Dirac Particles in Twisted Tubes

P. Ouyang, V. Mohta, and R.L. Jaffe

Center for Theoretical Physics and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Abstract

We consider the dynamics of a relativistic Dirac particle constrained to move in the interior of a twisted tube by confining boundary conditions, in the approximation that the curvature of the tube is small and slowly varying. In contrast with the nonrelativistic theory, which predicts that a particle’s spin does not change as the particle propagates along the tube, we find that the angular momentum eigenstates of a relativistic spin-$\frac{1}{2}$ particle may behave nontrivially. For example, a particle with its angular momentum initially polarized in the direction of propagation may acquire a nonzero component of angular momentum in the opposite direction on turning through $2\pi$ radians. Also, the usual nonrelativistic effective potential acquires an additional factor in the relativistic theory.

I. INTRODUCTION

Twisted tubes, in which particles are confined by an external potential, have proven to be excellent settings for the study of quantum mechanics in curved geometries. They have been studied extensively in the context of nonrelativistic quantum mechanics, with particular focus on the case of small and slowly varying curvature [1]. One important example is the limit of a very narrow tube, where the “confining potential” constrains a particle to move on a curve, just as a physical bead might be attached to a wire. Subsequent authors have considered the natural extension to a particle confined to a small neighborhood of an arbitrary Riemannian manifold embedded in a higher-dimensional Euclidean space and have applied their formalism to the energy spectra of molecules [2]. For a recent survey of the dynamics of a particle in various confining potentials in a loosely interpreted “tube” see Ref. [3]. However, analysis has been largely limited to nonrelativistic quantum mechanics, in which spin does not affect the dynamics; the relativistic theory has been thoroughly investigated only for confinement and quantization schemes other than the confining potential approach [4]. Here we present for the first time a treatment of a relativistic Dirac particle in the confining potential approach using confining boundary conditions.
The phenomena associated with the propagation of a relativistic fermion within a tube contrast with the simpler and well-known case of light propagating in an optical fiber, studied in the mid-1980’s by Chiao and Wu [5]. They assumed the fiber to have constant circular cross-section and small and smoothly varying curvature. In this approximation, they argued that the two states of circular polarization (helicity $\pm 1$) propagate along the fiber without mixing with one another and accumulate opposite Berry phases proportional to the integral of the torsion along the fiber. It is instructive to re-examine their argument that the helicity eigenstates do not mix [5]. First they use the adiabatic theorem to argue that modes with different transverse waveforms do not mix as they propagate through a smooth fiber. The lowest energy level is doubly degenerate, with total angular momentum component along the axis of the fiber $\hat{J} \cdot \hat{t} = \pm 1$. The longitudinal mode, $\hat{J} \cdot \hat{t} = 0$, is excluded by gauge invariance. Following Chiao and Wu we refer to these modes somewhat loosely as photon states of helicity $\pm 1$. In the limit of a thin fiber, the only surviving terms in the Hamiltonian affecting helicity are of the form $\vec{\kappa} \cdot \vec{S}$, where the “vector curvature”, $\vec{\kappa}$, is a vector in the direction of the principal normal to the fiber and of length $1/\kappa$, is the instantaneous radius of curvature of the fiber, and $\vec{S}$ is the photon’s spin. Since $\vec{\kappa} \cdot \vec{S}$ is a vector operator, the Wigner-Eckart theorem allows a change in helicity only by $\Delta h = \pm 1$. $\Delta h = \pm 2$ is forbidden, so the helicity states propagate without mixing, even though they are degenerate – a consequence of gauge invariance.

The point of departure for our paper is the observation that Chiao and Wu’s argument does not apply to a Dirac fermion, which has doubly degenerate modes with helicity $\hat{J} \cdot \hat{t} = \pm (n + \frac{1}{2})$. Once again we refer to these modes somewhat loosely as fermion states of helicity $\pm (n + \frac{1}{2})$. In particular, the ground state modes have $n = 0$, so $\hat{J} \cdot \hat{t} = \pm \frac{1}{2}$. The Wigner-Eckart theorem does not forbid transitions between these states because the eigenvalues of $\hat{J} \cdot \hat{t}$ differ by 1. We cannot exclude the possibility that the degenerate angular momentum states will mix as they propagate on the basis of symmetry. Therefore the Dirac equation demands more careful analysis.

As in Ref. [1], we consider a particle confined to the interior of a tube with circular cross-section. We work in the rest frame of the tube. The tube may be constructed by transporting a circular disc such that the center of the disc traces a smooth curve $\vec{X}(s)$ and the tangent to the curve is normal to the disc for all values of the arc-length parameter $s$. The region swept out by the disc is the tube. We model the potential that confines the particle to the tube by means of a boundary condition. The boundary condition must guarantee that the normal component of the probability current $\vec{j} = \psi^\dagger \alpha \psi$ vanishes on the surface of the tube. In the nonrelativistic problem one can simply impose Dirichlet boundary conditions. However, in the Dirac theory, the requirement that all components of the wavefunction vanish on the boundary is too stringent, for then in general the Dirac wavefunction will vanish everywhere. A less severe condition that will still confine probability is

$$-i \vec{\gamma} \cdot \hat{n} \psi = \psi$$

where $\hat{n}$ is the unit normal (in the outward sense) to the boundary of the tube [7]. That eq. (1) guarantees probability confinement follows from the commutation relations for the $\gamma$-matrices and the antihermiticity of $\vec{\gamma}$. Eq. (1) reduces to the usual Dirichlet boundary condition on the “large” components of the Dirac spinor in the nonrelativistic limit.

It is a general feature of the Dirac equation that spin and orbital angular momentum
are not separately conserved. It is not surprising, therefore, that the boundary condition, eq. (1) does not commute with the spin operator. The result is the nontrivial transport of angular momentum states that we will calculate. We could have chosen other boundary conditions, for example the chiral generalization of eq. (1),

\[-i\gamma^{5} \cdot \hat{n}\psi = e^{i\gamma^{5} \theta} \psi\]  

(2)

parameterized by the chiral angle $\theta$, with a slightly different phenomenology. We will explore some of the alternatives in the final section of this paper.

