2j + a_2k$, we find  
  $$\rho_3(i) = \rho_3'(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_3(j) = \rho_3'(j) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$  

and
while
\[ \rho_3(U_2) = -\frac{1}{2} \begin{pmatrix} 1 & -\tau & \tau^{-1} \\ -\tau & -\tau^{-1} & 1 \\ \tau^{-1} & 1 & \tau \end{pmatrix}, \quad \rho_\psi(U_2) = -\frac{1}{2} \begin{pmatrix} 1 & \tau^{-1} & -\tau \\ \tau^{-1} & \tau & 1 \\ -\tau & 1 & -\tau^{-1} \end{pmatrix}. \]

- Identifying \( H \) with \( \mathbb{R}^4 \) via the coordinates \( a_0 + a_1 i + a_2 j + a_3 k \), we obtain the action in the representation \( 4' \):

\[ \rho'_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho'_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]

and

\[ \rho'_4(U_2) = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{5} & -\sqrt{5} & -\sqrt{5} \\ \sqrt{5} & 3 & 1 & 1 \\ -\sqrt{5} & 1 & -1 & 3 \\ -\sqrt{5} & 1 & 3 & -1 \end{pmatrix}. \]

2.3 The ADHM description of an icosahedral 7-instanton

To construct the ADHM data for an icosahedral instanton of charge \( n \), we must choose a real representation of \( \tilde{G} \) of dimension \( n \) (the space \( W \)) and a 1-dimensional quaternionic representation (the space \( E_\infty \)). Having done so, we write down the most general \( \tilde{G} \)-invariant maps and attempt to use them to solve the ADHM constraints.

Notational remark In the rest of this paper, we shall use the term ‘\( \tilde{G} \)-map’ to denote any linear map between representation spaces of \( \tilde{G} \) that intertwines the \( \tilde{G} \)-actions. We shall also often use Schur’s lemma without comment.

Following the monopole situation, we suppose \( W = 7 = 3' \oplus 4' \). From McKay’s correspondence we have

\[ 3' \otimes 2 = 6, \quad 4' \otimes 2 = 6 \oplus 2'. \]

Since \( L \) is required to be a \( \tilde{G} \)-map from \( W \otimes 2 \) into \( E_\infty \), there is only one possibility for \( E_\infty \) which allows \( L \neq 0 \): the only 2-dimensional representation that occurs in \( W \otimes 2 \) is \( 2' \). Hence we take
\[ W = 3' \oplus 4', \quad E_\infty = 2'. \tag{2.15} \]

The quaternionic matrix \( M \) is naturally viewed as a map \( W \to W \otimes 2 \otimes 2 \). The McKay correspondence can be used also to calculate the space of such \( \tilde{G} \)-maps. Now \( M \) breaks up naturally into a real component and a pure imaginary component, corresponding to \( 2 \otimes 2 = 1 \oplus 3 \). The real component of \( M \) must be a multiple of the identity on the irreducible summands \( 3' \) and \( 4' \), yielding two free parameters. The pure imaginary part of \( M \) gives maps \( 3' \to 3' \otimes 3, \quad 3' \to 4' \otimes 3, \) and \( 4' \to 4' \otimes 3 \). From the McKay correspondence, there are no non-zero maps \( 3' \to 3' \otimes 3 \), but there is in each case a one-dimensional space of maps \( 3' \to 4' \otimes 3 \) and \( 4' \to 4' \otimes 3 \).

To sum up, we have the following parameter-count for \( \tilde{G} \)-symmetric ADHM data: one in the component of \( L \) that maps \( 4' \otimes 2 \to 2 \); two from the real part of \( M \); and three from the imaginary part of \( M \). In the following lemma we compute explicitly these invariant maps. Although MAPLE has been used to assist with these computations, they are quite straightforward to do if one first imposes invariance under the group \( \tilde{K} \subset \tilde{G} \) generated by \( i, j, k \). Using the above formulae,
• The most general $\tilde{G}$-map $4' \otimes 2 \rightarrow 2'$ is given by any real multiple of the row-matrix $l = (1, i, j, k)$.

