Coupling Impedances for Perforated Beam Pipes with General Shape from Impedance Boundary Condition

Stefania Petracca*

Abstract

We introduce an equivalent wall impedance to describe the electromagnetic boundary conditions at perforated pipe walls. The new impedance boundary condition, together with our general formulae for computing longitudinal and transverse beam coupling impedances in complex heterogeneous pipes, provides a good trade-off between computational accuracy and ease.

Geneva, Switzerland

February 3, 1999

On leave of absence from Università' del Sannio, Benevento and I.N.F.N. Salerno, Italy.

1 Introduction

In the foreseen LHC design [1], the stainless steel vacuum chamber (cold bore) will be kept at 1.9 K using superfluid Helium, and protected from synchrotron radiation by a beam screen cooled at some 4.5 to 20 K . Gas desorption due to synchrotron radiation, and subsequent surface deposition limits the pumping efficiency of the beam pipe vacuum system, unless many holes or slots are drilled in the beam screen wall, allowing for a transfer of the excess gas to the 1.9 K cold bore, where the pumping capacity is adequate. At the present stage of the project, the total number of holes or slots should be as large as $10^7 - 10^8$ ($10^2 - 10^3$ holes/slots per meter), with typical sizes of ~ 2 mm diameter (holes) or 1.5×8 mm (slots). The effect of so many holes/slots on the beam dynamics and stability, e.g. in terms of coupling impedances, is a fundamental issue and has been carefully investigated, both theoretically [2]-[6], and experimentally [7], [8].

In this paper we introduce a new (local) impedance boundary condition (b.c.), of the Leontóvich type [9], to describe *perforated* pipe walls. The latter can be used within the general framework presented in [10] and summarized in *Sect. 2* to obtain analytic estimates, based on reciprocity formulae, of the longitudinal and transverse coupling impedances for *heterogenous* beam pipes with *complex* geometry. The coupling impedances of a *simple, unperturbed* pipe assumed known, can be related to those of another pipe differing from the former by some *perturbation* in the boundary geometry and/or constitutive properties. The involved computations are straightforward (geometric and constitutive effects are embodied in separate factors) and reasonably accurate.

In the usual approach, based on the first order approximation of Bethe's theory [11], computation of the coupling impedances due to the presence of the holes in the pipe wall requires modal expansion of the field in the pipe geometry. This is possible only for geometries where the Helmholtz (potential) equation is separable. On the other hand, our (perturbative) approach (reciprocity formulae + impedance boundary condition + Bethe approximation) introduces an equivalent wall impedance to describe the electromagnetic boundary conditions at a perforated pipe wall. Therefore its advantage lies in the possibility of considering transverse pipe geometries where the modal analysis cannot be performed.

In *Sect. 3* we introduce an impedance boundary condition appropriate to a thin perfectly conducting pipe wall with many (non interacting) electrically *small* holes in free space starting from a *heuristic* argument. In *Sect. 4* we derive the same result by solving a rigorous boundary value problem. Possible model improvements are considered in *Sect. 5*, including: (i) holes in a thick wall, (ii) interacting holes, and (iii) perforated beam pipes in a coaxial lossy tube. In *Sect. 6* we apply our formalism to a possible *LHC* geometry and compare our results to those obtained by other Authors using a different approach, while in *Sect. 7* we compute the parasitic loss. Conclusions follow under *Sect. 8*. Relevant tools

Figure 1: Unit vectors $\hat{u}_c$, $\hat{u}_n$, $\hat{u}_z$ relevant to eqs. (1) and (2).

and definitions are collected in *Appendix A* to *D*.

2 Coupling Impedances in Complex Pipes

According to [10] the specific longitudinal and transverse beam coupling impedances $Z_{0,\parallel}(\omega)$ and $\bar{\bar{Z}}_{0,\perp}(\omega)$ of a *simple, unperturbed* pipe (e.g., circular, perfectly conducting) assumed known, can be related to those $Z_{\parallel}(\omega)$, $\bar{\bar{Z}}_{\perp}(\omega)$ of another pipe differing from the former by some *perturbation* in the boundary geometry and/or constitutive properties, as follows:

$$Z_{\parallel}(\omega) - Z_{0,\parallel}(\omega) = \frac{\epsilon_0}{\beta_0 c Q^2} \left\{ Y_0 \oint_{\partial S} Z_{wall} E_{0n}^{(irr.)*}(\vec{r}, 0) \right. \\ \left. \left[\beta_0 E_n^{(irr.)}(\vec{r}, 0) + \beta_0^{-1} E_n^{(sol.)}(\vec{r}, 0) \right] d\ell - \oint_{\partial S} E_{0z}^*(\vec{r}, 0) E_n^{(irr.)}(\vec{r}, 0) d\ell \right\}, \quad (1)$$

