KEK Preprint 98-189
November 1998
A

Simulation of Space Charge Effects in a Synchrotron

S. MACHIDA and M. IKEGAMI

SW9904

Workshop on Space Charge Physics in High Intensity Hadron Rings
Long Island, 4-7 May 1998

High Energy Accelerator Research Organization (KEK), 1998

KEK Reports are available from:

Information Resources Division
High Energy Accelerator Research Organization (KEK)
1-1 Oho, Tsukuba-shi
Ibaraki-ken, 305-0801
JAPAN

Phone: 0298-64-5137
Fax: 0298-64-4604
Cable: KEK OHO
E-mail: Library@kekvox.kek.jp
Internet: <http://www.kek.jp>

Simulation of Space Charge Effects in a Synchrotron

Shinji Machida

KEK-Tanashi

Midori-cho, Tanashi-shi, Tokyo, 188-8501 JAPAN, E-mail: shinji.machida@kek.jp

and

Masanori Ikegami

JAERI

Tokai-mura, Naka-gun, Ibaraki-ken, 319-11 JAPAN, E-mail: ikegami@linac.tokai.jaeri.go.jp

Abstract. We have studied space charge effects in a synchrotron with multi-particle tracking in 2-D and 3-D configuration space (4-D and 6-D phase space, respectively). First, we will describe the modelling of space charge fields in the simulation and a procedure of tracking. Several ways of presenting tracking results will be also mentioned. Secondly, it is discussed as a demonstration of the simulation study that coherent modes of a beam play a major role in beam stability and intensity limit. The incoherent tune in a resonance condition should be replaced by the coherent tune. Finally, we consider the coherent motion of a beam core as a driving force of halo formation. The mechanism is familiar in linac, and we apply it in a synchrotron.

INTRODUCTION

Space charge force is a major source of emittance increase of beam core, halo formation, and beam loss. Because of its nature of electromagnetic force proportional to $1/\beta\gamma^2$, it is significant only at low energy. In a ring accelerator, especially in a synchrotron, space charge effects are measured with tune shift as a result of its defocusing force against focusing one by an external magnetic lattice. A typical value of tune shift in a synchrotron at the low energy end is around -0.1 to -1.

Space charge force is a function of beam size in transverse and longitudinal planes as well as number of particles in a bunch. That makes the analysis of space charge effects difficult. At the low energy end of synchrotron, the number of particles is gradually increased as a function of turn number throughout an injection process. Once the beam emittance growth occurs due to the effects, the beam is no longer affected by the same space charge force as before. The transverse emittance is sometimes manipulated by the so-called phase space painting. The longitudinal bunch length and peak intensity strongly depend on an rf voltage pattern. Therefore, at this moment simulation study is the only way to model the effects and analyze beam behavior in a self-consistent way.

We have been developing a simulation code named Simpsons (Simulation of proton

synchrotrons). Simpsons tracks multi particles and a single particle in a synchrotron with space charge effects. There are two versions; one models a beam in 6-D phase space with acceleration and the other assumes a coasting beam and tracks only transverse 4-D phase space coordinates. Its ultimate goal is to simulate beams self-consistently in proton (or ion) synchrotrons where space charge effects play a significant role. It also includes time dependent parameters such as rf voltage, bending fields, injection orbit (for phase space painting), power supply ripple, etc.

In this paper, we first explain the way of simulation. Secondly, we discuss two subjects as a demonstration of Simpsons. One is a coherent motion and its role in beam stability. The other is halo formation due to coherent oscillations of the core.

MODELLING OF SPACE CHARGE EFFECTS

Field Calculation

A calculation of space charge fields is based on a particle-in-cell (PIC) method (1). As a coordinate system, cylindrical one (r, φ, z) for a 3-D beam and polar one (r, φ) for a 2-D beam are taken. Fractional charge of each macro particle is allocated to grid points nearby according to area weighting. Poisson equation is then solved with a boundary condition of circular cross sectional beam pipe. A contribution from the magnetic field is simply taken into account with a $1/\gamma^2$ factor.