• The most general $\tilde{G}$-map $3' \rightarrow 4' \otimes 3$ is given by any real multiple of

$$B = \begin{pmatrix} i & j & k \\ 0 & \tau k & \tau^{-1} j \\ \tau^{-1} k & 0 & \tau i \\ \tau j & \tau^{-1} i & 0 \end{pmatrix}.$$ 

More specifically, this means that for each $g \in \tilde{G}$, we have

$$g' l \rho_4(g)^{-1} g^{-1} = l,$$

and

$$\rho_4(g) B \rho_3(g)^{-1} g^{-1} = B.$$ 

Taking the quaternionic conjugate of this,

$$\rho_3(g) g B^\dagger \rho_4(g)^{-1} g^{-1} = B^\dagger.$$ 

It follows that

$$BB^\dagger = \begin{pmatrix} 3 & -i & -j & -k \\ i & 3 & k & -j \\ j & -k & 3 & i \\ k & j & -i & 3 \end{pmatrix}$$

satisfies $\rho_4(g) BB^\dagger \rho_4(g)^{-1} g^{-1} = BB^\dagger$, so its imaginary part gives the unique $\tilde{G}$-map $4' \otimes 3 \rightarrow 4'$. Since this is skew symmetric, it cannot be used as a diagonal block in the matrix $M$ from the ADHM data, since $M$ is required to be symmetric. It follows that the most general $\tilde{G}$-invariant ADHM data, with $W = 3' \oplus 4'$, is given by

$$\hat{M} = \begin{pmatrix} a l & 0 & 0 & 0 \\ b \mathbb{I}_4 & c B & 0 & 0 \\ -c B^\dagger & d \mathbb{I}_3 & 0 & 0 \end{pmatrix}$$

(2.16)

where $a, b, c, d$ are real numbers and we have used the fact that $B$ is pure imaginary to write $B^\dagger = -B^\dagger$. One computes

$$B^\dagger B = 4 \mathbb{I}_3, \quad l^\dagger l = 4 \mathbb{I}_4 - BB^\dagger$$

so that

$$\hat{M}^\dagger \hat{M} = \begin{pmatrix} (4a^2 + b^2) \mathbb{I}_4 & (c^2 - a^2)BB^\dagger & c(b - d)B \\ (c^2 - a^2)BB^\dagger & (c^2 + d^2) \mathbb{I}_3 \end{pmatrix}.$$ 

Thus the ADHM constraints are satisfied iff $a^2 = c^2$ and $c(b - d) = 0$. If $c = 0$, then $a = 0$ and the top row of $\hat{M}$ is identically zero. This yields a singular instanton and hence is not allowable. Hence $c \neq 0$, $b = d$ and $a = \pm c$. Since, moreover,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a l & 0 & 0 \\ b \mathbb{I}_4 & -a B \\ a B^\dagger & b \mathbb{I}_3 \end{pmatrix} = \begin{pmatrix} a l & 0 & 0 \\ b \mathbb{I}_4 & a B \\ -a B^\dagger & b \mathbb{I}_3 \end{pmatrix}$$

9
the two choices of sign lead to gauge-equivalent ADHM data and hence to gauge-equivalent
instantons.

What we have found is a charge-7 icosahedral instanton that is unique up to the ob-
vious freedom to translate along the $x_4$ axis (the parameter $b$) and overall scale ($(a, b) \mapsto (\lambda a, \lambda b)$). If we centre the instanton at the origin of $\mathbb{R}^4$ then the ADHM data are given
by any real multiple of

$$
\begin{pmatrix}
1 & i & j & k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & j & k \\
0 & 0 & 0 & 0 & 0 & \tau k & \tau^{-1} j \\
0 & 0 & 0 & 0 & \tau^{-1} k & 0 & \tau i \\
0 & 0 & 0 & 0 & \tau j & \tau^{-1} i & 0 \\
i & 0 & \tau^{-1} k & \tau j & 0 & 0 & 0 \\
j & \tau k & 0 & \tau^{-1} i & 0 & 0 & 0 \\
k & \tau^{-1} j & \tau i & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(2.17)

A Maple calculation shows that $\Delta(x)^\dagger \Delta(x)$ is invertible for every $x$, so that this is a non-
singular instanton.

