Conformal anomaly for 2d and 4d dilaton coupled spinors

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ABSTRACT

We study quantum dilaton coupled spinors in two and four dimensions. Making classical transformation of metric, dilaton coupled spinor theory is transformed to minimal spinor theory with another metric and in case of 4d spinor also in the background of the non-trivial vector field. This gives the possibility to calculate 2d and 4d dilaton dependent conformal (or Weyl) anomaly in easy way. Anomaly induced effective action for such spinors is derived. In case of 2d, the effective action reproduces, without any extra terms, the term added by hands in the quantum correction for RST model, which is exactly solvable. For 4d spinor the chiral anomaly which depends explicitly from dilaton is also found. As some application we discuss SUSY Black Holes in dilatonic supergravity with WZ type matter and Hawking radiation in the same theory. As another application we investigate spherically reduced Einstein gravity with 2d dilaton coupled fermion anomaly induced effective action and show the existence of quantum corrected Schwarzschild-de Sitter (SdS) (Nariai) BH with multiple horizon.

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1 Introduction

Spherical reduction of Einstein gravity (see, for example [1]) with minimal matter leads to some 2d dilatonic gravity (for most general action, see [2]) with 2d dilaton coupled matter. Then, applying s-wave approximation one is forced to study 2d quantum dilaton coupled theory. Having in mind mainly cosmological applications, conformal anomaly may be one of the most important objects to study in such theory. For 2d dilaton coupled scalar the conformal (or Weyl) anomaly has been found in ref.[3] and later in refs.[4, 5, 6, 7]. Using it, one can find anomaly induced, dilaton dependent effective action [4, 5, 6] (see also later work [8] where this effective action has been rederived). Anomaly induced effective action has been applied to investigate the following remarkable problems: Hawking radiation [9] in dilatonic supergravity [6], anti-evaporation of multiple horizon black holes [10, 11], discussion of semi-classical energy-momentum tensor in the presence of dilaton [12], etc.

In the present paper we investigate the conformal anomaly for 2d and 4d dilaton coupled spinors. Naive transformation of quantum spinor (inclusion of dilatonic function to fermion) seems to indicate that no dilaton dependent terms appear in conformal anomaly for dilaton coupled spinor [3, 6, 13]. Unfortunately it is not quite correct and Jacobian of quantum transformation is lost in this way. In this work, making only classical transformation of external classical gravitational field, we map dilaton coupled fermion to minimal fermion in the another classical gravitational background. (No transformation of quantum fields is made). Then, calculation of conformal anomaly may be done in quite straightforward way. In case of 2d, the anomaly induced effective action is derived. The effective action reproduces the term added by hands in the quantum correction for RST model[14], which is exactly solvable. Adding such effective action to reduced Einstein gravity, i.e. working in large $N$ and s-wave approximation we derive quantum corrected SdS (Nariai) BH. The structure of theory under investigation shows the possibility of anti-evaporation of such BH in analogy with results of refs. [10, 11].

The paper is organised as follows. In the next section we calculate the conformal anomaly and induced action for 2d dilaton coupled fermion. Comparison with 2d scalar is made. The possibility of realization for quantum part of RST model as fermion anomaly induced effective action is shown. Quantum corrected SdS BH is also constructed.

Section 3 is devoted to short discussion of BHs in dilatonic supergravity
with WZ type matter. Energy-momentum tensor components are found and Hawking radiation is derived. In section 4 we evaluate conformal and chiral anomaly as well as induced action for 4d dilaton coupled fermion. Its explicit dilaton dependence is demonstrated. Finally, some discussion is presented in last section.

2 Conformal anomaly for 2d dilaton coupled spinor

Let us start from 2d dilaton coupled spinor Lagrangian:

\[ L = \sqrt{-g} f(\phi) \bar{\psi} \gamma^\mu \partial_\mu \psi \]  

where \( \psi \) is 2d Majorana spinor, \( f(\phi) \) is an arbitrary function and \( \phi \) is dilaton.