In Section II we establish coordinates in the interior of the tube that will be convenient for our purposes. Section III is a construction of the dynamical equations of our system in the parallel transport frame. We write the dynamical equations as those for propagation in a straight tube with an interaction. In Section IV we calculate the necessary basis states for a Dirac particle in a straight tube. In Section V we specialize to the approximation that the curvature is small and slowly varying. This approximation appears to capture most of the interesting physics. We then expand the Hamiltonian to lowest nontrivial order in small parameters. Using the resulting Hamiltonian, we compute in Section VI the relevant dynamical equations. Section VII contains our conclusions and also briefly discusses possibilities for boundary conditions other than eq. (1).

II. ELEMENTARY FORMALISM

We construct a coordinate system for the tube described in Section I by erecting a right-handed orthonormal frame at each point on the curve. We parameterize the curve as $\hat{X}(s)$ where $s$ is the arc-length parameter. Since the tangent to the curve is always normal to the cross-sectional disc that sweeps out the tube, a natural choice of frame is $\{\hat{e}_{1}, \hat{e}_{2}, \hat{e}_{3}\}$ where $\hat{e}_{1}$ and $\hat{e}_{2}$ are orthogonal unit vectors in the plane of the disc and $\hat{e}_{3}$ is the unit tangent to the disc in the direction of increasing $s$. For clarity we adopt a summation convention where Greek indices, $\mu$, $\nu$, etc., are summed over 1, 2, and 3, whereas Latin indices, $i$, $j$, etc., are summed only over the “transverse” directions 1 and 2. To completely define the frame, we must specify the $s$-dependence of $\hat{e}_{1}$ and $\hat{e}_{2}$. We require that the frame rotates smoothly as the disc sweeps out the tube. That is,

\[\frac{d\hat{e}_{\mu}}{ds} = \vec{\omega} \times \hat{e}_{\mu}\]  

(3)

where $\vec{\omega}(s)$ is a smoothly varying instantaneous “angular velocity” associated with the rotation of the frame.

Since the curve uniquely determines $\hat{e}_{3}(s)$, it determines $\vec{\omega}(s)$ up to the addition of a constant multiple of $\hat{e}_{3}(s)$, corresponding to an additional rotation of the transverse axes about the tangent. For simplicity, we adopt the “parallel transport” frame, in which $\vec{\omega}$ has no component along the tangent. There is a simple visual model of the parallel transport frame. Imagine a rigid wire with a disc attached to it by a frictionless sleeve that keeps the disc normal to the sleeve. Without friction all the torques act in the plane of the disc so the angular momentum about the tangent direction is conserved. If the disc has no angular velocity about $\hat{e}_{3}$ initially, it will not develop any. In this case, vectors drawn on the disc
are said to execute parallel transport along the wire. A simple calculation shows that the instantaneous angular velocity of this frame is

\[ \vec{\omega} = \hat{e}_3 \times \frac{d\hat{e}_3}{ds}. \] (4)

A new coordinate system for points inside the tube may now be specified in terms of the parallel transport frame by

\[ \vec{x}(s, \xi^1, \xi^2) = \vec{X}(s) + \xi^1 \hat{e}_1 + \xi^2 \hat{e}_2. \] (5)

In this coordinate system, the spatial metric tensor is diagonal

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & (1 - \vec{\kappa} \cdot \vec{\xi})^2 \end{pmatrix} \] (6)

where the vector curvature \( \vec{\kappa} \) of \( \vec{X}(s) \) is defined as

\[ \vec{\kappa} \equiv \frac{d\hat{e}_3}{ds} \equiv \kappa^1 \hat{e}_1 + \kappa^2 \hat{e}_2. \] (7)

Comparison with the definition of \( \vec{\omega} \) shows

\[ \vec{\omega} = \hat{e}_3 \times \vec{\kappa} \] or
\[ \omega^1 = \kappa^1 \quad \text{and} \quad \omega^2 = -\kappa^2. \] (8)

The diagonal metric tensor yields a simple gradient operator:

\[ \vec{\nabla} = \hat{e}_1 \frac{\partial}{\partial \xi^1} + \hat{e}_2 \frac{\partial}{\partial \xi^2} + \hat{e}_3 \frac{1}{1 - \vec{\kappa} \cdot \vec{\xi}} \frac{\partial}{\partial s}. \] (9)

The volume element for integration in these coordinates is \( \sqrt{\det g_{jk}} dsd^2\xi = (1 - \vec{\kappa} \cdot \vec{\xi}) dsd^2\xi. \)

Since the tube intersects itself if \( \vec{\kappa} \cdot \vec{\xi} \) equals one, we require that \( \kappa R < 1 \) where \( R \) is the radius of the disc.

An important subtlety is that while the choice of the parallel transport frame is always valid locally, it is not always possible to adopt it globally. In particular, if the curve is closed, parallel transport of the frame through one circuit of the curve need not return the frame to its original orientation. This geometric feature manifests itself in classical physics as the well-known Hannay’s angle [8,1]. For the purposes of this paper, we assume that our tube continues indefinitely without intersecting itself.

