Instability of two-dimensional heterotic stringy black holes

Mustapha Azreg-Aïnou*

Eastern Mediterranean University, Department of Physics,
Gazimagusa, North Cyprus,
(Via Mersin 10, Turkey)

Abstract

We solve the eigenvalue problem of general relativity for the case of charged black holes in two-dimensional heterotic string theory, derived by McGuigan et al. For the case of $m^2 > q^2$, we find a physically acceptable time-dependent growing mode; thus the black hole is unstable. The extremal case $m^2 = q^2$ is stable.

1 Introduction

Two-dimensional low-energy string theory admits several black hole solutions. Some of them are neutral, like the black hole solution to the bosonic closed string theory on the sphere [1, 2, 3] and [4]. The others are charged [5], open string Born-Infeld black holes or heterotic string black holes. These two classes of charged black holes correspond to two different alternative ways to couple a gauge field to string theory (with a non-minimal coupling of the scalar dilaton field to gravity). They have very interesting and complex spacetime geometries with more than two horizons depending on the number of string loop corrections included [5].

Their mass, charge, temperature and entropy have been evaluated in different ways [5, 6, 7]. In [6], and then elegantly in [7], the authors used thermodynamic arguments, but different methods, to compute the entropy and other thermodynamic quantities. While, recently, Teo [8] has extended the work of Hyun [9], and by Sfetsos & Skenderis [10], and derived the entropy for the two-dimensional black hole by explicitly establishing the U-duality between the two-dimensional black hole and the five-dimensional one. Very recently, a similar work [11], calling on, this time, a sequence of S and T-S-T duality transformations in four dimensions, has been performed leading to the same expressions of the entropy for two-dimensional black holes.

It is now obvious that the thermodynamics of higher-dimensional black holes may be described by that of lower-dimensional black holes and vice versa. And one question arises: Is this observation extendible to other physical properties? In other words: May the much simpler physics of lower-dimensional black holes describe faithfully that of higher-dimensional black holes? Suggestions that higher-dimensional black holes might be governed by the same conformal field theories as lower-dimensional ones have been put forward [8, 12, 13].

*E-mail address: azreg.as@mozart.emu.edu.tr, azreg@hermes.unice.fr
Studying the propagation of scalars, Satoh [13] has shown some common features between higher- and lower-dimensional black holes. Almost similar studies carried out in [14] for four-dimensional Einstein-Maxwell-Dilaton black holes (without potential for the dilatonic field) and in [15] for four-dimensional cold scalar-tensor black holes, respectively, showed that for some arrangement of the parameters (infinite temperature and vanishing entropy), the black holes even become repulsive. We will prove almost similar properties for two-dimensional black holes [16]. In [14], the authors went further in their conclusions and argued that certain classes of four-dimensional Einstein-Maxwell-Dilaton black holes behave as follows: Some, like extended objects (liquid drop), exhibit time delay for the re-emission, and others, like elementary particles, exhibit no time delay for the re-emission of low-energy quanta. But, before an interpretation in terms of liquid drop or elementary particles can be achieved, and before any further development, the condition of stability must be fulfilled.

In this paper we address the question of stability of a class of two-dimensional stringy black holes. We mention that the small fluctuations of the Witten solution [1] and other solutions have been studied by some authors who gave no conclusion regarding the stability of the solutions investigated [17]. While in [18] the authors, investigating the linearized time-dependent perturbations of two-dimensional stringy black holes in the presence of real tachyon field, concluded that there is no time-dependent solutions with horizons shrinking to a point; rather they expand with increasing time, and are thus unstable solutions. For the first time, to our knowledge, the stability of two-dimensional heterotic stringy black holes has been studied, so far, explicitly by Hsu et al [19]. The authors’ analysis, which we believe to be flawed, does not deal with the problem of stability as it is defined in general relativity, especially in the presence of dilatonic field.

Motivated by all these developments and analyses mentioned above, we intend to re-examine the case of heterotic stringy black holes by completing their stability analysis, on the one hand, and clarifying the eigenvalue problem of general relativity on the other hand. The paper is organized as follows. In section 2 we review the black hole heterotic string solution. In section 3 we write the linearized equations governing the evolution of small perturbations and the resulting Schrödinger-like equation for a specific function related to the small perturbations, together with some useful relationships between them. In section 4 we define the general-relativity eigenvalue problem corresponding to the stability analysis, which is, in general, different from the Schrödinger eigenvalue problem obtained in section 3. We solve the former and draw a parallel with the stability investigation given by Hsu et al [19].