2.4 The associated Skyrmion

Having constructed the ADHM data for an icosahedral instanton we now wish to make
use of this to compute a Skyrme field. Recall the proposal of Atiyah and Manton[2] which
results in the following explicit prescription for the Skyrme field

$$
U(x) = P \exp \left( - \int_{-\infty}^{+\infty} A_4(x, x_4) \, dx_4 \right).
$$

(2.18)

Here $U(x)$ is the $SU(2)$-valued Skyrme field in $\mathbb{R}^3$, $P$ denotes path ordering and $A_\mu$ is the
gauge potential of a Yang-Mills instanton field in $\mathbb{R}^4$.

For an instanton of charge $n$ this holonomy produces a Skyrme field with baryon number
$n$. Although this procedure does not give exact solutions to the Skyrme model it does give
fields which are good approximations to important Skyrmion configurations, in the sense of
having not only the correct symmetries but also energies which are only a few percent
above those of the numerically known Skyrmion solutions. For example, the 1-instanton
generates a hedgehog Skyrme field and by adjusting the scale of the instanton, which
may be regarded as a free parameter in the approximation, it is possible to obtain an
approximation whose energy is only 1% above that of the true solution (which is known
only numerically).

The minimal energy Skyrmions of charge two, three and four have axial, tetrahedral
and cubic symmetry respectively[7]. In each of these cases instantons have been found
with the correct symmetries, so that computing their holonomies produces Skyrme fields
which are good approximations to these Skyrmions[2, 15]. It should be noted that in these
multi-instanton examples the holonomy can not be computed analytically and therefore
numerical methods must be employed. Specifically, if $\tilde{U}(x, x_4)$ denotes the solution of the
matrix ordinary differential equation

$$
\partial_4 \tilde{U} = -A_4 \tilde{U}
$$

(2.19)
with the initial condition \( \tilde{U}(x, -\infty) = 1 \), then \( U(x) = \tilde{U}(x, \infty) \). The above set of ordinary differential equations, with \( x \) regarded as a parameter, need to be solved numerically, for which we employ a standard Runge-Kutta method. In principle, since we have the ADHM data explicitly, it is possible to obtain an exact analytic expression for \( A_4 \) by performing some quaternionic linear algebra. However, this is a non-trivial computation, even with the use of a symbolic computer algebra package, and is of little practical use. Furthermore, since we are employing a numerical algorithm to solve equation (2.19) then it is sensible to compute \( A_4 \) numerically also. To achieve this, note that equation (2.4) states that the vector \( N(x) \) is orthogonal to each of the columns of the matrix \( \Delta(x) \). Thus we can compute \( N(x) \) using a quaternionic Gram-Schmidt orthonormalization process and hence \( A_4(x) \) by using a finite difference approximation to equation (2.5).

Applying our numerical scheme to the ADHM data of the icosahedral 7-instanton derived above we obtain a Skyrme field whose baryon density isosurface is displayed in Figure 1.11. This surface, which looks identical to that obtained from full field simulations of the Skyrme model[4], resembles a dodecahedron, with the baryon density being maximal at the vertices of the dodecahedron and holes at the centre of each face. We have not attempted to find the instanton scale at which the energy of this Skyrme field is minimized since to perform an accurate calculation would require a more substantial amount of computing time. The important point is that we have demonstrated that an icosahedrally symmetric 7-instanton exists whose holonomy produces a good approximation to the minimal energy 7-Skyrmion.

3 Seven Skyrmion scattering

The aim of this section is to obtain a family of instantons which describes a seven Skyrmion scattering process in which seven well separated Skyrmions merge to form the dodecahedral 7-Skyrmion. In order to understand this scattering it is useful to first consider the analogous situation for monopoles, since from numerical simulations it appears that many Skyrmion scattering events are remarkably similar to known monopole scatterings[5].