### III. DIRAC EQUATION IN THE PARALLEL TRANSPORT FRAME

In free space, with units such that \( \hbar = c = 1 \), the Hamiltonian in the Dirac equation \( \mathcal{H}\Psi = E\Psi \) is

\[ \mathcal{H} = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m + V). \] (10)
We define $\alpha$ and $\beta$ as usual by $\alpha^i = \gamma^0 \gamma^i$ and $\beta = \gamma^0$; in the chiral basis that we employ,
\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11} \]

In the coordinate system defined in Section II, the Dirac Hamiltonian takes the form
\[ H = -i \vec{\alpha} \cdot \hat{e}_j \frac{\partial}{\partial \xi^j} - \frac{i}{\Delta} \vec{\alpha} \cdot \hat{e}_3 \frac{\partial}{\partial s} + \beta m \tag{12} \]
where $\Delta = 1 - \vec{\kappa} \cdot \vec{\xi}$. We normalize our wavefunctions $\Psi$ so that
\[ \int d^3x \Psi^\dagger \Psi = \int ds d^2 \xi \Delta \Psi^\dagger \Psi = 1. \tag{13} \]
The factor of $\Delta$ in the integration measure complicates calculations and questions of hermiticity. Therefore we define new wavefunctions $\psi$ by
\[ \psi = \Delta \frac{1}{2} \Psi \tag{14} \]
which have the normalization $\int ds d^2 \xi \psi^\dagger \psi = 1$. The original Dirac equation can be recast in terms of a Hamiltonian operator $\tilde{H}$, defined by $\tilde{H} = \Delta \frac{1}{2} \mathcal{H} \Delta^{-\frac{1}{2}}$, which acts on the wavefunctions $\psi$,
\[ \tilde{H} = -i \vec{\alpha} \cdot \hat{e}_j \frac{\partial}{\partial \xi^j} - \frac{i}{2 \Delta} \vec{\alpha} \cdot \vec{\kappa} - \frac{i}{\Delta} \vec{\alpha} \cdot \hat{e}_3 \left( \frac{\partial}{\partial s} + \frac{1}{2 \Delta} \frac{\partial (\vec{\kappa} \cdot \vec{\xi})}{\partial s} \right) + \beta m . \tag{15} \]

This Hamiltonian is difficult to calculate with because the matrices $\vec{\alpha} \cdot \hat{e}_\mu$ are $s$-dependent. That is, the momentum operators are expressed with respect to the rotating coordinate frame while the spin operators are expressed with respect to some fixed rectilinear frame. To remedy this problem, we change the spin basis. We define a unitary transformation $\Omega$ by
\[ \Omega(s) \vec{\alpha} \cdot \hat{e}_\mu(s) \Omega^\dagger(s) = \alpha_\mu \tag{16} \]
for all $s$. Here we have made the $s$ dependence of the unit vectors $\hat{e}_\mu$ explicit.

Differentiating the above condition with respect to $s$ and using eq. (3) for $d\hat{e}_\mu/ds$, we find
\[ \frac{d\Omega}{ds} = \frac{i}{2} \vec{\Sigma} \cdot \vec{\omega} \Omega \tag{17} \]
where $\Sigma^i$ are the $4 \times 4$ Dirac spin matrices. Therefore,
\[ \Omega(s) = \mathcal{P} \exp \left[ \frac{i}{2} \int \! ds' \vec{\Sigma} \cdot \vec{\omega}(s') \right] \tag{18} \]
where $\mathcal{P}$ denotes path-ordering along the tube.

To take advantage of this transformation, we once again define new wavefunctions,
\[ \chi(s, \vec{\xi}) = \Omega(s) \psi(s, \vec{\xi}) \tag{19} \]
and a new Dirac Hamiltonian, $H$,

$$H = \Omega \bar{H} \Omega^\dagger.$$  

(20)

The form of eq. (18) indicates that the transformation $\Omega(s)$ just represents a change of spin basis to the spinors representing spin up and down in the $\hat{e}_3(s)$ direction. Although this transformation replaces the $\alpha$-matrices as intended, it generates a “gauge term” in $H$,

$$-i \frac{\Delta}{2} \bar{\alpha} \cdot \hat{e}_3 \frac{\partial \Omega^\dagger}{\partial s} = -i \frac{\Delta}{2} \alpha^3 \left( \frac{1}{2} \bar{\Sigma} \cdot \vec{\omega} \right).$$

(21)

Simplifying $\alpha^3 \bar{\Sigma} \cdot \vec{\omega}$ yields $-i \alpha \cdot \vec{\kappa}$. Thus, this gauge term cancels the second term in eq. (15) giving the following relatively simple Hamiltonian for $\chi$:

$$H = -i \alpha^j \frac{\partial}{\partial \xi^j} - i \alpha^3 \frac{\partial}{\partial s} + \beta m - i \alpha^3 \left( \frac{\vec{\kappa} \cdot \vec{\xi}}{\Delta} \frac{\partial}{\partial s} + \frac{1}{2 \Delta} \frac{\partial (\vec{\kappa} \cdot \vec{\xi})}{\partial s} \right).$$

(22)

The transformations we have performed reduce our problem to the study of this perturbed Dirac Hamiltonian in a right circular cylinder, $(\xi^1)^2 + (\xi^2)^2 \leq R^2$, $-\infty < s < \infty$, with respect to a fixed rectilinear frame. The price of this simplification is the variety of curvature dependent terms in eq. (22).

IV. DECOMPOSITION INTO TRANSVERSE EIGENSTATES

We look for eigenstates of the Hamiltonian in eq. (22) of the form

$$\chi(s, \vec{\xi}) = \sum_{\{\rho\}} c_{\{\rho\}}(s) \eta_{\{\rho\}}(\vec{\xi})$$

(23)

where the $\{\eta_{\{\rho\}}\}$ are eigenstates of the transverse Hamiltonian,

$$H_\perp = -i \alpha^j \frac{\partial}{\partial \xi^j} + \beta m$$

(24)

satisfying the boundary condition

$$-i \gamma^j \cdot \hat{r} \eta = \eta|_{r=R}.$$  

(25)

Here we have introduced plane polar coordinates, $r = |\vec{\xi}|$ and $\varphi = \tan^{-1} \xi^2/\xi^1$. The labels $\{\rho\}$ denote the quantum numbers necessary to specify the eigenstates of $H_\perp$ completely.