2 Black holes in two-dimensional heterotic string theory

The two-dimensional metric of heterotic stringy black hole can be parametrized by [5]
\[
\begin{align*}
\text{ds}^2 &= g(r) \, dt^2 - \frac{1}{g(r)} \, dr^2 \\
&\quad \text{(1)}
\end{align*}
\]

with
\[
\begin{align*}
g(r) &= 1 - 2 m e^{-Q r} + q^2 e^{-2 Q r} \\
&\quad \text{(2)}
\end{align*}
\]

the dilaton field background is that one of a neutral stringy black hole,
\[
\phi(r) = \phi_0 - \frac{Q}{2} r \\
&\quad \text{(3)}
\]

while the gauge field \( F_{tr} \equiv f \) reads
\[
f(r) = \sqrt{2} Q q e^{-Q r}.
\]

The metric, dilatonic and gauge fields given by equations (1 \( \rightarrow \) 4) are solutions to the equations of motion
\[
\begin{align*}
R_{\mu\nu} - 2 \phi_{;\mu;\nu} - \frac{1}{2} F_{\mu\rho} F^\rho_{\ \nu} &= 0 \\
\left( e^{-2\phi} F_{\mu\nu} \right)_{;\nu} &= 0 \quad \text{(5)}
\end{align*}
\]

in our conventions \( R_{\mu\nu\rho\sigma} \) is defined so that \( R_{trtr} = -g_{r,r}/2 \) or, equivalently, \( R_{rr} = g_{r,r}/2g \), which are derived from the effective action [5]
\[
S = \int d^2 x \sqrt{-G} e^{-2\phi} \left[ R - 4 \left( \nabla \phi \right)^2 - c - \frac{1}{4} F^2 \right].
\]

The parameters \( m, q \) are related to the mass, \( \mathcal{M} \), and the electric charge, \( \mathcal{Q} \), respectively, of the black hole by
\[
\begin{align*}
\mathcal{M} &= 2 Q m e^{-2\phi_0} \\
\mathcal{Q} &= \sqrt{2} Q q e^{-2\phi_0}
\end{align*}
\]

while \( Q, \phi_0 \) are just integration constants, and \( c \) is the central charge [4]. The sign of \( Q > 0 \) is fixed by the asymptotic flatness condition at \( r = +\infty^* \), while its value is related to \( c \) since the third equation in (5) requires
\[
c = -Q^2.
\]

The entropy and the temperature are given in terms of \( \mathcal{M}, \mathcal{Q} \), respectively by [7]
\[
S = \frac{2 \pi}{\mathcal{Q}} \left( \mathcal{M} + \sqrt{\mathcal{M}^2 - 2 \mathcal{Q}^2} \right)
\]

\*Note that the case \( c = 0 \) is trivial.
\[
T = \frac{Q}{2\pi} \frac{\sqrt{M^2 - 2Q^2}}{M + \sqrt{M^2 - 2Q^2}}.
\]

According to [14], the condition for the thermal description to break down is

\[
\left( \frac{\partial T}{\partial M} \right)_Q \equiv \frac{Q}{4\pi Q^2} \left[ \frac{2(M^2 - Q^2)}{\sqrt{M^2 - 2Q^2}} - 2M \right] \gg 1
\]

is largely fulfilled for the extremal black hole \( m^2 = q^2 \) \((M^2 = 2Q^2)\); a small change in the mass is accompanied by a huge change in the temperature of the hole, the thermal description becomes ambiguous.

The curvature singularity is at \( r = -\infty \) since \( R = -g_{rr} \). Putting \( R(r) = e^{-Qr} \), the roots of the equation \( g(R) = 0 \) read

\[
R_{\pm} = \frac{m \pm \sqrt{m^2 - q^2}}{q^2}.
\]

There are event horizons (apparent singularities) at \( r_{\pm} \) related to \( R_{\pm} \) by

\[
R_+ = e^{-Qr_-} \quad R_- = e^{-Qr_+}.
\]

From now on we shall consider the case \( m^2 \geq q^2, m > 0 \) and \( q \) real, corresponding to two real values \( r_+ \geq r_- > 0 \) with two event horizons.