$$\bar{\bar{Z}}_{\perp}(\omega) - \bar{\bar{Z}}_{0,\perp}(\omega) = \frac{\epsilon_0}{\beta_0 c Q^2 k} \left\{ Y_0 \oint_{\partial S} Z_{wall} \nabla_{\vec{r}_0} E_{0n}^{(irr.)*}(\vec{r}, \vec{r}_0) \otimes \right. \\ \left. \otimes \nabla_{\vec{r}_1} \left[\beta_0 E_n^{(irr.)}(\vec{r}, \vec{r}_1) + \beta_0^{-1} E_n^{(sol.)}(\vec{r}, \vec{r}_1) \right] d\ell + \right. \\ \left. - \oint_{\partial S} \nabla_{\vec{r}_0} E_{0z}^*(\vec{r}, \vec{r}_0) \otimes \nabla_{\vec{r}_1} E_n^{(irr.)}(\vec{r}, 1) d\ell \right\}_{\vec{r}_1=\vec{r}_0=0}, \quad (2)$$

where $c = (\epsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum, $Y_0 = (\epsilon_0 / \mu_0)^{1/2}$ is the vacuum characteristic admittance, ϵ_0 and μ_0 being the vacuum permittivity and permeability, β_0 is the relativistic factor, Q is the total beam charge, $\vec{E}_0^{(sol.)}$, $\vec{E}_0^{(irr.)}$ are the solenoidal and irrotational parts of the electric field in the *unperturbed* pipe, the unit vectors $\hat{u}_c$, $\hat{u}_n$, $\hat{u}_z$ are defined in Fig. 1, $\bar{\bar{Z}}_{\perp}(\omega)$ is a tensor and one assumes an impedance (Leontovich) boundary condition to hold at the pipe wall ∂S :

$$\left| (\bar{\vec{I}} - \hat{u}_n \hat{u}_n) \cdot \vec{E} - Z_{wall} \hat{u}_n \times \vec{H} \right|_{\partial S} = 0, \quad (3)$$

where Z_{wall} is the pipe-wall complex characteristic impedance.

The first integral term on the r.h.s of (1) accounts for the effect of the finite wall conductivity, and is nonzero if and only if Z_{wall} is not identically zero on ∂S . The second integral on the r.h.s. of (1), on the other hand, accounts for the effect of the geometrical perturbation of the boundary, and is non-zero if and only if the *unperturbed* axial field component E_{0z} is not identically zero on ∂S ¹.

For the simplest case of a circular pipe with radius b and uniform wall impedance Z_{wall} one readily obtains from eq. (1):

$$Z_{\parallel} = \frac{Z_{wall}}{2\pi b}, \quad \bar{Z}_{\perp} = \frac{Z_{wall}}{\pi k_0 b^3} (\hat{u}_x \hat{u}_x + \hat{u}_y \hat{u}_y), \quad (4)$$

in agreement with the known exact result [12].

3 Impedance Boundary Conditions at Perforated Beam Pipe Wall: Heuristics

Extensive calculations and estimates for the longitudinal and transverse impedances per unit length in perforated beam pipes have been performed by S. Kurennoy² and R. Gluckstern [2]-[6].

According to these Authors, the longitudinal impedance per unit length of a circular beam pipe with radius b carrying N_{λ} holes per unit length is³:

$$Z_{\parallel} = -j Z_0 k_0 \frac{(\alpha_e + \alpha_m)}{4\pi^2 b^2} N_{\lambda}, \quad (5)$$

where $k_0 = \omega/c$ is the free-space wavenumber, c the light velocity in vacuum, and $\alpha_{e,m}$ the electric and magnetic polarizabilities of each hole⁴. The result (5) is independent of the azimuthal position of the holes, such being the field produced by an axial beam in a circular pipe. Thus, letting:

$$n_{\sigma} = \frac{N_{\lambda}}{2\pi b} \quad (6)$$

represent the number of holes per unit wall area, eq. (5) can be written:

¹The beam impedances are obviously independent of the total beam charge, as the field in (1), (2) are proportional to Q .

²A notable result of Kurennoy's analysis is related to the occurrence of *trapped modes* in the neighbourhood of pipe discontinuities, including holes [13].

³Kurennoy has also computed the (negligible) real correction to (5) resulting from radiation leakage through the holes [3].