$H_\perp$ is just the free Dirac Hamiltonian in one lower dimension. $S_3$ and $L_3$, the tangential components of the spin and orbital angular momentum respectively, do not separately commute with $H_\perp$ but $J_3 = L_3 + S_3$ does. $J_3$ also commutes with the operator $\gamma^j \cdot \hat{r}$ in the boundary condition. In addition, the operator $\gamma^0 S_3$ commutes with $H_\perp$, $J_3$, and $\gamma^j \cdot \hat{r}$. We define the “transverse energy” as the eigenvalue $\lambda$ of $H_\perp$. An eigenstate of $H_\perp$ with eigenvalue $\lambda$ may thus be chosen to be an eigenstate of the helicity operator $J_3$ with eigenvalue $\nu$ ($\nu = n - 1/2$ for some integer $n$) and of the operator $\gamma^0 S_3$ with eigenvalue $\varepsilon$ ($\varepsilon = \pm \frac{1}{2}$). For simplicity, we use the notation $\{\nu^\pm, \lambda\}$ when specifying these quantum numbers.
\( H_\perp \) has the usual Dirac charge conjugation symmetry, so each eigenstate with eigenvalue \(+|\lambda|\) is paired with another with eigenvalue \(-|\lambda|\). Since \( \alpha^3 \) anticommutes with \( H_\perp \) and \( \gamma^0 S_3 \) and commutes with \( J_3 \) and \( \vec{\gamma} \cdot \vec{r} \), \( \alpha^3 \eta_{\nu^\pm,\lambda} = \eta_{\nu^\mp,\lambda} \). The transformation that sends \( J_3 \) to \(-J_3\) commutes with \( H_\perp \) and \( \vec{\gamma} \cdot \vec{r} \) and anticommutes with \( \gamma^0 S_3 \). Thus, the states with \( \nu^+ \) have the same eigenvalue \( \lambda \) as states with \(-\nu^-\). For each value of \( \nu \) and \( \varepsilon \), there are towers of radial excitations, one for positive \( \lambda \) and one for negative \( \lambda \), obtained by adding more nodes to the radial wavefunction. A summary of the eigenstates of \( H_\perp \) for small \( \nu \) and \( \lambda \) is given in Fig. 1.

\[ \text{FIG. 1. Summary of the low-lying transverse energy eigenstates in the extreme relativistic limit, } m \to 0. \text{ Only positive } \lambda \text{ are shown. Each level is twofold degenerate corresponding to } \nu^+ \leftrightarrow -\nu^-\text{. Radial excitations are displayed with dashed lines to correlate with Fig. } 3. \]

With these general considerations in mind, we construct the explicit representation of the wavefunction \( \eta_{\nu^\pm,\lambda} \) for positive \( \lambda \):

\[
\eta_{\nu^\pm,\lambda}(r, \varphi) = N_{\nu^\pm,\lambda} \begin{pmatrix} \zeta^{\pm 1} J_{n-1}(qr) e^{i(n-1)\varphi} \\ -i J_n(qr) e^{in\varphi} \\ \pm \zeta^{\pm 1} J_{n-1}(qr) e^{i(n-1)\varphi} \\ \pm i J_n(qr) e^{in\varphi} \end{pmatrix}
\] (26)

where \( n \) is an integer with \( \nu = n - \frac{1}{2} \), \( J_n \) is a Bessel function of the first kind, \( q = +\sqrt{\lambda^2 - m^2} \), \( \zeta \equiv \frac{m + \sqrt{q^2 + m^2}}{q} \), and \( q \) satisfies the following equation obtained from the boundary condition:

\[
J_n(qR) = \pm \zeta^{\pm 1} J_{n-1}(qR) \ .
\] (27)

The normalization \( N \) is fixed up to a phase by the condition

\[
\int d^2 \xi \eta^\dagger \eta = 1 .
\] (28)

The unusual form of the spinors in the nonrelativistic limit is due to our choice of the chiral representation of the Dirac matrices (\( \gamma_5 = \text{diag}(-1, -1, 1, 1) \)). Transforming the eigenstates to the Dirac basis (\( \gamma_0 = \text{diag}(1, 1, -1, -1) \)), we find that only one of the two
upper components and one of the two lower components of each wavefunction are nonzero. In the nonrelativistic limit, for the states with positive $\lambda$, the lower components in the Dirac basis tend to zero. These are eigenstates of $S_3$ and $L_3$, in agreement with the fact that $\gamma^0 S_3$ is just $S_3$ on the upper components in the Dirac basis. We use the chiral basis because the wavefunctions can be expressed more compactly away from the nonrelativistic limit and because computation of matrix elements is simpler.

For a straight tube, the energy eigenstates with helicity $\nu$ and longitudinal momentum $k$ are obtained from the ansatz, eq. (23):

\[ \chi(s, \vec{\xi}) = \exp ik s \left( \eta_{\nu^+, \lambda}(\vec{\xi}) + \frac{k \text{sgn}(\lambda)}{\sqrt{\lambda^2 + k^2 + |\lambda|}} \eta_{\nu^-, -\lambda}(\vec{\xi}) \right). \] (29)

In a twisting tube, the energy eigenstates could be superpositions of several $\{\eta_{\nu^\pm, \lambda}\}$ with nontrivial $\{c_{\nu^\pm, \lambda}(s)\}$ determined by eq. (22).