3 The linearized field equations for small perturbations

In the standard terminology employed by Chandrasekhar [20], the perturbation functions of the background fields are grouped into two sets, polar and axial perturbations. In our case, the polar set is that one which preserves the “initial” diagonal form of the background metric (1). Now, for any two-dimensional initial metric configuration, the most sufficiently general form of the associated perturbed metric can always be brought to a diagonal form. The proof is given in the chapter 2 of [20] in the case of a metric with a positive- or negative-definite signature (+, + or −, −) and is manifestly generalized to our case with a signature (+, −). Consequently, two-dimensional metric perturbations are entirely polar. In our case the perturbed metric can be written as

\[
ds^2 = [g(r) + T(r,t)] dt^2 - \left( \frac{1}{g(r)} + Y(r,t) \right) dr^2
\]

and the perturbations of the dilatonic and Maxwell fields are introduced by

\[
\begin{align*}
\phi(r,t) &= \phi(r) + D(r,t) \\
f(r,t) &= f(r) + M(r,t)
\end{align*}
\]

where \( T, Y, D \) and \( M \) are considered to be “relatively” small quantities compared to the background fields. So, this yields an inventory of four variables.
Notice that it is still possible to fixe the gauge and supply the three linearized field equations (see Eqs. (13 → 14) below) with an extra equation, which yields an inventory of four equations. As shown in the appendix, we can always bring the diagonal sufficiently generalized two-dimensional metric to a Schwarzschild-like form where \( g_{rr} = -1/g_{tt} \). In our case, we have \([g(r)+T(r,t)][(1/g(r))+Y(r,t)] = 1\). Keeping only linear terms, we get

\[
g Y + \frac{T}{g} = 0.
\] (12)

The basic three linearized field equations read\(^\dagger\) (substitute (10), (11) into (5))

\[
\ell^{(G)}_{tr} = g \phi_r Y - 2 D_r + \frac{g_x}{g} D = 0 , \quad \bar{\ell}^{(A)}_r = -2 f D + M = 0 \tag{13}
\]

\[
\ell_{rr} + \frac{\ell^{(g)}}{2g} + \frac{f}{2} \ell^{(A)}_r + \frac{g_x}{g} \ell_{tr}^{(G)} - 2 \phi_r \ell_{tr}^{(G)} - \bar{\ell}_{tr,r}^{(G)} = - \frac{2}{g^2} D_{,t,t} - 2 D_{,r,r} = 0 \tag{14}
\]

where equation (12) has been used\(^\dagger\).

In terms of the tortoise \( r^* \)-coordinate, the equation (14) can be brought to Schrödinger-like equation via the transformations \( dr^* \equiv (1/g(r)) dr, \psi(r^*, t) \equiv D/\sqrt{g} \). This yields [19]

\[
-\psi_{,r^*,r^*} + \left( \frac{3}{4} g^2 (g_{r^*})^2 - \frac{1}{2 g} g_{r^*,r^*} \right) \psi = \psi_{,t,t}
\] (15)

where the function between round brackets is the potential \( V(r^*) \). It can be put on the following form which we shall use in section 4

\[
V = \frac{1}{2} \left( \frac{1}{2} (g_r)^2 - g g_{r,r} \right).
\] (16)

### 4 Stability analysis

Starting with small perturbations of the background fields at \( t = 0 \), we examine if they may increase indefinitely in time. For stationary perturbations of the form \( D(r, t) = D(r) e^{i \omega t} \), etc, this means searching for physically acceptable solutions to the linearized field equations for \( \omega \) imaginary, i.e \( k^2 \equiv -\omega^2 \). If such solutions exit, then an initially small perturbation grows exponentially in time, and the background solution is unstable. Conversely, the solution is stable if all the eigenvalues \( \omega \) are real.

Imposing that the perturbed fields should be regular is a natural definition of physically acceptable perturbations of regular background fields. This is not the case of the static component \( g_{rr} \) of the metric which diverges at \( r = r_\pm \). In this

\(^\dagger\)The Eqs. (13 → 14) have been given in [19], where “\( \partial/\partial t \)” has been replaced by “\( i \sigma \)”, but without specifying the combinations in the l.h.s leading to these equations.