Let us make now the following classical transformation of background field \( g_{\mu\nu} \):

\[ g_{\mu\nu} \rightarrow f^{-2}(\phi) \tilde{g}_{\mu\nu} \]  

Then it is easy to see that \( \gamma^\mu(x) \rightarrow f(\phi) \tilde{\gamma}^\mu(x) \) and in terms of new classical metric we obtain usual, non-coupled with dilaton (minimal) Lagrangian for 2d spinor:

\[ L = \sqrt{-\tilde{g}} \bar{\psi} \tilde{\gamma}^\mu \partial_\mu \psi . \]  

The conformal anomaly for Lagrangian (3) is well-known:

\[ \sqrt{-\tilde{g}} T = \frac{1}{24\pi} \left\{ \frac{1}{2} \tilde{R} \right\} . \]  

Now, transforming Eq.(4) to original variables:

\[ \tilde{g}_{\mu\nu} = f^2(\phi) g_{\mu\nu} , \quad \tilde{R} = f^{-2}(\phi) (R - 2\Delta \ln f) \]

we get the following conformal anomaly for dilaton coupled Majorana spinor (1):

\[ \sqrt{-g} T = \frac{\sqrt{-g}}{24\pi} \left[ \frac{1}{2} R - \Delta \ln f \right] \]

\[ = \frac{\sqrt{-g}}{24\pi} \left[ \frac{1}{2} R - \frac{f'}{f} \Delta f - \frac{(f'' f - f'^2)}{f^2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] . \]
Hence, we found the conformal anomaly for 2d dilaton coupled Majorana spinor.

Above result corrects the error in refs.[3, 6, 13] where conclusion was made that for dilaton coupled spinor no new dilaton dependent terms appear in conformal anomaly if compare with minimal spinor. Note also that for 2d dilaton coupled Dirac spinor the conformal anomaly is twice of eq.(6).

In the conformal anomaly (6) dilaton dependent terms appear in the form of total derivative. In principle, it means that this term is ambiguous by physical reasons. Indeed, in two dimensions the analogue of Einstein action looks like:

\[ S = \frac{1}{G} \int d^2x \sqrt{-g} R f(\phi) . \]  

(7)

Now, there exists the following relation

\[ g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^2x \sqrt{-g} R f(\phi) = \Delta f(\phi) . \]  

(8)

In other words, by finite renormalization of gravitational action (7), we can always change the coefficient of \( \Delta f \) term in conformal anomaly. So in 2d gravity this term is only fixed by the physical renormalization condition. That is why total derivative term of conformal anomaly for dilaton coupled scalar is ambiguous.

Now we discuss anomaly induced effective action for dilaton coupled spinor. The derivation goes in the same way as it was for dilaton coupled scalar [4, 5]. Making the conformal transformation of the metric \( g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \) in the trace anomaly, and using relation

\[ \sqrt{-g} T = \frac{\delta}{\delta \sigma} W[\sigma] \]  

(9)

one can find the anomaly induced effective action \( W \). In the covariant, non-local form it may be written as following:

\[ W = -\frac{1}{2} \int d^2x \sqrt{-g} \left\{ \frac{1}{96\pi} R \frac{1}{\Delta} R \right. \\
+ \left( F_2(\phi) - \frac{\partial F_3(\phi)}{\partial \phi} \right) (\nabla^\lambda \phi)(\nabla_\lambda \phi) \frac{1}{\Delta} R + R \int F_3(\phi) d\phi \right\} \]  

(10)

where

\[ F_2(\phi) = -\frac{f'' f - f'^2}{24\pi f^2} , \quad F_3(\phi) = -\frac{f'}{24\pi f} . \]  

(11)
Note that coefficient of second term is actually zero as is easy to check. Hence, we got anomaly induced effective action for dilaton coupled spinor.