In later sections, we will study the propagation and mixing of the ground states of the transverse Hamiltonian, i.e., those with the smallest possible value ($\lambda_0$) for $|\lambda|$. For concreteness, we present the explicit form of the wavefunctions for this fourfold degenerate set of states:

\begin{align*}
\eta_1 &\equiv \eta_{\frac{1}{2}^+, \lambda_0} = N \left( \begin{array}{c}
\zeta J_0(qr) \\
-iJ_1(qr)e^{i\varphi}
\end{array} \right) \\
\eta_2 &\equiv \eta_{-\frac{1}{2}^-, \lambda_0} = N \left( \begin{array}{c}
-iJ_1(qr)e^{-i\varphi} \\
\zeta J_0(qr)
\end{array} \right) \\
\eta_3 &\equiv \eta_{\frac{1}{2}^-, -\lambda_0} = N \left( \begin{array}{c}
\zeta J_0(qr) \\
-iJ_1(qr)e^{i\varphi}
\end{array} \right) \\
\eta_4 &\equiv \eta_{-\frac{1}{2}^+, -\lambda_0} = N \left( \begin{array}{c}
-iJ_1(qr)e^{-i\varphi} \\
\zeta J_0(qr)
\end{array} \right)
\end{align*}

where we have moved some of the factors in the representation in eq. (26) into the normalization for later convenience and used $J_{-n} = (-1)^n J_n$. The eigenvalue condition

\[ J_0(qR) = \frac{1}{\zeta} J_1(qR) \] (31)

determines $\lambda_0$, and the normalization condition eq. (28) then fixes $N$:

\[ \frac{1}{N^2} = 2\pi R^2 \left( \frac{(\zeta^2 + 1)^2}{\zeta^2} - \frac{2}{\zeta q R} \right) J_1^2(qR). \] (32)

In the nonrelativistic limit ($m \gg q$), eq. (31) reduces to $J_0(qR) = 0$, a familiar result, and in the ultrarelativistic limit ($m \rightarrow 0$) it becomes $J_0(qR) = J_1(qR)$.

V. ADIABATIC CONDITION

In addition to assuming that the vector curvature, $\vec{\kappa}$, is a smooth function of $s$, we assume that it is nonzero only in a finite region. Thus the tube consists of two infinitely long straight regions joined by a curved region of finite length. Our goal is to propagate the
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unperturbed solutions to the Dirac equation, labelled by their longitudinal momentum, $k$, and their transverse quantum numbers, $\{\nu, \lambda\}$, through the region of nonzero curvature. We expect both scattering states ($k$ real) and bound states ($k$ imaginary). By the “transverse quantum numbers” of a tube eigenstate, we mean the quantum numbers associated with the first term of its representation in the form given by eq. (29).

We restrict our analysis to the case of adiabatic propagation in a mildly curved tube that dominates the work in this field. This amounts to the conditions

\[ |kR| \ll 1 \]
\[ \left| \frac{\partial \kappa^j}{\partial s} R^2 \right| \ll 1. \]  

(33)

In this approximation to the nonrelativistic problem, states with different transverse waveforms do not mix [9]. Because of the analogy between time development with a slowly varying $V(t)$ and $s$-development with a slowly varying curvature, this is known as the “adiabatic approximation”. The generalization of the adiabatic approximation to the Dirac case is that only states with the same values of $|\lambda|$ can mix when propagating in a smooth enough tube. For example, the four states with $|\nu| = \frac{1}{2}$ given explicitly in eq. (30) can mix adiabatically. Since this subspace includes states of different $\vec{J} \cdot \hat{e}_3$, interesting angular momentum transport is still possible in the adiabatic limit.

For simplicity, we consider the case of small longitudinal momentum. We assume that the square of the longitudinal momentum of a state under consideration is much smaller than the difference between the square of its transverse energy and that of other states with differing transverse energy. For a particular state with a set of transverse quantum numbers $\rho$, this amounts to the condition

\[ |kR| \ll d_\rho \]  

(34)

where $d_\rho = \min_{\rho'} \sqrt{|(q_{\rho'} R)^2 - (q_{\rho} R)^2|}$ and $\rho'$ runs over all sets of transverse quantum numbers such that $q_{\rho'} \neq q_\rho$.

For a physical tube, these conditions are easily satisfied. For a fixed mass $m$ and a given curve with smoothly varying vector curvature $\kappa(s)$ which is nonvanishing only in a finite region, we can always choose a nonzero $R$ small enough to satisfy each of the above conditions. There are, however, some subtleties associated with this condition. For a fixed tube radius $R$, making $m$ arbitrarily large makes $d_\rho$ arbitrarily small for most $\rho$. The additional degeneracies that develop as $mR \rightarrow \infty$ originate in the decoupling of spin and orbital angular momentum. Thus, pairs of states with $L_3 = 1$ and $S_3 = \pm \frac{1}{2}$, which are split by relativistic effects, become degenerate as $mR \rightarrow \infty$ as shown in Fig. (2). The cases $\rho = \{\frac{1}{2}^+, \lambda\}$ or $\{-\frac{1}{2}^-, \lambda\}$ are an exception. These states, which have $L_3 = 0$ and $S_3 = \pm \frac{1}{2}$ are degenerate for all $mR$, and no new degeneracies arise as $mR \rightarrow \infty$. Again, see Fig. (2).

Of course, this is not problematic because we know exactly how spin and orbital angular momentum eigenstates behave in the nonrelativistic limit: the Dirac problem reduces to the scalar problem solved, for example, in Ref. [1], accompanied by a dynamically trivial spin label.

Each of the conditions in eqs. (33) and (34) has a small parameter associated with it. We let $\delta$ denote the largest of these parameters. To simplify the analysis of the relative
FIG. 2. Dependence of \( qR \) on \( mR \) for the five lowest levels in Fig. 1

importance of terms in the Hamiltonian, we introduce scaled coordinates \( u^j \equiv \xi^j/R \) and multiply the Dirac equation by \( R \) to obtain \( H' \chi = E' \chi \) where \( H' = RH \) and \( E' = RE \) are the scaled Hamiltonian and energy respectively. From eq. (22), we obtain

\[
H' = -i\alpha^j \frac{\partial}{\partial u^j} + \beta mR - i\alpha^3 R \left( \frac{R \vec{\kappa} \cdot \vec{u}}{\Delta} \frac{\partial}{\partial s} + \frac{R}{2\Delta^2} \frac{\partial}{\partial s} \left( R \vec{\kappa} \cdot \vec{u} \right) \right) + O(\delta^3). \tag{35}
\]