\(^\dagger\)In Eqs. (13 → 14), \( \ell^{(G)}_{\mu\nu}, \ell^{(A)}_\mu, \ell^{(\phi)}_\phi \) are the linear variations of the l.h.s of Eqs. (5), respectively, and \( \bar{\ell}^{(G)}_{\mu\nu}, \bar{\ell}^{(A)}_\mu, \bar{\ell}^{(\phi)}_\phi \) are their primitives with respect to the time coordinate.
case, we suppose that the relative perturbations $T(r)/g(r), D(r)/\phi(r), M(r)/f(r)$ and $Y(r)g(r) = (Y/(1/g))$ are bounded [21, 22] in the range $]r_+, +\infty[$, which is the physically interesting region

$$|gY| = |T/g| < \infty \quad |D/\phi| < \infty \quad |M/f| < \infty \quad (r \in ]r_+, +\infty[). \quad (17)$$

This minimal requirement – referred to as the weak boundary condition in [21] – defines the general-relativity eigenvalue problem, which is, as we shall see, not necessarily equivalent [23] to the requirement that the auxiliary function $\psi$, the solution to equation (15), should be square integrable in the range $]r_+, +\infty[$, which is the Schrödinger eigenvalue problem.

For a time dependence of the form $T(r,t) = T(r)e^{i\omega t}, Y(r,t) = Y(r)e^{i\omega t}, D(r,t) = D(r)e^{i\omega t}, M(r,t) = M(r)e^{i\omega t}$, with $\omega = -ik$ and $k > 0$, the three equations (13), (14) take the form

$$gY = \frac{1}{\phi,\rho} \left( 2D_\rho - \frac{g_\rho}{g} D \right) \quad (18)$$

$$\frac{M}{f} = 2D \quad (19)$$

$$D_{\rho,\rho} + \frac{k^2}{g^2} D = 0 \quad (20)$$

where the new variable $\rho \in ]0, +\infty[$ is a translation of $r : \rho \equiv r - r_+$.

For $m^2 > q^2$, consider the case where

$$k = k_0 \equiv \sqrt{m^2 - q^2} Q R_- \quad (21)$$

(it is straightforward to show that $k_0 = 2\pi T$). The general solution to the equation (20) can be put into the form

$$D(\rho) = D_0(\rho) \left( a + b \int^\rho \frac{d\rho'}{D_0^2(\rho')} \right) \quad (22)$$

where $a, b$ are real constants. The function $D_0(\rho)$ is a particular solution to the equation (20) developed by

$$D_0(\rho) \equiv \sqrt{\rho} \left( 1 + \frac{L}{2} \rho + \cdots \right) \quad (23)$$

where $L \equiv Q^2(3mR_- - 2)/(2k_0)$ and the function between round brackets is an integer series in $\rho$.

The asymptotic behaviour for $\rho \to +\infty$ of the function $D(\rho)$ can be obtained by directly considering the equation (20), while its behaviour at $\rho = 0_-$ can be deduced from equation (22). For $\rho \to +\infty$, equation (20) reads

$$D_{\rho,\rho} + k_0^2 D \simeq 0 \quad (24)$$
which is solved by
\[
D(\rho) \simeq c_1 \cos k_0 \rho + c_2 \sin k_0 \rho
\]  \hspace{1cm} (25)

where \(c_1, c_2\) are two arbitrarily real constants. This \textit{harmonic} behaviour for \(\rho \to +\infty\) is transmitted via equations (19), (18) and (12) to the relative perturbations \(M/f, gY = -T/g\). Hence, the relative perturbations are bounded at \(\rho = +\infty\).

Substituting (23) into (22) and keeping only up to the first powers in \(\rho\), we get for \(\rho \to 0_\pm\)
\[
D(\rho) \simeq \sqrt{\rho} \left( a + \left( a - 2b \right) \frac{L \rho + \cdots}{2} \right) + \sqrt{\rho} \left( b + \frac{b}{2} L \rho + \cdots \right) \ln \rho
\]  \hspace{1cm} (26)

which is bounded. Upon substituting this function into equation (18), we obtain the following expression for \(gY = -T/g\) as \(\rho \to 0_\pm\)
\[
gY(\rho) \simeq -2 \frac{b}{Q} \frac{1}{\sqrt{\rho}} (2 - L \rho + \cdots)
\]  \hspace{1cm} (27)

which diverges, while the ratio \(M/f (= 2D)\) remains bounded. To deal with this situation we have to choose \(b = 0\) in equation (26), which is possible by suitably fixing the ratio \(c_1/c_2\) in equation (25).