An interesting thing is that the effective action (10) exactly reproduces the effective action of RST model [14], which is exactly solvable. The RST model is given by adding the quantum correction

\[
W_{RST} = -\frac{1}{2} \int d^2 x \sqrt{-g} \left\{ \frac{N}{48\pi} R \frac{1}{\Delta} R + \frac{N}{24\pi} \phi R \right\}
\]

(12)
to the action of the CGHS model [15]

\[
S_{CGHS} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} e^{-2\phi} \left( R + 4 \nabla_\mu \phi \nabla^\mu \phi + 4\lambda^2 \right).
\]

(13)

In [14], the second term in \(W_{RST}\) (12) is added by hands. In the work by Bousso-Hawking [4], it has been found that there appears the third term in (10), which corresponds to the second term in (12), from the conformal anomaly of the dilaton coupled scalar fields but there always appears the second term in (10), which makes the model not to be exactly solvable, again. As found here, however, if we consider only dilaton coupled (2\(N\)) spinor fields as a matter to CGHS model [15], the model becomes exactly solvable even when we include quantum corrections and has the exact quantum solutions, in the conformal gauge:

\[
g_{\pm\mp} = -\frac{1}{2} e^{2\rho}, \quad g_{\pm\pm} = 0,
\]

(14)

including the following quantum black hole solution

\[
\Omega = \chi = -\frac{\lambda^2}{\sqrt{\kappa}} x^+ x^- - \frac{\sqrt{\kappa}}{4} \ln \left( \frac{\lambda^2 x^+ x^-}{4} \right),
\]

\[
\Omega \equiv \frac{\sqrt{\kappa}}{2} \phi + \frac{e^{-2\phi}}{\sqrt{\kappa}}, \quad \chi \equiv \sqrt{\kappa} \rho - \frac{\sqrt{\kappa}}{2} \phi + \frac{e^{-2\phi}}{\sqrt{\kappa}}, \quad \kappa \equiv \frac{N}{12}.
\]

(15)

Here we neglect the quantum correction from the dilaton and graviton. Similarly, changing classical part of the model by the spherically reduced action of Einstein gravity one can find 2d BHs in such model with quantum correction. However, now such 2d BHs will get 4d interpretation.

As an example we consider spherically reduced Einstein gravity with the quantum correction (12) from 2\(N\) (\(N\) in 4d) spinors. The equations of motion
are given by the variation over $g_{\pm\pm}, g_{+-}$ and $\phi$. They have the following form in the conformal gauge (14):

$$0 = e^{-2\phi} \left( 2\partial_r \rho \partial_r \phi + (\partial_r \phi)^2 - \partial_r^2 \phi \right)$$

$$- \frac{N}{12} \left( \partial_r^2 \rho - (\partial_r \rho)^2 \right) - \frac{N}{6} \left( 2\partial_r \rho \partial_r \phi - \partial_r^2 \phi \right) + Nt_0$$  \hspace{1cm} (16)

$$0 = e^{-2\phi} \left( \partial_r^2 \phi - 2(\partial_r \phi)^2 - \Lambda e^{2\rho} + e^{2\rho + 2\phi} \right)$$

$$+ \frac{N}{12} \partial_r^2 \rho - \frac{N}{6} \partial_r^2 \phi \hspace{1cm} (17)

$$0 = -e^{-2\phi} \left( -\partial_r^2 \phi + (\partial_r \phi)^2 + \partial_r^2 \rho + \Lambda e^{2\rho} \right) + \frac{N}{6} \partial_r^2 \rho.$$  \hspace{1cm} (18)

Now we investigate if there exists a Nariai or SdS type solution [16] for above equations. For this purpose, we assume $\phi$ is a constant: $\phi = \phi_0$. Then Eqs.(17) and (18) can be rewritten as follows:

$$0 = \frac{3}{NG} \left( -\Lambda e^{-2\phi_0} + 1 \right) e^{2\rho} + \partial_r^2 \rho \hspace{1cm} (19)

$$0 = \frac{3\Lambda e^{-2\phi_0}}{2NG} \left( \frac{3e^{-2\phi_0}}{2NG} - 1 \right)^{-1} e^{2\rho} + \partial_r^2 \rho.$$  \hspace{1cm} (20)