Under the Lorentz gauge, a scalar potential satisfies,

$$\Delta\phi - \epsilon_0\mu_0 \frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad (1)$$

We introduce the ordering (2),

$$\frac{\partial^2}{\partial z^2}\phi \sim \epsilon_0\mu_0 \frac{\partial^2\phi}{\partial t^2} \ll \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \phi \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \phi \quad (2)$$

Then,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \phi \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \phi = -\frac{\rho}{\epsilon_0} \quad (3)$$

Once a charge distribution at grid points is obtained, it is Fourier transformed in the azimuthal direction.

$$\phi = \sum_m \phi_m \exp(im\varphi) \quad (4)$$

$$\frac{\rho}{\epsilon_0} (\equiv n) = \sum_m n_m \exp(im\varphi) \quad (5)$$

with the inverse transform,

$$\phi_m = \frac{1}{2\pi} \int_0^{2\pi} \phi(r, z, \varphi) \exp(-im\varphi) d\varphi \quad (6)$$

$$n_m = \frac{1}{2\pi} \int_0^{2\pi} n(r, z, \varphi) \exp(-im\varphi) d\varphi \quad (7)$$

Note that, $n_{-m} = n_m^*$. Then,

$$n_m(r, z) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \phi_m(r, z) \right) - \frac{m^2}{r^2} \phi_m(r, z) \quad (8)$$

The general solution to the equation ($m \geq 0$) is,

$$\phi_m = \alpha \left(\frac{r}{b} \right)^m + W(r) - \left(\frac{r}{b} \right)^m W(b) \quad (9)$$

On the boundary condition, $\phi_m = 0$ at $r = b$ (b is the beam pipe radius), it becomes

$$\phi_m = W(r) - \left(\frac{r}{b} \right)^m W(b) \quad (10)$$

where

$$W(r, z) = \int_0^r \ln \frac{r}{r'} n_m(r', z) r' dr' \quad m = 0 \quad (11)$$

$$= \frac{r^m}{2m} \int_0^r r'^{(1-m)} n_m(r', z) dr' - \frac{r^{-m}}{2m} \int_0^r r'^{(1+m)} n_m(r', z) dr' \quad m \neq 0 \quad (12)$$

where $\phi_{-m} = \phi_m^*$ is used to compute ϕ_m for $m < 0$.

We replace the integral with summation at grid points. The electric fields are

$$E_r = \sum_m \frac{\partial}{\partial r} \phi_m \exp(im\varphi) \quad (13)$$

$$E_\varphi = \sum_m \frac{1}{r} \frac{\partial}{\partial \varphi} \phi_m \exp(im\varphi) \quad (14)$$

$$E_z = \sum_m \frac{\partial}{\partial z} \phi_m \exp(im\varphi) \quad (15)$$

The differentiation is done analytically beforehand except with z .

Tracking Procedures

The independent variable is *time* so that a snapshot of a beam in configuration space is obtained. At each time step, space charge fields are calculated and applied to each macro particle as a impulse kick. Then, the particle position is advanced with newly calculated momentum. Those two computations are repeated until as many turns as specified initially.

The size of grids is recalculated at each time step so that all the macro particles are included with the minimum required grid size. Once a particle amplitude becomes larger than the pipe radius, it is regarded as a lost particle. The number of grid points is optimized with some tests. For example, radial grid points are fixed as 40 in order to reproduce a smooth field curve.

The number of longitudinal grids is determined in such a way that a half FODO cell is divided into at least 5 grids. Since a beam stretches over several transverse focusing periods and transverse beam size is accordingly modulated (3), it is necessary to divide a

beam in longitudinal direction such that harmonic components driven by transverse space charge are included. Once the longitudinal grid size is determined, a time step is calculated as,

$$dt \sim \frac{dz}{\beta c} \quad (16)$$

The number of azimuthal modes is usually taken up to four, which is enough to model octupolar shape in configuration space.

The space charge defocusing force modifies the lattice functions and an optical matching condition. If the force is linear, all the particles feel the same focusing force and therefore the modified matching condition with space charge applies to a whole beam. In reality, however, a distribution is not uniform and nonlinear space charge force cannot make the whole beam matched. In the simulation, the rms envelope equations including space charge force are solved to find the matched optical condition for the rms emittance. In the 3-D simulation, a peak current is assumed to solve the envelope equations so that only the center of a bunch is matched.

As an initial particle distribution, one can specify several kinds, such as K-V, waterbag, parabolic, and gaussian for transverse planes, and uniform, parabolic, and gaussian for longitudinal one. We adopt a concept of *equivalent beam* so that the rms emittance is fixed when we compare two or more distributions.