So far, this equation is exact. The first two terms are zeroth order in \( \delta \), the third term is first order in \( \delta \), and the last two terms are second order in \( \delta \). To second order in \( \delta \), we can replace \( \Delta^{-1} \) by 1:

\[
H' = -i\alpha^j \frac{\partial}{\partial u^j} + \beta mR - i\alpha^3 \left( (1 + R \vec{\kappa} \cdot \vec{u}) R \frac{\partial}{\partial s} + \frac{R}{2} \frac{\partial}{\partial s} \left( R \vec{\kappa} \cdot \vec{u} \right) \right) + O(\delta^3). \tag{36}
\]

For later use, we also compute \( H'^2 \) to second order in \( \delta \):

\[
H'^2 = -\nabla_u^2 + R^2 m^2 - R^2 \frac{\partial^2}{\partial s^2} - iR^2 \vec{\Sigma} \cdot \vec{\omega} \frac{\partial}{\partial s} - \frac{R^2}{2} \frac{\partial}{\partial s} \left( \vec{\Sigma} \cdot \vec{\omega} \right) + O(\delta^3). \tag{37}
\]

Here \( \Sigma \) is the Dirac spin matrix, \( \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \).

Having systematically expanded the Hamiltonian to second order in small quantities, we may now return to unscaled coordinates. The adiabatic condition implies that the effective potential mixes two transverse eigenstates \( \eta_\rho \) and \( \eta_\rho' \) to second order in \( \delta \) only if \( |\lambda_\rho| = |\lambda_\rho'| \), or equivalently if \( q_\rho = q_{\rho'} \). Thus, to second order in \( \delta \), the mixing amplitudes outside of the four dimensional subspaces of fixed \( q \) are negligible.

As is often the case with the Dirac equation, it is easier to work with the square of the Dirac operator instead of the first order form. Since the curvature vanishes for \( s \leq s_0 \) for some \( s_0 \), the solution \( \chi(s, \xi) \) is given by the straight cylinder problem for \( s < s_0 \). This determines both \( \chi(s_0, \xi) \) and \( d\chi(s, \xi)/ds|_{s=s_0} \). To solve the Dirac equation to the desired order, we project the equation onto the four-dimensional subspace containing \( \chi(s_0, \xi) \) to obtain four coupled first order equations. Equivalently, any solution of the projected
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equation must also satisfy the projection of the equation \( H^2 \chi = E^2 \chi \) to second order in \( \delta \).
The initial conditions on \( \chi \) and \( d\chi/ds \) at \( s = s_0 \) determine a unique solution of the second order equation, which must also be the unique solution of the first order equation. From now on, we study the second order equation, eq. (37). We complete the square and divide by \( R^2 \), to obtain

\[
H^2 = -\nabla^2 + m^2 - \left( \frac{\partial}{\partial s} - \frac{i}{2} \vec{\Sigma} \cdot \vec{\omega} \right)^2 - \frac{\kappa^2}{4} + O(\delta^3/R^2).
\] (38)

Notice that the terms of order \( \delta^3/R^2 \), can be neglected in the adiabatic approximation. \( H^2 \) is simpler to study than \( H \) because it only mixes states with the same \( \lambda \). Thus the mixing problem has been reduced from a \( 4 \times 4 \) to a \( 2 \times 2 \) problem. For simplicity, we refer to the two dimensional space of states (with \( \vec{J} \cdot \hat{t} = \pm \nu \)) that can mix under \( H^2 \) as the “adiabatic subspace”. [This simplification is analogous to the Foldy-Wouthuysen transformation that eliminates the lower components of the Dirac spinor in the nonrelativistic limit.] This remarkably simple result is interpreted in the following section.

VI. PROPAGATING MODES AND SPIN EVOLUTION

This Hamiltonian (squared) of eq. (37) appears to be the same one that one would obtain in a nonrelativistic theory. The curvature induced “potential”, \( -\kappa^2/4 \), is well known. The “gauge term”, \( -\frac{i}{2} \vec{\Sigma} \cdot \vec{\omega} \), looks like it undoes the rotation, \( \Omega(s) \), that put us into the moving frame. If it did, a stationary experimenter outside the tube would observe a spin vector that always points in the same direction as the particle propagates in \( s \). However, this description of propagation by this Hamiltonian must not be correct: Because \( \vec{\Sigma} \cdot \vec{\omega} \) fails to commute with the boundary condition eq. (1), it cannot in general be diagonalized with respect to the transverse basis states identified in Section IV.

To obtain the correct physical description, we must remember that \( H^2 \) is to be projected onto the adiabatic subspace containing two states of opposite helicity \( (\nu = \pm \frac{1}{2}) \). The externally prescribed vector curvature, \( \kappa \), explicitly breaks rotation invariance and allows states of different helicity to mix. However, the symmetry violating terms in \( H^2 \) are at most vector operators (note \( (\vec{\Sigma} \cdot \vec{\omega})^2 = \kappa^2 \)), and by the Wigner-Eckart theorem can only induce transitions between states with helicity differing by unity. Thus mixing of different helicity states can only occur if \( |\nu| = \frac{1}{2} \). For all the states with \( |\nu| > \frac{1}{2} \), helicity, and in fact all transverse quantum numbers, are conserved. This is exactly analogous to the propagation of photons discussed by Chiao and Wu [5].