A dodecahedral 7-monopole exists[12] whose energy density isosurface looks very similar to the baryon density isosurface shown in Figure 1.11. In fact it was known earlier that a tetrahedral 3-monopole and a cubic 4-monopole also exist[10] and again they resemble the corresponding Skyrmions[13]. In studying symmetric monopoles it is useful to consider rational maps, which we outline below.

The \( n \)-monopole moduli space is diffeomorphic to the space of degree \( n \) rational maps between Riemann spheres, with the equivalence relation that two maps that are equal after a rotation of the target sphere are identified. The rational map arises from the monopole as the scattering data along a half-line emanating from a chosen origin[14]. Explicitly, let \( z \) be a point on the Riemann sphere and consider Hitchin’s equation

\[
(D_r - i\Phi)s = 0
\]

for the two-component field \( s \), along the radial half-line through the point \( z \). Here \( D_r \) is the covariant derivative in the radial direction and \( \Phi \) is the Higgs field. Up to a constant multiple, there is a unique solution \( s = (s_1, s_2)^t \) which decays as \( r \to \infty \). Let \( R \) be the ratio of the components of this solution evaluated at the origin, that is, \( R = \frac{s_1}{s_2} |_{r=0} \). Now consider how \( R \) varies as we choose a new direction for the half-line by changing the value
of $z$. Then, as proved by Jarvis[14], $R$ is a holomorphic function of $z$ of degree $n$, where $n$ is the charge of the monopole fields occurring in (3.1). The effect of a gauge transformation is to transform $R$ by an $SU(2)$ Möbius transformation, and after taking this equivalence into account, there is a one-to-one correspondence between $n$-monopoles and rational maps $R(z)$ of degree $n$.

If a map $R(z)$, of degree $n$, is $G$-invariant (up to Möbius transformations) then there is an $n$-monopole with symmetry $G$, and vice versa. In ref.[11] many symmetric maps are presented but the one of relevance here is the following degree seven map which arises after the imposition of the symmetry $T_h$

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)}.$$  

(3.2)

$T_h$ is the group of rotations of a tetrahedron extended by inversion symmetry and, after a choice of orientation, the above one-parameter family, with $b$ real, gives all such maps. Since this one-parameter family is the fixed point set of a group action then it is a geodesic in the 7-monopole moduli space. Using the geodesic approximation[17] this family describes a low energy seven monopole scattering process as $b$ varies along the real line from $-\infty$ to $\infty$. Changing the sign of $b$ can be undone with a Möbius transformation plus the replacement $z \mapsto iz$, which corresponds to a rotation by $90^\circ$ about the $x_1$-axis. If $b = \pm 7/\sqrt{5}$ then the map has icosahedral symmetry, and represents the dodecahedral 7-monopole and its dual, whereas at $b = 0$, which is the midpoint of the scattering process, the map has cubic symmetry. In the limit as $b \to \infty$ the map degenerates to $R(z) = z$, which represents a single monopole at the origin, the other six monopoles having moved off to infinity along the Cartesian axes.

In summary this geodesic models a scattering event where six monopoles, moving in along the Cartesian axes, merge with a single monopole at the origin to form first a dodecahedron and then a cube, after which the process reverses but with a $90^\circ$ rotation. The purpose of the remainder of this section is to present a one-parameter family of Skyrme fields which describe a similar scattering of Skyrmions. At this point it is important to note that although an ansatz exists for Skyrme fields in terms of rational maps[11], this approximation only works for Skyrme fields which have a shell-like structure. Thus, for example, using the rational map (3.2) with $b = 7/\sqrt{5}$ gives a good approximation to the dodecahedral 7-Skyrmion, but the rational map ansatz breaks down for large $b$ and can not be used to describe well separated Skyrmions. Thus we need to turn to the instanton approximation to attempt to produce Skyrme fields which describe this process.

The upshot of the above discussion is that we now want to consider 7-instantons with symmetry $T_h$.