Presentation of Simulation Results

Simulation results are presented in several ways. Plotting of macro particles in phase space every n turns (n should be reasonably large unless a particle distribution changes very quickly) is most common one. With its projection to position and gradient axes, it shows the particle distribution and its evolution in the most primitive way.

Some statistical quantities, such as second or higher order moments, can be derived from the particle ensemble. An example is the rms emittance defined by the second moments and their cross term. When a machine has its physical acceptance, we can define a fraction of particles which fits in the acceptance. A fraction of particles outside of the acceptance can be regarded as particle loss. Then, one can trace particle loss as a function of time or turn.

Instead of dealing with all the macro particles, we also see a single particle trajectory, so-called Poincare map. Several particle trajectories with different initial coordinates reveal phase space structure. In a synchrotron, it is common to look at the single particle coordinates at one fixed point in a ring. As a tracking study in linac, however, it is also possible to plot coordinates with other periodicity such as the one introduced by initial mismatch or a coherent mode oscillations.

A single particle trajectory gives a tune of individual particle. It is defined as a frequency at which the following quantity has a peak as shown in Fig. 1 (left) (4).

$$m_j^{klm}(\nu) = \frac{1}{n_{\max}} \sum_n^{n_{\max}} x_j^k(n) y_j^l(n) z_j^m(n) \exp(-2\pi i \nu n) \quad (17)$$

FIGURE 1. Fourier spectrum of single particle motion (left) and of second moment (right).

Each individual particle has its own incoherence tune. That results in an incoherent tune spread in a tune diagram, called 'necktie diagram'.

A characteristic frequency of the whole beam is also calculated. One example is a coherent frequency defined in the following. First, a moment of the particle coordinates is calculated every turn,

$$M_{coh}^{klm}(n) = \frac{1}{N} \sum_j^N x_j^k(n) y_j^l(n) z_j^m(n) \quad (18)$$

Then, the frequency spectrum of the moment is

$$M_{coh}^{klm}(\nu) = \frac{1}{n_{max}} \sum_n^{n_{max}} M_{coh}^{klm}(n) \exp(-2\pi i \nu n) \quad (19)$$

The frequency at which the above quantity has a peak is the coherent tune as shown in Fig. 1 (right). The sum of $k+l+m$ defines the order of the coherent modes.

Validation of a Code

The code was tested in several ways in the 2-D simulation. First, the necessary number

FIGURE 2. Number of macro particle dependence on the results in 2-D calculation. When it is more than 10⁴, the rms emittance converges.

FIGURE 3. Evolution of rms emittance and β -functions with initial distribution of K-V. With (left) and without (right) rms matching at injection. The bare tune is (6.84,5.81) and incoherent tune shift is -0.81.

of macro particles are examined as shown in Fig. 2. When one chooses more than 10^4 macro particles, tracking results do not depend on the number. Another check of the code is tracking of the uniform (K-V) distribution. As shown in Fig. 3 (left), the rms emittance and β -functions stay constant as long as the initial matching condition is satisfied. The initial mismatch introduces oscillations of β -functions. The rms emittance growth occurs since the K-V distribution cannot remain the K-V for a numerical reason (Fig. 3 (right)).

When the beam intensity is increased, even the matched K-V distribution becomes unstable as shown in Fig. 4. It is not clear if it becomes unstable due to numerical (artificial) reasons or it is a physical phenomenon.

For the 3-D simulation, validation of the code is not established. We need further study.

FIGURE 4. Unstable K-V beam. For first 300 turns, the beam is stable. Then, suddenly envelopes start to oscillate and rms emittance growth occurs. The bare tune is (5.84,5.81) and incoherent tune shift is -1.22. It is not clear if it is due to numerical reasons.

FIGURE 5. Incoherent tune as a function of beam intensity. Left figure shows horizontal result and right figure does vertical. Particle distribution is uniform (K-V) so that all the particles have the same tune.