Comparing (19) with (20), we obtain

$$e^{-2\phi_0} = \frac{NG}{6} + \frac{1}{2\Lambda} \pm \sqrt{\frac{1}{4\Lambda^2} - \frac{NG}{2\Lambda} + \frac{N^2G^2}{36}}.$$  \hspace{1cm} (21)

The sign $\pm$ in (21) should be $+$ if we require the solution coincides with the classical one $e^{-2\phi_0} = \frac{1}{\Lambda}$ in the classical limit of $N \rightarrow 0$. Then from (19) or (20), we find

$$e^{2\rho} = \frac{2C}{R_0 \cosh^2 \left( \sqrt{C} \right)}.$$  \hspace{1cm} (22)

Here $C > 0$ is a constant of the integration and $R_0$ is 2d scalar curvature, which is given by

$$R_0 = -2e^{-2\rho} \partial_r \rho = \Lambda - \frac{3}{NG} \pm \frac{6\Lambda}{NG} \sqrt{\frac{1}{4\Lambda^2} - \frac{NG}{2\Lambda} + \frac{N^2G^2}{36}}.$$  \hspace{1cm} (23)
It is straightforward to check that the solution in (21) and (22) satisfies Eq. (16). The 4d curvature $R_4$ is also given by

$$R_4 = R_0 + 2e^{2\phi_0} = \frac{3\Lambda}{2} - \frac{5}{2NG} \pm \frac{3\Lambda}{NG} \sqrt{\frac{1}{4\Lambda^2} - \frac{NG}{2\Lambda} + \frac{N^2G^2}{36}}.$$  

(24)

Above spherically reduced Einstein gravity with fermion quantum correction (in large $N$ and $s$-wave approximation) acquires the structure carefully studied in the model by Bousso-Hawking [11]. It is easy to repeat literally the same investigation as in ref.[11] and to show the possibility of anti-evaporation of Nariai BH due to quantum spinors. (Note that ref.[11] is dealing with only quantum dilaton coupled scalars). This study is even simpler as two terms of induced effective action for 2d dilaton coupled scalars simply do not appear for the spinor case. So it is not necessary to adopt approximation where these two terms disappear for the study of quantum equations of motion.

For comparison we give here also conformal anomaly and induced effective action for dilaton coupled scalar $a$ with Lagrangian:

$$L = f_s(\phi)g^{\mu\nu} \partial_\mu a \partial_\nu a.$$  

(25)

Using results of refs.[3, 4], one can write down:

$$\sqrt{-g}T = \frac{\sqrt{-g}}{24\pi} \left \{ R - 3 \left ( \frac{f''}{f_s} - \frac{f'^2}{2f_s^2} \right ) (\nabla^\lambda \phi)(\nabla_\lambda \phi) - 3 \frac{f''}{f_s} \Delta \phi \right \}.$$  

(26)

Anomaly induced effective action is given by Eq.(10), where coefficient of the first term is $\frac{1}{48\pi}$ (i.e. twice bigger), and

$$F_2(\phi) = -\frac{1}{8\pi} \left ( \frac{f''}{f_s} - \frac{f'^2}{2f_s^2} \right ), \quad F_3(\phi) = -\frac{1}{8\pi} \frac{f''}{f_s}.$$  

(27)

Now, having at hands conformal anomaly for 2d dilaton coupled spinor and scalar we may discuss specific models.

### 3 SUSY Black Holes and Hawking radiation

We will start from the theory of the dilatonic supergravity with WZ type matter. The corresponding action has been constructed in ref.[6] using modified form of tensor calculus (for the introduction, see [17]).
Working on purely bosonic background (but still keeping fermions in matter sector) we may write the different versions of above dilatonic SG with matter. One of them is actually SUSY generalization [18, 19] of CGHS model [15]:

\[
L = -\left(\tilde{V} + \frac{C(\phi)R}{2} + (Z + 3\phi Z')(\nabla^\mu \phi)(\nabla_\mu \phi) - f(\phi) \sum_{i=1}^{N} \left[(\nabla_\mu a_i)(\nabla^\mu a_i) + \bar{\xi}^i \gamma^\mu \partial_\mu \xi^i\right]\right)
\]  