### 3.1 Tetrahedral deformations of the icosahedral instanton

We have described the general framework for the construction of symmetric ADHM data. What we seek now is the most general family of tetrahedral 7-instantons that contains the icosahedral 7-instanton. Denote by $\tilde{T}$ the binary tetrahedral group, i.e. the double cover in $SU(2)$ of the rotation group of the tetrahedron. To study tetrahedral deformations, we realize $\tilde{T}$ as a subgroup of $\tilde{G}$ generated by $i, j, k$ and the three-fold rotation $(1 + i + j + k)/2$. (We continue to take $\tilde{G}$ to be generated by the $U_i$ of (2.14).) Conjugation by this quaternion gives a $120^\circ$-rotation about the axis in the direction $(1, 1, 1)$ in $\mathbb{R}^3$. Then the
given representations of \( \tilde{G} \) yield representations of \( \tilde{T} \) and we can attempt to follow the procedure described above. It is preferable first to impose the inversion from the subgroup \( T_h \), however, since this forces the two diagonal blocks in \( M \) to be zero.

To explain this, we must first show in what sense the data (2.17) are inversion symmetric. For this we must again find compensating gauge transformations \( J_1 \) and \( J_2 \) such that

\[
\Delta(-x) = J_1 \Delta(x) J_2^{-1}.
\]

It is easy to see that the essentially unique choice for this, in the case of the ADHM data (2.17), is given by

\[
J_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathbb{I}_4 & 0 \\ 0 & 0 & -\mathbb{I}_3 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -\mathbb{I}_4 & 0 \\ 0 & \mathbb{I}_3 \end{pmatrix}.
\]

The reason for the simple form of these matrices is that the inversion \( x \mapsto -x \) is central and so must act by \( \pm \) the identity in any irreducible representation of \( \tilde{G} \).

Now any ADHM data that is inversion-symmetric in this sense must have the form

\[
\hat{M} = \begin{pmatrix} v & 0 \\ 0 & C \\ C^t & 0 \end{pmatrix}
\]

relative to the usual block-decomposition. The row-vector \( v \) and \( 4 \times 3 \) matrix \( C \) are constrained by having to be symmetric under \( \tilde{T} \), where this acts by restriction of the given \( \tilde{G} \)-representations. Since \( 2 = 2' \) and \( 3 = 3' \) as \( \tilde{T} \) representations, we have that \( 4' = 2 \otimes 2 = 1 \oplus 3 \) as \( \tilde{T} \)-representations. One may check that \( v \) is invariant iff it has the form \((\lambda_0, \lambda_1i, \lambda_1j, \lambda_1k)\) and that \( C \) is invariant if it has the form

\[
C = \begin{pmatrix} \lambda_2i & \lambda_2j & \lambda_2k \\ p & qk & rj \\ rk & p & qi \\ qj & ri & p \end{pmatrix}
\]

where \( \lambda_0, \lambda_1, \lambda_2, p, q, r \) are real numbers.

The next task is to impose the ADHM constraints; these yield the equations

\[
\text{Im}(v^t v + \bar{C}C^t) = 0, \quad \text{Im}(C^t C) = 0.
\]

It is straightforward to show that these reduce to the three equations

\[
\lambda_0 \lambda_1 = \lambda_2(p + q - r), \quad \lambda_1^2 = qr - p(q - r), \quad \lambda_2^2 = qr + p(q - r).
\]

Solving for the \( \lambda \)'s in terms of \( p, q, r \), we conclude that the general \( T_h \) invariant ADHM data are given by

\[
\hat{M} = \begin{pmatrix}
\lambda_0 & \lambda_1i & \lambda_1j & \lambda_1k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_2i & \lambda_2j & \lambda_2k \\
0 & 0 & 0 & 0 & p & qk & rj \\
0 & 0 & 0 & 0 & rk & p & qi \\
0 & 0 & 0 & 0 & qj & ri & p \\
\lambda_2i & p & rk & qj & 0 & 0 & 0 \\
\lambda_2j & qk & p & ri & 0 & 0 & 0 \\
\lambda_2k & rj & qi & p & 0 & 0 & 0
\end{pmatrix}
\]