Hence, for \(k = k_0\) and for a suitable value of the ratio \(c_1/c_2\), not only are the relative perturbations \(D/\phi, M/f, gY, T/g\) bounded at both \(\rho = 0_\pm, \rho = +\infty\) but the perturbations themselves as well. These perturbations are also bounded everywhere in the range \([0, +\infty]\) of the variable \(\rho\). This can be shown by developing the perturbations around any point \(\rho_0 \in [0, +\infty]\). Since the background fields \(g, \phi, f\) are regular functions for all \(\rho \in [0, +\infty]\), the general behaviour –depending on two arbitrary constants– of the perturbations given by the Frobenius Method is bounded in the neighbourhood of \(\rho_0\). This mode of perturbation is physically acceptable. Since it grows in time like \(e^{k_0 t}\), the static background black hole solution is unstable. The function \(D_0\) with the corresponding functions \(T_0, Y_0, M_0\) and the eigenvalue \(k_0\) constitute the solution to the general-relativity eigenvalue problem.

The case \(m^2 = q^2\) is quite different. For \(\rho \to 0_\pm\), the equation (20) behaves as
\[
D_{,\rho,\rho} + \frac{k^2}{Q^4 \rho^4} D \simeq 0.
\]  \hspace{1cm} (28)

The general solution to the exact equation –replacing “\(\simeq\)” by “=”– is given by the r.h.s of the following equation expressing the behaviour of \(D\) when \(\rho \to 0_\pm\)
\[
D(\rho) \simeq a \rho \cos \left( \frac{k}{Q^2 \rho} \right) + b \rho \sin \left( \frac{k}{Q^2 \rho} \right)
\]  \hspace{1cm} (29)

\((a, b\) are real constants), which is well bounded \(\forall k > 0\), giving rise to a bounded behaviour of the ratio \(M/f (= 2D)\) but to an unbounded ratio \(gY = -T/g\) \(\forall a, b, k > 0\), as shown by
\[
gY = -\frac{T}{g} \simeq \frac{4}{Q^2} \frac{1}{\rho} \left[ b \cos \left( \frac{k}{Q^2 \rho} \right) - a \sin \left( \frac{k}{Q^2 \rho} \right) \right].
\] (30)

Consequently, there are no physically acceptable growing modes for the case \( m^2 = q^2 \); thus it is stable.

Notice that the uncharged case \( q = 0, m > 0 \), which is a special case of \( m^2 > q^2 \), is unstable, for the equations (22 \( \rightarrow \) 27), as well as the following discussion, are still valid with, in this situation, \( k_0 = Q/2, L = -Q/2 \).

We now consider the Schrödinger-like equation (15). Putting \( \psi(r^*, t) = \psi(r^*) e^{kt} (k > 0) \) in the equation, we get

\[
-\psi_{,r^*} + (V(r^*) - k^2) \psi = 0
\]

where \( r^* \in ]-\infty, +\infty[ \), the horizon \( r_+ \) is at \( r^* = -\infty \) and the flat region is at \( r^* = +\infty \). We can show analytically that \( V(r^*) > 0 \) for all \( r^* \in ]-\infty, +\infty[ \). In fact, we have \( V(r^* = -\infty) = k_0^2, V(r^* = +\infty) = 0 \). From (16), we obtain

\[
V_{,r} = -\frac{g}{2} g_{,r,r} = Q^2 \left( 4q^2 e^{-Qr} - m \right) g(r) e^{-Qr}
\]

with \( g(r) > 0 \) for \( r_+ < r < +\infty \). Hence, if \( 16q^2/7 \geq m^2 \), the potential has a maximum at \( r_{\text{max}} \geq r_+ \), it is then positive. If \( m^2 > 16q^2/7 \), the potential is a monotonic function decreasing from \( k_0^2 \rightarrow 0 \) when \( r \) runs from \( r_+ \rightarrow +\infty \).

Since \( V(r^*) > 0 \) for all \( r^* \in ]-\infty, +\infty[ \), Schrödinger equation (31) does not admit bound states (with \( k^2 > 0 \)) vanishing at both \( r^* = -\infty, r^* = +\infty \). This has led Hsu et al [19] to the conclusion that the background solution is stable, they have then studied the Schrödinger eigenvalue problem for the auxiliary function \( \psi \), and not the general-relativity eigenvalue problem we have formulated for the relative perturbations. We notice that Schrödinger equation (31) admits, for all \( 0 < k^2 < k_0^2 \), finite unbound states vanishing exponentially at \( r^* = -\infty \) and oscillating at \( r^* = +\infty \), which are thus non-square integrable but can be normalized [24]. It also admits a finite unbound state for \( k^2 = k_0^2 \) behaving as a constant at \( r^* = -\infty \) and oscillating at \( r^* = +\infty \).