(28)

where \(\tilde{V} = -CS^2 - C'FS - ZF^2 - V'F\), \(C(\phi) = 2e^{-2\phi}\), \(Z(\phi) = 4e^{-2\phi}\), \(V(\phi) = 4e^{-2\phi}\). Using the equations of motion with respect to the auxiliary fields, we obtain

\[
S = 0, \quad F = -\lambda.
\]  

(29)

We consider the situation where metric and dilaton are background fields and matter is quantized. Then, one can work in large \(N\) approximation. The anomaly induced effective action due to \(N\) scalars and spinors may be derived from Eqs.(10), (27) as the following:

\[
W = -\frac{1}{2} \int d^2x \sqrt{-g} \left\{ \frac{N}{32\pi} R \frac{1}{\Delta} R - \frac{N}{16\pi} \frac{f'^2}{f^2} (\nabla^\lambda \phi)(\nabla_\lambda \phi) \frac{1}{\Delta} R - \frac{N}{6\pi} R \ln f \right\}.
\]  

(30)

Due to missing fermionic contribution the coefficient of last term in (30) is slightly different if compare with the result written in [6].

The total effective action is given by sum of anomaly induced effective action \(W\) and some conformally invariant functional which may be defined in Schwinger-De Witt expansion. The leading term of this expansion for fermion is actually zero unlike the scalar. There are indications that this conformal invariant functional is exactly zero for dilaton coupled spinor, as it is impossible to construct conformal invariant combinations from dilaton and flat derivatives (no massive coupling constants) with more than two derivatives. The corresponding SD coefficient on flat background for spinor is total derivative as is shown above. The exact solvability of RST model indicates the same.
We investigate the correction from the previous results in Ref.[6] due to the missing fermion contribution. As a matter fields, we use dilaton coupled matter supermultiplet, which is natural as a toy model of the 4d models but is different from the original CGHS model. We are interested in the change of the estimation of the back-reaction from such a matter supermultiplet to black holes and Hawking radiation working in large-$N$ approximation. Since we are interested in the vacuum (black hole) solution, we consider the background where matter fields, the Rarita-Schwinger field and dilatino vanish.

In the superconformal gauge, the equations of motion in [6] obtained by the variation over $g^{±±}$, $g^{±∓}$ and $φ$ are slightly changed. We should also note that there is, in general, a contribution from the auxiliary fields to $T^{±∓}$ besides the contribution from the trace anomaly.

In large-$N$ limit, where classical part can be ignored, field equations become simpler and we can find the explicit solutions:

$$h(φ) = \int dρ \frac{4}{3} ± \sqrt{\frac{16}{9} + \frac{ρ}{ρ}}$$

$$ρ = -\frac{16}{9} + \left(ρ^+(x^+) + ρ^-(x^-)\right)\frac{2}{3}.$$  \hspace{1cm} (31)

Here $ρ^±$ is an arbitrary function of $x^± = t ± x$. Comparing with the results in [6], the coefficients in (31) are slightly changed but the essential behavior does not change. For example, the scalar curvature is given by

$$R = 8e^{-2ρ} \partial_+ \partial_- ρ$$

$$= -4e^{-2\left\{-\frac{16}{9} + (ρ^+(x^+) + ρ^-(x^-))\frac{2}{3}\right\}} \left(ρ^+(x^+) + ρ^-(x^-)\right)\frac{4}{3} ρ^{+′}(x^+) ρ^{-′}(x^-).$$  \hspace{1cm} (32)