(3.3)
where
\[\lambda_0 = (p + q - r) \sqrt{\frac{qr + p(q - r)}{qr - p(q - r)}}, \quad \lambda_1 = \sqrt{qr - p(q - r)}, \quad \lambda_2 = \sqrt{qr + p(q - r)}. \quad (3.4)\]

Here we have three free parameters, \(p, q, r\), but we require only a one-parameter family for our application. One of the free parameters is an overall scale factor, which corresponds to the multiplication of \(\hat{M}\) by a scale, and this can only be determined by computing the energy of the associated Skyrme field. We will therefore set this scale to one, as it can be easily reintroduced later. As \(p, q, r\) are homogeneous coordinates we can fix this scale by setting \(r = q - 1\).

To get some insight into the interpretation of the parameters, let us consider the special case \(p = 0\). Then from (3.4), \(\lambda_0 = (q - q - 1)\), \(\lambda_1 = \lambda_2 = 1\); denote the corresponding matrix (3.3) by \(\hat{M}(q)\). If \(q = \tau\), then \(\lambda_0 = 1\) and \(\hat{M}(\tau)\) coincides with the icosahedral matrix (2.17). If \(q\) is interpreted as the exponential of a ‘time parameter’ \(t\), then \(q\) runs from 0 to \(\infty\) as \(t\) runs from \(-\infty\) to \(\infty\), and time-reversal is the transformation \(q \to q^{-1}\). This transformation is equivalent to a \(90^\circ\) rotation about the \(x_1\) axis. Indeed if we denote by \(\hat{N}(q)\) the ADHM data obtained from \(\hat{M}(q)\) by replacing \(q\) by \(q^{-1}\), \(i\) by \(i\), \(j\) by \(k\) and \(k\) by \(-j\), then we have
\[\hat{N}(q) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_2 \end{pmatrix} \hat{M}(q) \begin{pmatrix} R_1 & 0 & 0 \\ 0 & 0 & R_2 \end{pmatrix}^{-1} \quad (3.5)\]
if
\[R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (3.6)\]

We conclude that the value \(q = \tau^{-1}\) is also a dodecahedral instanton (obtained from (2.17) by a \(90^\circ\) rotation) and that at \(q = 0\) we have an instanton symmetric under the symmetry group of the cube (since this group is generated by the tetrahedral group together with any \(90^\circ\) rotation about one of the coordinate axes).

This one-parameter family thus has all the symmetry properties expected of the seven Skyrmion scattering process. However it turns out that it does not have the correct asymptotic behaviour as \(q\) goes to 0 or \(\infty\). This was verified both by numerical work and asymptotic analysis. The problem can be traced to the fact that the first entry of the top row \((q - q^{-1})\) blows up as \(q \to 0, \infty\).

Therefore we try to find a more general family with the same symmetry properties but allow \(p \neq 0\) to improve the asymptotic behaviour. In particular we want time-reversal to continue to correspond to the replacement \(q \mapsto q^{-1}\). Then it can be seen that if we make the replacements
\[q \mapsto q^{-1}, p \mapsto -p, i \mapsto i, j \mapsto k, k \mapsto -j \quad (3.7)\]
which, because of (3.4), result in
\[\lambda_0 \mapsto -\lambda_0, \lambda_1 \mapsto \lambda_1, \lambda_2 \mapsto \lambda_2, \quad (3.8)\]
equivalent ADHM data are obtained. (The ‘compensating gauge transformation’ is as in (3.5) and (3.6).) Note that the fixed point set of the time reversal transformation, in other
words the midpoint of the scattering process, is given by \( q = 1, p = 0 \), which indeed has cubic symmetry as it should.

These arguments show that if \( p \) is any function of \( q \) with the properties

\[
p(q^{-1}) = -p(q), \quad p(\tau) = 0
\]

then the corresponding family of ADHM matrices will have the correct symmetry properties: the first condition gives that the data at \( q \) and at \( q^{-1} \) are equivalent up to a 90°-rotation; the second ensures that the data at \( \tau \) reduce to the icosahedral data (2.17).