5 Conclusion

Our analysis, given in section 4, has led to the conclusion that the two-dimensional heterotic stringy black holes are unstable \( (m^2 > q^2) \), except the extremal case \( (m^2 = q^2) \), which is stable. For \( m^2 > q^2 \) \( (M^2 > 2Q^2) \), we have found a bounded mode, solution to the general-relativity eigenvalue problem, growing with time as \( \exp(2\pi T t) \), where \( T \) is the temperature of the hole. We have seen that the perturbations of the physical background fields associated with this mode are all bounded; there is then no objection that this mode is well accepted physically. We have also seen that the general-relativity eigenvalue problem, we have solved in section 4, is not equivalent to the Schrödinger

\[§\text{This has been done numerically in [19].}\]
eigenvalue problem. The latter has been outlined in the last paragraph of section 4 of the present paper, as well as in [19], where the stability analysis has rested on an auxiliary function $\psi$, whose direct physical meaning is not transparent, and led to the conclusion that all the black holes ($m^2 \geq q^2$) are stable, which agrees partially with our conclusion. In a subsequent work, we shall investigate the stability of all the two-dimensional stringy black holes.

The case of four-dimensional stringy black holes has been investigated in [25, 26]. The results of the stability analysis given in [25] are consistent with those of the present case, and different from those given in [26].

Appendix

To make this article self-contained we show in this appendix how to bring a two-dimensional diagonal metric to Schwarzschild-like metric. The functions introduced here have nothing to do with those of the same notation introduced in the text. We start with the metric

$$\begin{align*}
ds^2 &= g_{tt}(r,t) \, dt^2 + g_{rr}(r,t) \, dr^2 \\
    &= F(r,t) \, dt^2 - H(r,t) \, dr^2
\end{align*}
$$

(A.1)

and introduce new coordinates $t', r'$ by

$$
t' = T(t,r) \quad r' = R(t,r)
$$

so that the new metric preserves its diagonality, $g'_{\nu\nu'} \equiv 0$, and is Schwarzschild-like metric, $g'_{\nu\nu'}g'_{\nu'\nu} = -1$. These two conditions are given respectively by

$$
HR_tT_t - FR_rT_r = 0 \quad \text{(A.2)}
$$

and

$$
[F(T_r)^2 - H(T_t)^2][H(R_t)^2 - F(R_r)^2] = \delta^4 \quad \text{(A.3)}
$$

where

$$\delta \equiv T_t R_r - T_r R_t (\neq 0). \quad \text{(A.4)}$$

Developing (A.3) with the help of (A.2), we arrive at $\delta^2 = HF$, or equivalently

$$T_t R_r - T_r R_t = \pm \sqrt{HF}. \quad \text{(A.5)}
$$

Combining (A.2), (A.5) we obtain

$$
(HR_t).T_t - (FR_r).T_r = 0
$$

$$
R_r.T_t - R_t.T_r = \pm \sqrt{HF}. \quad \text{(A.6)}
$$

The system (A.6) is solved with respect to $T_t, T_r$ by
\[ T_t = \pm \frac{F \sqrt{HF} R_t}{F (R_t)^2 - H (R_t)^2} \quad T_r = \pm \frac{H \sqrt{HF} R_t}{F (R_t)^2 - H (R_t)^2}. \] (A.7)

The integrability condition of the system (A.7) yields
\[ \Delta R = [\ln |R_\rho R^\rho|]_{,\mu} R^\mu \] (A.8)

where
\[ \Delta R \equiv \frac{1}{\sqrt{G}} \left( \sqrt{G} \ g^{\mu \nu} R_{,\rho} \right)_{,\rho} \]
\[ R^\mu \equiv g^{\mu \nu} R_{,\nu} \]

with \( G \equiv |\det g_{\mu \nu}| = FH \). Similarly, we get by symmetry
\[ \Delta T = [\ln |T_\rho T^\rho|]_{,\mu} T^\mu. \] (A.9)

It is then possible to bring the metric (A.1) to a diagonal Schwarzschild-like form by a coordinate transformation \( t' = T(t, r), \ r' = R(t, r) \), where \( R(t, r), \ T(t, r) \) are solutions to the differential equation
\[ \Delta \Psi = [\ln |\Psi_\rho \Psi^\rho|]_{,\mu} \Psi^\mu. \] (A.10)

**Acknowledgements**

I would like to thank Yost and Carlini for sending copies of the reprints [5, 25].

**References**


Cvetič M and Larsen F 1997 *Nucl. Phys.* B **506** 107


[16] Azreg-Aïnou M *in preparation*