Note that when $ρ^+(x^+) + ρ^-(x^-) = 0$, there is a curvature singularity. Especially if we choose

$$ρ^+(x^+) = \frac{r_0}{x^+}, \quad ρ^-(x^-) = -\frac{x^-}{r_0},$$  \hspace{1cm} (33)

there are curvature singularities at $x^+x^- = r_0^2$ and horizon at $x^+ = 0$ or $x^- = 0$. Hence we got black hole solution in the model under discussion. The asymptotically flat regions are given by $x^+ → +∞$ ($x^- < 0$) or $x^- → -∞$ ($x^+ > 0$).
We now consider Hawking radiation. We investigate the case that \( f(\phi) = e^{\alpha \phi} (h(\phi) = \alpha \phi) \). Hawking radiation can be obtained by substituting the classical black hole solution which appeared in the original CGHS model \([15]\)

\[
\rho = -\frac{1}{2} \ln \left( 1 + \frac{M}{\lambda} e^{\lambda (\sigma^- - \sigma^+)} \right), \quad \phi = -\frac{1}{2} \ln \left( \frac{M}{\lambda} + e^{\lambda (\sigma^+ - \sigma^-)} \right)
\]

(34)

(where \( M \) is the mass of the black hole and we used asymptotic flat coordinates) into the quantum part of the energy momentum tensor. We use eq.(29). Then we find that when \( \sigma^+ \to +\infty \), the energy momentum tensor behaves as

\[
T^q_{\pm} \to 0, \quad T^q_{\pm \pm} \to \frac{N \lambda^2}{16} \alpha^2 + t^\pm(\sigma^\pm).
\]

(35)

Here \( t^\pm(\sigma^\pm) \) is a function which is determined by the boundary condition. In order to evaluate \( t^\pm(\sigma^\pm) \), we impose a boundary condition that there is no incoming energy. This condition requires that \( T^q_{++} \) should vanish at the past null infinity (\( \sigma^- \to -\infty \)) and if we assume \( t^-(\sigma^-) \) is black hole mass independent, \( T^q_{--} \) should also vanish at the past horizon (\( \sigma^+ \to -\infty \)) after taking \( M \to 0 \) limit. Then we find

\[
t^-(\sigma^-) = -\frac{N \lambda^2 \alpha^2}{16}
\]

(36)

and one obtains

\[
T^q_{--} \to 0
\]

(37)

at the future null infinity (\( \sigma^+ \to +\infty \)). Eqs.(35) and (37) might tell that there is no Hawking radiation in the dilatonic supergravity model under discussion when quantum back-reaction of matter supermultiplet in large-\( N \) approach is taken into account as in \([6]\). (That indicates that above black hole is extremal one). From another side since we work in strong coupling regime it could be that new methods to study Hawking radiation should be developed.

4 Conformal and chiral anomaly for 4d dilaton coupled spinor

We consider now 4d dilaton coupled fermion which may appear as the result of spherical reduction of higher dimensional minimal spinor. The correspond-
The Lagrangian may be taken as follows:

\[ L = \sqrt{-g} f(\phi) \bar{\psi} \gamma^\mu(x) \nabla_\mu \psi \]  

(38)

where \( \nabla_\mu = \partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \) unlike the case of 2d Majorana spinor. Note that the action (38) is conformally invariant.

Let us make the following transformation of background gravitational field:

\[ g_{\mu\nu} = e^{2\sigma(x)} \tilde{g}_{\mu\nu} , \quad \gamma^\mu(x) = e^{-\sigma(x)} \tilde{\gamma}^\mu(x) , \quad \sqrt{-g} = e^{4\sigma(x)} \sqrt{-\tilde{g}} . \]  

(39)

It is easy to check then that

\[ \gamma^\mu \nabla_\mu = \gamma^\mu \left( \partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \right) = e^{-\sigma(x)} \tilde{\gamma}^\mu \left( \partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} + \frac{3}{2} \partial_\mu \sigma(x) \right) . \]  

(40)

(Note that review of conformal transformations in 4d gravity may be found in ref.[20].) Let us select \( \sigma(x) \) to satisfy

\[ e^{3\sigma(x)} f(\phi) = 1 . \]  

(41)

Then, Lagrangian (38) after transformation (39) takes the form:

\[ L = \sqrt{-\tilde{g}} \left[ \bar{\psi} \tilde{\gamma}^\mu(x) \tilde{\nabla}_\mu \psi + \bar{\psi} \tilde{\gamma}^\mu \tilde{A}_\mu \psi \right] \]  

(42)

where we used \( \partial_\mu \equiv \tilde{\partial}_\mu \) and \( \tilde{A}_\mu = \frac{3}{2} \partial_\mu \sigma(x) \). Note that field strength for above vector field is equal to zero, that is why no terms of the sort-square of field strength for above vector appear in conformal anomaly. Hence, the calculation of \( a_2 \) Schwinger-De Witt coefficient in theory (38) in curved spacetime with nontrivial dilaton is equivalent to the calculation of \( a_2 \) is an external gravitational field \( \tilde{g}_{\mu\nu} \) (but no dilaton) and external vector field \( \tilde{A}_\mu \).