States with \( |\nu| = \frac{1}{2} \) mix as they propagate. The crucial feature that distinguishes the relativistic and nonrelativistic situations is that the states with \( \nu = \pm \frac{1}{2} \) are not eigenstates of \( \Sigma_3 \). Instead they are eigenstates of \( J_3 \) and \( \gamma^0 S_3 \). Since the ground states do not acquire any additional degeneracies in the nonrelativistic limit, we can apply our formalism to them in this limit. The ground states become \( L_3 = 0 \) and \( S_3 = \pm \frac{1}{2} \) states. The gauge term in eq. (38) exactly undoes the rotation \( \Omega(s) \) and leads to trivial spin transport along the tube. However, in the relativistic case, the matrix elements of \( \vec{\Sigma} \) between states with \( \nu = \pm \frac{1}{2} \) are reduced by a “depolarizing factor” which reflects the mixing of spin and orbital angular momentum.
We study the spin transport in the interesting case of $\nu = \pm \frac{1}{2}$. For concreteness, we calculate the matrix elements of the operator $\vec{\Sigma} \cdot \vec{\omega}$ in the lowest adiabatic subspace, \{\eta_j\}, enumerated in eq. (30)

$$\langle j | \vec{\Sigma} \cdot \vec{\omega} | k \rangle = \int d^2 \xi \, \eta_j^\dagger \vec{\Sigma} \cdot \vec{\omega} \eta_k.$$  

(39)

For positive $\lambda$, only the (12) and (21) matrix elements are nonzero, and for negative $\lambda$, only (34) and (43) matrix elements are nonzero,

$$\int d^2 \xi \, \eta_1^\dagger \vec{\Sigma} \cdot \vec{\omega} \eta_2 = \int d^2 \xi \, \eta_3^\dagger \vec{\Sigma} \cdot \vec{\omega} \eta_4$$

$$= 2N^2(\omega^1 - i\omega^2) \zeta^2 \int_0^R \int_0^{2\pi} rdr d\varphi J_0(qr)^2$$

$$= 2\pi N^2 R^2 (\omega^1 - i\omega^2)(\zeta^2 + 1) J_1(qR)^2$$

$$= (\omega^1 - i\omega^2)D$$  

(40)

where we have defined the “depolarizing factor” $D$ as

$$D \equiv \frac{\zeta^2(\zeta^2 + 1)}{(\zeta^2 + 1)^2 - \frac{2\omega}{qR}}.$$  

(41)

A similar expression holds for each excited subspace with $|\nu| = \frac{1}{2}$. For a subspace containing a state in the tower over $\eta_1 ^{\pm, \lambda_0}$, one just changes $q$ accordingly in the expression for $D$. For a subspace containing a state $\eta_2 ^{\pm, \lambda}$, one needs to change $q$ and replace $\zeta$ by $-\zeta^{-1}$, in the expression for $D$. The general expression is

$$D(q, \pm) \equiv \frac{\zeta^{\pm 2}(\zeta^{\pm 2} + 1)}{(\zeta^{\pm 2} + 1)^2 - \frac{2\omega}{qR}}.$$  

(42)

where the sign $\pm$ is the product of the signs of the transverse quantum numbers and is the same for the 4 basis states in each adiabatic subspace. An important caveat is that this formula fails to hold in the nonrelativistic limit for the lower choice of sign. The states with $\nu = \frac{1}{2}$ and those with $\nu = \frac{3}{2}$ both reduce to $L = 1$ and have the same transverse energy in the nonrelativistic limit; the additional degeneracies signal that these subspaces do not evolve adiabatically. However, as remarked earlier, we know that in this limit $D = 1$ and a straightforward calculation extending to a 16-dimensional subspace of states verifies this fact.

The factor $(\omega^1 - i\omega^2)$ in eq. (40) shows the parallel between the matrix elements of $\vec{\Sigma} \cdot \vec{\omega}$ in the \{\eta_j\} basis and its matrix in the basis of eigenstates of $\Sigma_3$. The constant of proportionality, $D$, measures the rate at which the spin vector rotates in the parallel transport frame as the wave propagates. In the nonrelativistic limit, $\zeta \rightarrow \infty$ so $D \rightarrow 1$, and the spin completely decouples from the evolution of the tube. If $D$ were zero for some choice of parameters, the spin would remain fixed in the parallel transport frame as for the photon. In the ultrarelativistic limit, $\zeta = 1$ and $qR \approx 1.434$, the lowest root of the equation $J_0(qR) = J_1(qR)$. Substitution gives $D \simeq 0.767$, so that even in the extreme relativistic limit helicity is not conserved.
We now project the second-order Dirac equation, \( H^2 \chi = E^2 \chi \), onto the lowest multiplet, \( \{ \eta_j \} \). Writing \( \chi = \sum_{j=1}^4 c_j(\xi) \eta_j(\xi) \), the \( 4 \times 4 \) matrix equation for the four component vector \( C(s) = (c_1, c_2, c_3, c_4) \) is
\[
\left[ -\left( \frac{\partial}{\partial s} - \frac{i}{2} D \vec{T} \cdot \vec{\omega}(s') \right)^2 + q^2 - \frac{\kappa^2 D^2}{4} + m^2 \right] C(s) = E^2 C(s) \quad (43)
\]
where \( \vec{T} \) is the vector of \( (4 \times 4) \) Pauli matrices, \( \vec{T} = \begin{pmatrix} \vec{\tau}_0 & 0 \\ 0 & \vec{\tau}_0 \end{pmatrix} \), that generates the rotations of the \( J_3 = \pm \frac{1}{2} \) eigenstates. Just as the Pauli matrices, \( S_j \), rotate \( S_3 = \frac{1}{2} \) states into \( S_3 = -\frac{1}{2} \) states, the matrices \( T_j \) rotate the \( J_3 = \frac{1}{2} \) states into \( J_3 = -\frac{1}{2} \) states.

Eq. (43) is our fundamental result. Relativistic effects appear in two places. First, as pointed out earlier, the rotation back to the laboratory fixed frame is only complete in the nonrelativistic limit, when \( D = 1 \). As \( q \) increases relative to \( m \), \( D \) decreases. The ideal case of \( D = 0 \), in which the helicity is conserved, is not reached even in the extreme relativistic limit. The following section discusses opportunities presented by other choices of boundary conditions. Second, the depolarizing factor appears in the effective scalar potential, \( -\frac{\kappa^2 D^2}{4} \). If we apply the obvious unitary transformation \( C_D(s) = \Omega_D(s) C(s) \) (see eq. (18)), where
\[
\Omega_D(s) \equiv \mathcal{P} \exp \left[ -\frac{i}{2} D \int ds' \vec{T} \cdot \vec{\omega}(s') \right] ,
\]
then the Dirac equation for \( C_D(s) \) reduces to the scalar case
\[
\left( -\frac{\partial^2}{\partial s^2} - \frac{\kappa^2 D^2}{4} \right) C_D = k^2 C_D . \quad (45)
\]
This equation can be solved numerically for a given tube, with the usual phenomenology of bound and scattering states.