SIMULATION OF COHERENT MODE

A study of beam stability with the envelope equations had been made by L. Smith and F. Sacherer in 1960s (5,6). Their essential statement is that amplitude growth occurs when an envelope mode frequency coincides with quadrupole field harmonics. Although their analysis was based on the uniform (K-V) distribution when it first appeared, Sacherer himself found in 1971 that the same equations should be valid if one substitutes rms beam size for sharp edge beam size of the K-V distribution (7). Extension of their analysis to non-uniform particle distributions and also to nonlinear error fields is not trivial,

FIGURE 6. Tune of coherent quadrupole mode (left) and rms emittance at 512 turns after injection (center and right). Upper figures show horizontal results and lower ones vertical. In both planes, rms emittance growth is observed when the coherent quadrupole tune becomes integer.

FIGURE 7. RMS emittance growth before and after harmonic correction. The growth is suppressed with the correction. Horizontal (left) and vertical (right) planes.

but can be done numerically.

Let us start from the coherent quadrupole mode, whose answer we knew by Sacherer. Figure 6 (left) shows the intensity dependence of the coherent quadrupole mode. At zero current, it is just twice as much as the bare tune, that is 6.84 for horizontal and 5.81 for vertical. When the intensity is increased, the coherent tune decreases linearly.

If one compares the slope of the quadrupole mode with the slope of the incoherent tune of Fig. 5, the quadrupole tune shift is about 60% of twice of the incoherent tune shift, that agrees with the analytical prediction by Sacherer, namely 63% (or 5/8).

In addition to the uniform (K-V) distribution, the equivalent beams with different distributions; waterbag and parabolic, follow the same intensity dependence (Fig. 6 (left)). In either particle distribution, the emittance growth and particle loss is observed when the coherent mode frequency becomes integer where quadrupole error harmonics is excited. Since the incoherent tune is already well below the half integer when the coherent one hits the integer, one can conclude that the resonance occurs when the coherent mode frequency satisfies the resonance condition. The emittance growth can be suppressed by harmonic correction of quadrupole error fields (Fig. 7). The beam loss is shown in Fig. 8.

Now, we look at higher order modes and their resonance behavior with external magnetic error fields. Figure 9 (left) shows the coherent mode frequency of sextupole as a function of intensity. At zero current, the coherent mode frequency is three times as much

FIGURE 8. Beam loss due to resonance of coherent quadrupole modes. Results of several initial distributions (left and center) and those with harmonic correction (right).

FIGURE 9. Tune of coherent sextupole mode (left), rms emittance growth (center), and beam loss.

as the bare tune. We assume non-uniform initial distributions: waterbag and parabolic.

When the intensity is increased, the coherent mode frequency decreases. The slope of intensity dependence of the coherent tune is about 72% of the three times of the incoherent tune. With sextupole error fields, the rms emittance growth and beam loss is observed at the intensity where the coherent mode frequency of sextupole becomes integer. Similar to the quadrupole mode, those resonance is explained by the resonance condition of the coherent sextupole mode. Those can be suppressed by harmonic correction of sextupole error fields (Fig. 10). Similarly the coherent octupole mode is shown in Fig. 11.

Until here, image charges play a minor role. Once a beam travels off-center of a beam pipe, however, it becomes resonance driving force similar to the external error fields (8). Figure 12 shows that the emittance growth and beam loss occur when the coherent quadrupole mode becomes integer. Although there is no external field error, the closed orbit distortions destroy the symmetry of image charges and excite the resonance.

BEAM HALO FORMATION IN A SYNCHROTRON

One of trends in linac beam dynamics for last several years is a study of beam halo formation based on the particle-core model (9). The particle-core model assumes time dependent oscillations of a beam core. In a linac case, initial mismatch is regarded as a major source of the core oscillations. When the frequency of core oscillations becomes as twice as that of single particle oscillations, a parametric resonance occurs and the particle at the tail may have a large amplitude, becoming halo.

FIGURE 10. RMS emittance growth (left) and beam loss before and after harmonic correction.

FIGURE 11. Tune of coherent octupole mode (left), rms emittance growth (center), and beam loss.

Although the injection mismatch may be minimized in a synchrotron, there are other sources which introduce time dependent core oscillations. A coherent mode oscillation discussed above is one of them. We have investigated the possible halo formation mechanism in a synchrotron based on the particle-core model and Simpsons.