We are now left with finding the variable \( p \) as a function of \( q \); this cannot be determined from symmetry arguments alone. A simple function satisfying (3.9) is given by

\[
p = -\frac{(q - q^{-1})((q - q^{-1})^2 - 1)}{(q - q^{-1})^4 + 1}.
\]

The numerator of \( p \) is the simplest function having the required zeros and symmetry properties, whereas at this stage the only fact we know about the denominator is that it must be a symmetric function of \((q - q^{-1})\). However, as we shall now see, the form of the denominator is highly constrained by examining the asymptotic limit of the scattering process, where all seven instantons are well separated, and this leads naturally to the given solution.

The asymptotic out state corresponds to the limit \( q \to \infty \), and in this limit the leading order behaviour of \( p \) is

\[
p = -q^{-1} + O(q^{-5}).
\]

Hence in this limit we find that

\[
\lambda_0 = \frac{1}{\sqrt{2}} + O(q^{-2}), \quad \lambda_1 = \sqrt{2} + O(q^{-2}), \quad \lambda_2 = q^{-1} + O(q^{-3}).
\]

Neglecting negative powers of \( q \) we thus arrive at the asymptotic ADHM data

\[
\hat{M}_\infty = \begin{pmatrix}
1/\sqrt{2} & \sqrt{2}i & \sqrt{2}j & \sqrt{2}k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & gk & 0 & 0 \\
0 & 0 & 0 & 0 & qi & 0 & 0 \\
0 & 0 & gk & 0 & 0 & 0 & 0 \\
0 & 0 & qi & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

After a gauge transformation by the matrix

\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\]
we obtain
\[
\begin{pmatrix}
1 & 0 \\
0 & T
\end{pmatrix}
\hat{M}_\infty T^{-1} = \begin{pmatrix}
1 & i & j & k & k & i & j \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -qk & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -qi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -qj & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & qj & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & qk & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & qi
\end{pmatrix}.
\]

This matrix is of the form identified by Christ, Stanton and Weinberg[8] as representing well separated instantons. The leading order terms lie on the diagonal of the square part of the matrix and determine the instanton positions, giving one instanton at the origin and the other six on the Cartesian axes at a distance \( q \) from the origin. The terms of next order all lie on the top row of the matrix and these give the scales and \( SU(2) \) orientations of the instantons. The fact that all the entries on the top row have unit length means that each instanton has scale one. Since the first entry in the top row is one, then on computing the holonomy the Skyrmion at the origin will be in standard orientation, whereas, for example, the fact that the second and sixth entries in the top row are \( i \), means that the Skyrmions located on the \( x_3 \)-axis have an orientation which is obtained from the standard one by rotating the Skyrmion by 180° around the \( x_1 \)-axis. These kinds of configurations, that is three collinear Skyrmions such that the outer Skyrmions are rotated by 180° about a line perpendicular to the line joining the inner Skyrmion, are known to give attractive initial conditions for full field simulations[5]. Thus we see that our ADHM data has the correct asymptotic properties to produce the configuration which we require.

Now we address the possible freedom in the choice (3.10) that we have made. The main effect of changing the denominator in (3.10) is to alter the scale of the instanton at the origin in the limit in which the other six are far from it. The requirement that this scale is finite as \( q \to \infty \) determines that the leading term in the denominator of \( p \) must be \((q - q^{-1})^4\) with coefficient one. Only even powers of \((q - q^{-1})\) are allowed by symmetry and it can be shown that the coefficient of \((q - q^{-1})^2\) must be zero if the scale of the instanton at the origin is to be not only finite but equal in value to the scale of the instantons which are moving along the Cartesian axes. There remains the freedom to change the value of the constant in the denominator of (3.10) but clearly this has little effect since it is only relevant for small values of \((q - q^{-1})\) and the numerator contains \((q - q^{-1})\) as a factor.

Using this ADHM data for increasing values of \( q \) we compute Skyrme fields whose baryon density isosurfaces are shown in Figure 1. The various values of \( q \) corresponding to each figure are given in the table below.