Let us write the above Lagrangian as the following:

\[ L = \sqrt{-\tilde{g}} \left[ \bar{\psi} \tilde{\gamma}^\mu \tilde{D}_\mu \psi \right] \]  

(43)

where \( \tilde{D}_\mu = \tilde{\nabla}_\mu + \tilde{A}_\mu \). We got the usual system: fermion in curved spacetime with the abelian external vector field. Conformal anomaly for such quantum (Dirac) fermion is well-known:

\[ \frac{\sqrt{-\tilde{g}}}{(4\pi)^2} \left\{ \frac{1}{20} \left( \tilde{F} + \frac{2}{3} \Box \tilde{R} \right) - \frac{11}{360} \tilde{G} \right\} \]  

(44)
where $\tilde{F} = \tilde{R}_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} - 2 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \frac{1}{3} \tilde{R}^2$, $\tilde{G} = \tilde{R}_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} - 4 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \tilde{R}^2$. One can also present coefficients of conformal anomaly as $b = \frac{1}{20(4\pi)^2}$ and $b' = -\frac{11}{360(4\pi)^2}$. And in principle, one can add $\Box \tilde{R}$ (total derivative) term with arbitrary coefficient to Eq.(44). It is known that coefficient of this term may be changed by finite renormalization of $\tilde{R}^2$-term in gravitational action, so it is ambiguous.

Now, one can transform the relation (44) back to original metric tensor quantities:

$$\tilde{g}_{\mu\nu} \rightarrow e^{-2\sigma} g_{\mu\nu} .$$  \hspace{1cm} (45)

Then $\sqrt{-\tilde{g}} \tilde{F} \rightarrow \sqrt{-g} F$, $\tilde{R} \rightarrow e^{2\sigma} [ R + 6 \Box \sigma - 6 (\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma)]$, $\sqrt{-\tilde{g}} (\tilde{G} - \frac{2}{3} \Box \tilde{R}) \rightarrow \sqrt{-g} (G - \frac{2}{3} R - 4 \Box^{2} \sigma - 8 \tilde{R}^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \sigma + \frac{2}{3} R \Box \sigma - \frac{4}{3} (\nabla^{\mu} R)(\nabla_{\mu} \sigma)]$. Using the above relations, we may transform conformal anomaly to the original metric variables:

$$\sqrt{-g} T = \frac{\sqrt{-\tilde{g}}}{(4\pi)^2} \left[ \frac{1}{20} F - \frac{11}{360} G - \frac{2}{3} \Box R - 4 \Box^{2} \sigma 
- 8 \tilde{R}_{\mu\nu} \nabla_{\mu} \nabla_{\nu} \sigma + \frac{8}{3} R \Box \sigma - \frac{4}{3} (\nabla^{\mu} R)(\nabla_{\mu} \sigma) 
+ \frac{2}{3} \left( \frac{1}{20} - \frac{11}{360} \right) \{ \Box R + 6 \Box^{2} \sigma - 6 \Box ( (\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma) ) 
+ 2 \Box \sigma R + 12 (\Box \sigma)^2 - 36 \Box \sigma (\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma) 
+ 2 \nabla_{\mu} \sigma \nabla^{\mu} R + 12 \nabla_{\mu} \sigma \nabla^{\mu} \Box \sigma - 12 \nabla_{\mu} \sigma \nabla^{\mu} ((\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma)) 
- 4 (\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma) R + 24 ((\nabla_{\mu} \sigma)(\nabla^{\mu} \sigma))^2 \} \right]$$ (46)

where $\sigma = -\frac{1}{3} \ln f$ depends explicitly from dilaton. Note that for Majorana spinor the conformal anomaly is given by $\frac{1}{2}$ of above expression.