Let us first look at how the coherent mode oscillations can excite parametric resonance of tail particles. As we have mentioned, the coherent tune shift is smaller than the twice of the incoherent tune shift. Figure 13 (left) schematically depicts the intensity dependence of coherent (quadrupole in this example) and incoherent tune as a function of intensity. Now let us plot both tune as a function of particle amplitude (Fig. 13 (right)). Here we assume a 2-D uniform (K-V) distribution. In the beam core region, the incoherent tune is constant and increasing outside approaching the bare tune. If one compares that with the coherent quadrupole tune in the core, the incoherent tune is always smaller than the half of the coherent one. In the limit of large amplitude, however, the incoherent tune is larger than the half of the coherent one. Therefore, there is a certain amplitude where the incoherent tune is exactly the half of the coherent one.

In order to see the phase space structure in more detail, the particle-core model is applied with the JHF booster parameters. The ring consists of 28 FODO cells. The horizontal and vertical bare tune are 6.8. The beam intensity is chosen so that the coherent quadrupole tune is 13. Since there is no external field error, beam blow up should not

FIGURE 12. Tune of coherent quadrupole mode (left), rms emittance growth (center), and beam loss. There is no quadrupole errors. However, the image charge due to closed orbit distortion excites the quadrupole mode.

FIGURE 13. Incoherent tune in the core and coherent quadrupole tune as a function of beam intensity (left), and as a function of particle amplitude (right). Since twice of the incoherent tune is always smaller than the coherent tune in the core (left), there should be an amplitude outside of the core, where 2 x incoherent tune is equal to coherent one. Fixed point of the parametric resonance locates there.

occur. The coherent tune is adjusted at integer simply because Poincare map can be obtained by looking at particle trajectories at a fixed point in a ring. Otherwise, one has to look at it at an interval according to the periodicity of the coherent mode, which may not coincide with the lattice periodicity.

FIGURE 14. Beam envelope modulations (upper) and phase space structure (lower) based on the particle-core model. Modulation amplitude is 2-5% (left), 15% (center), and 27% (right). We assume that a ring lattice consists of 28 FODO cells and the envelope oscillates 13 times per ring.

FIGURE 15. Phase space trajectory of a particle initially located near the edge of the core. When it is observed, the coherent quadrupole frequency of the core is almost 13 and the incoherent tune of the particle is 6.5.

An amplitude of the coherent oscillations is an unknown parameter. That should depend on the strength of external error fields, distance to a resonance, beam intensity, etc. We have looked at three cases; 2-5%, 15%, and 25%. Phase space structure together with a modulation of horizontal and vertical beam size along the ring are shown in Fig. 14. With relatively low intensity compared with linac (but with typical magnitude in a synchrotron), the particle-core model predicts beam halo due to the coherent oscillations.

We, then, tracked multi particles to see if we can observe the same resonance island. Now, we need to setup external quadrupole error fields to excite the coherent quadrupole mode at $\nu=13$. At the same time, the strength of errors should be small enough so that the oscillations of core should not become unstable in a time scale of the halo formation.

In Fig. 15, we show a single particle trajectory which indicates fixed point in the phase space. The particle is initially located near the edge of the beam core. When the island is observed, the coherent tune is almost 13 and incoherent tune of that particle is 6.5. In conclusion, the parametric resonance and halo formation induced by the coherent quadrupole oscillations are indeed observed in the multi-particle tracking.

REFERENCES

1. For example, Birdsall, C. K. and Langdon, A. B., *Plasma Physics via Computer Simulation*, New York: MacGraw-Hill Book Company.
2. This was suggested by Sloan, M. L.
3. That is true for spallation source type of a synchrotron, where rf harmonic number is a few, whereas the number of focusing period is a few tens.
4. Orlov, Y. and Soffer, A., "Fourier Analysis of Higher Order Coherent and Incoherent Resonances in Beam-Beam Interaction", CLNS 92/1178.
5. Smith, L., "Effects of Gradient Errors in the Presence of Space Charge Forces", Proc. of International Conference on High Energy Accelerators, 897 (1963).
6. Sacherer, F. J., "Transverse Space-Charge Effects in Circular Accelerators", Ph.D. Thesis, 1968.
7. Sacherer, F. J., "RMS Envelope Equations with Space Charge", IEEE Trans. Nucl. Sci. **18**, 1105 (1971).
8. Baartman, R., in this proceedings.
9. The following reference summarizes the literature on the particle-core model, Okamoto, H. and Ikegami, M., "Simulation Study of Halo Formation in Breathing Round Beams", Physical Review E **55**, No.4, 4694 (1997).