<table>
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<th>Fig</th>
<th>1</th>
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<th>3</th>
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<tbody>
<tr>
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<td>2.00</td>
<td>2.10</td>
<td>2.25</td>
<td>2.50</td>
</tr>
</tbody>
</table>

Table 1. Parameter values for the scattering shown in Figure 1.

From Figure 1 we see that indeed our family of instantons describes the sought after scattering process. Figure 1.1 clearly shows the six Skyrmions on the Cartesian axes and
a Skyrmion at the origin. As the Skyrmions approach, Figure 1.2, the one at the origin shrinks until it disappears completely, Figure 1.3. The Skyrmions then merge until the dodecahedron is formed, Figure 1.5, after which the configuration twists until it turns into a cube, Figure 1.8. This process is then reversed but accompanied by a 90° rotation around the \( x_1 \)-axis, so that for example the dual dodecahedron is formed, Figure 1.11, and the Skyrmions finally separate again along the Cartesian axes, Figure 1.15.

An important difference between monopoles and Skyrmions is that monopoles are BPS solitons, and therefore all configurations of the same charge have equal energy, whereas for Skyrmions this is certainly not the case and the potential energy of different configurations is an important factor in considering the dynamics. We have not computed the scale factor of each configuration to minimize the energy of our Skyrme fields since, although this could be done if required, it would involve a substantial amount of computing time. However, we have ensured that the scale factor is the order of unity for all \( q \), so that we get an accurate representation of the baryon density isosurfaces. Qualitatively we know that for all the family of Skyrme fields the energy per Skyrmion is less than it is for seven well separated Skyrmions, and that the minimum energy configurations are the two dodecahedra. From numerical simulations\[6\] it is known that the energy of the cubic 7-Skyrmion is above that of the dodecahedron but it is still substantially less than that of seven well separated Skyrmions. Thus the true dynamical evolution depends upon the initial speeds of the incoming Skyrmions, together with the amount of energy lost through radiation as the process evolves. However, it is expected that if the incoming speeds are great enough then the whole scattering process displayed in Figure 1 would take place. Radiation effects will mean that for most speeds the incoming Skyrmions will eventually get trapped at one of the dodecahedra, and perhaps if the Skyrmions are initially static then only the first portion of the scattering process will occur and the cube may never be formed, but this depends upon the amount of radiation generated. By performing full field simulations, using the numerical code described in [5, 6], with initial conditions given by the instanton generated Skyrme field, it has been verified that the true dynamical evolution does follow the sequence described above and the family of instanton generated Skyrme fields provides an accurate approximation to the Skyrmion scattering process.

In current approaches to the quantization of Skyrmions a first step is to examine the vibrational modes of the minimal energy Skyrmion at each charge\[3\]. Thus for charge seven it is the vibrational modes of the dodecahedral Skyrmion which need to be studied and clearly one mode is the tetrahedral deformation we have displayed. Thus if the amplitude of the deformation is sufficient then one of the important vibrational modes will be the one considered here where the dodecahedron deforms to its dual via a cube.

4 Conclusion

We have used the ADHM construction to present a charge seven instanton with icosahedral symmetry whose holonomy generates a Skyrme field which approximates the minimal energy dodecahedral 7-Skyrmion. Furthermore, by imposing tetrahedral symmetry we have found a family of ADHM data which we used to generate Skyrme fields that model a seven Skyrmion scattering process that results in the formation of the dodecahedral 7-Skyrmion. There are a number of other highly symmetric Skyrmions, such as the icosahedrally symmetric charge seventeen Skyrmion which resembles a buckyball\[11\], and the methods used here could also be applied to construct the ADHM data of the predicted correspond-
ing instantons. In particular it would be interesting if the instanton approach led to an understanding of the formation of the buckyball 17-Skyrmion from individual Skyrmions.

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Figure Caption

Fig. 1. Baryon density isosurfaces for a family of charge seven Skyrme fields obtained from 7-instantons with tetrahedral symmetry.
References