Hence, we found conformal anomaly for 4d dilaton coupled spinor. It is easy to see that there are dilaton dependent contributions to conformal anomaly. One can also find anomaly induced effective action for dilaton coupled 4d spinor. Starting from Eq.(44) for conformal anomaly in terms of tilded metric the corresponding anomaly induced action in this case is quite known (see [21]):

$$W = -\frac{1}{4b'} \int d^{4}x \sqrt{-\tilde{g}} \int d^{4}x' \sqrt{-\tilde{g}'} \left[ b \tilde{F} + b' \left( \tilde{G} - \frac{2}{3} \Box \tilde{R} \right) \right]$$
\begin{equation}
\times \left[ 2\Box^2 + 4\tilde{R}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu - \frac{4}{3} \tilde{R} \Box + \frac{2}{3} \left( \tilde{\nabla}^\mu \tilde{R} \right) \tilde{\nabla}_\mu \right]^{-1} \times \left[ b\tilde{F} + b' \left( \tilde{G} - \frac{2}{3} \Box \tilde{R} \right) \right]_x' - \frac{1}{18} (b + b') \int d^4x \sqrt{-\tilde{g}} R^2 .
\end{equation}

It is trivial to substitute metric \( \tilde{g}_{\mu\nu} = e^{-2\sigma} g_{\mu\nu} \), \( \sigma = -\frac{1}{3} \ln f(\phi) \) and rewrite above equation in terms of original metric and dilaton function. We do not do this in order to save the place.

As final remark, we note that similarly one can calculate the chiral anomaly for dilaton coupled spinor. Chiral anomaly for theory (42) is known [22, 17, 23]:

\begin{equation}
A_1^2 = \frac{2i}{(4\pi)^2} \left[ -\frac{1}{48} e^{\mu\nu\rho\sigma} \tilde{R}^{\xi}_{\mu\nu} \tilde{R}^{\zeta}_{\rho\sigma} \right] = \frac{2i}{(4\pi)^2} \left[ -\frac{1}{48} e^{\mu\nu\rho\sigma} \left\{ R^{\xi}_{\mu\nu} R^{\zeta}_{\rho\sigma} - 16 R^{\xi}_{\rho\mu\nu} \nabla_\sigma \nabla_\xi \right. \right.
\end{equation}

Hence we found explicitly dilaton dependent corrections for chiral anomaly in the theory of 4d dilaton coupled spinor.

## 5 Discussion

In summary, we found the conformal anomaly and induced effective action for 2d and 4d dilaton coupled spinor as well as 4d chiral anomaly. The dilaton dependent part of conformal anomaly for 2d non-minimal spinor is shown to be the total derivative. As the result corresponding induced effective action has very simple form. It consists of two terms: Polyakov term and RST term which has been suggested as phenomenological, ad hoc term some time ago [14]. Hence, well-known RST model or, more exactly, some its simple modifications (in classical part) which were thought to be just 2d toy models acquire new remarkable interpretation. This model may be considered now as spherically reduced 4d gravity with spherically reduced minimal spinor (s-wave and large \( N \) approximation is used). In other words, RST-like model may be applied to study 4d gravity (in above approximation). We give the
example of such sort showing the possibility to realize quantum corrected SdS (Nariai) BH in reduced Einstein gravity with quantum correction due to large $N$ spinors. Actually we work with 2d dilatonic BH which may be re-interpreted as 4d BH without dilaton. It is very interesting that this solution may be also rewritten as cosmological solution which describes 4d Kantowski-Sachs Universe. Corresponding investigation will be presented elsewhere.

Similarly, one can consider anomaly induced effective action for 4d dilaton coupled spinor to investigate quantum dilaton cosmology (no s-wave approximation then). In particular, it would be very interesting to answer: can such fermions help in resolution of singularity problem via realization of non-singular Universes with non-trivial dilaton.

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References


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