\section*{VII. DISCUSSION}

It is helpful to interpret our results first in the nonrelativistic limit. We expect to find no coupling to the spin because the nonrelativistic boundary conditions do not involve spin at all and the exact Hamiltonian commutes with \( \vec{S} \). It is reassuring to see that our formalism recovers this result: in the nonrelativistic limit, \( D \rightarrow 1 \), so that in the parallel transport frame, the spin appears to rotate with angular frequency \( -\vec{\omega} \), where the minus sign indicates that the sense of the rotation is opposite that of the coordinate frame. In other words, the spin vector maintains its orientation in the three-dimensional Euclidean space in which the tube is embedded (which corresponds to the coordinate frame of some three-dimensional Euclidean experimenter). Away from the nonrelativistic limit, \( D \neq 1 \), so the variation of spin is no longer exactly opposite to the rotation of the parallel transport frame.

Now if \( D \) were to go to zero, the vector \( \vec{J} \) would not rotate relative to the axis of the tube. This would parallel the phenomenon described by Chiao and Wu. However, \( D \) is never zero in the lowest energy subspace. Even for \( m = 0 \), \( D \approx 0.767 \). We obtain a complete description of \( D \) as a function of \( mR \) by solving for the lowest root of the eigenvalue condition.
eq. (27) as a function of $m$ and substituting in the expression eq. (41) for $D$. A plot of $D$ versus $mR$ appears in Fig. (3). The states with $\rho = \{\frac{1}{2}^+, \lambda\}$ do have $D \to 1$ as $m \to \infty$, as we expect. However, the states for which $\rho = \{\frac{1}{2}^-, \lambda\}$ have $D \to 0$ in the nonrelativistic limit. This “unphysical” result appears as an artifact of the adiabatic approximation, which breaks down for these states in the nonrelativistic limit, as explained in Section VI.

![Fig. 3. Depolarizing factor $D$ as a function of $mR$ for the $\frac{1}{2}^+$ and $\frac{1}{2}^-$ towers of states in Fig. 1.](image)

The effect of the factor $D$ on the vector $\vec{J}$ has a simple geometric interpretation. As an example, consider a tube whose defining curve lies in a plane and such that it has two straight sections, $|s| \geq s_0$, joined by a section with constant curvature $\kappa$ and center of curvature $P$. In addition, construct a complementary tube parameterized by the same arc-length parameter $s$ satisfying the following conditions: it is parallel to the first for $s \leq -s_0$; it has a section of constant curvature $D\kappa$ with center of curvature $P$ for $|s| < s_0$; and it has another straight section for $s \geq s_0$. The smaller curvature implies a larger radius of curvature. The effect of the depolarizing factor can be summarized by the following construction: as the Dirac particle traverses the curved section of the tube, the vector $\vec{J}$ remains fixed relative to the tangent of the complementary curve at the corresponding value of $s$. For $m = 0$, if the tube curves through $\Delta \Theta = 2\pi$, then $\vec{J}$ falls behind the tangent by approximately $\frac{1}{534}\pi$.

In our formalism, we have used the boundary condition eq. (1) to simulate a hard-wall potential barrier. Many of the qualitative features of our results follow from the fact that spin is not a good quantum number for the Dirac equation subject to such a boundary condition. It is natural to ask how sensitive these results are to the specific choice of confining boundary condition. An obvious generalization given in eq. (2) is characterized by a chiral angle $\theta$. We have chosen $\theta = 0$ throughout; let us now take $\theta \neq 0$. If we set $\psi' = e^{i\kappa\theta/2}\psi$, we can map this version back to the original boundary condition at the cost of introducing a “chiral” mass term into the Dirac equation:

1Actually, we require the transition from zero to nonzero curvature to occur smoothly over a long enough interval that the adiabatic approximation is applicable.
Repeating the calculations in the earlier sections shows that the sole effect of the chiral angle is to replace $m$ by $m \cos \theta$ in the expression for the parameter $\zeta$ and thus implicitly in the boundary conditions and depolarizing factor. Thus the analysis above, with an effective mass $m \cos \theta$, applies to this family of problems. In fact, for $\cos \theta \geq 0$, the plot in Fig. (2) describes the depolarizing factor as a function of $mR \cos \theta$. When $mR \cos \theta$ becomes negative new phenomena arise. In particular, the boundary condition eq. (2) acts like an effective attractive potential at $r = R$. As a result for sufficiently negative $mR \cos \theta$ a qualitatively new solution appears in each partial wave, which is “bound to” $r = R$ and decays exponentially as $r$ decreases. The detailed study of these states is beyond the scope of this paper.

In addition to these boundary conditions, there are other families of boundary conditions that confine the probability current but which we have not yet considered in detail. Specifically one can generalize eq. (2) by the condition

$$-i \hat{\gamma} \cdot \hat{n} \psi = e^\Gamma \psi$$

where $\Gamma$ is any antihermitian Dirac matrix that anticommutes with $\hat{\gamma} \cdot \hat{n}$. $\Gamma$ need not be constant, as the demonstration of probability confinement involves only algebraic manipulations. Interesting alternatives include $\Gamma_1 = \gamma_5 \theta(s)$, $\Gamma_2 = \hat{\gamma} \cdot \hat{t}$ and $\Gamma_3 = \hat{\gamma} \cdot \hat{t} \times \hat{r}$. Each of these choices will generate different, nontrivial spin effects. Thus we believe that the essential feature of nontrivial spin transport should not be unique to our choice of boundary condition. Another direction of possible future work is the generalization of our approach to higher-dimensional spaces, as has been done with the nonrelativistic theory.

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