⁴Polarizabilities for most hole shapes can be found in [14]-[16].

$$Z_{\parallel} = -jZ_0 k_0 \frac{(\alpha_e + \alpha_m)}{2\pi b} n_{\sigma}. \quad (7)$$

By comparing eq.s (7) and (4), one is led to the heuristic conclusion that a perforated wall could be described by an impedance boundary condition with:

$$Z_{wall} = -jZ_0 k_0 (\alpha_e + \alpha_m) n_{\sigma}. \quad (8)$$

Using (8) in (1) and (2) would allow to estimate the longitudinal and transverse beam impedances under very general assumptions, including e.g. pipes with *unevenly* distributed holes, and complicated transverse geometries. This is indeed the case⁵, as can be seen by considering a pipe with *general* transverse geometry carrying N_{λ} (uniformly spaced) holes per unit length, located at $\ell = \ell_h$, ℓ being the arc-length along the pipe cross-section contour ∂S , for which:

$$n_{\sigma} = N_{\lambda} \delta(\ell - \ell_h). \quad (9)$$

Using (9) in (1), gives

$$Z_{\parallel} = -jZ_0 k_0 (\alpha_e + \alpha_m) e_n(\ell_h) e_n^*(\ell_h), \quad (10)$$

where $e_n(\ell_h) = (Q/\epsilon_0)^{-1} E_n(\ell_h)$, $E_n(\ell_h)$ being the normal electric field at the hole position produced by an axial beam with total charge Q . Equation (10) reproduces exactly Kurennoy's result valid for this most general case [17].

In the next section we shall further support the heuristic result (8) by solving a rigorous EM boundary value problem.

4 Impedance Boundary Conditions at Perforated Beam Pipe Wall: Boundary Value Approach

In this section we consider a TM plane wave $(\vec{E}_i, \vec{H}_i)$:

$$\begin{aligned} \vec{k}^{(i)} &= k_0 (\sin\theta \hat{u}_x + \cos\theta \hat{u}_z), \\ \vec{H}^{(i)} &= H_0 \hat{u}_y e^{j\vec{k}^{(i)} \cdot \vec{r}} = H_0 \hat{u}_y e^{-jk_0(z\cos\theta + x\sin\theta)}, \\ \vec{E}^{(i)} &= Z_0 \vec{H}^{(i)} \times \vec{k}^{(i)} = (-\sin\theta \hat{u}_z + \cos\theta \hat{u}_x) Z_0 H_0 e^{j\vec{k}^{(i)} \cdot \vec{r}}, \end{aligned} \quad (11)$$

incident with an angle θ on a perfectly conducting plane at $z = 0$ bearing a regular array of holes at $x = na_h$, $y = mb_h$, $m, n = -\infty, \dots, \infty$ (see Fig. 2). In (11) $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the free space impedance. Note that we use the $\exp(-j\omega t)$ time dependance (which is rather unusual in electromagnetics), to comply with the Particle Accelerator Literature.

⁵The only restrictive underlying assumption is that the holes should be (piecewise) uniformly-distributed in the longitudinal direction, which is needed for the very definition of beam impedance per unit length to apply.

Figure 2: Regular 2D array of circular holes of radius r_0 on a conducting plane at $x = na_h$, $y = mb_h$, $m, n = -\infty, \dots, \infty$.

In the limit of near-grazing incidence $\theta \rightarrow \pi/2$ the primary field (holes suppressed) has the same local structure at $z = 0$ as the field of a relativistic particle beam at a perfectly conducting pipe wall: the magnetic field is nearly-tangent to the wall, and the electric field nearly normal. Thus, following [18] we expect to be able to deduce an appropriate (local) impedance boundary condition from the asymptotic ($\theta \rightarrow \pi/2$) form of the TM plane wave reflection coefficient.

If the holes were absent the reflected field $(\vec{E}^{(r)}, \vec{H}^{(r)})$ would be:

$$\begin{aligned} \vec{k}^{(r)} &= k_0 (\sin\theta \hat{u}_x - \cos\theta \hat{u}_z), \\ \vec{H}^{(r)} &= H_0 \hat{u}_y e^{j\vec{k}^{(r)} \cdot \vec{r}} = H_0 \hat{u}_y e^{jk_0(-z\cos\theta + x\sin\theta)}, \\ \vec{E}^{(r)} &= Z_0 \vec{H}^{(r)} \times \vec{k}^{(r)} = (-\sin\theta \hat{u}_z - \cos\theta \hat{u}_x) Z_0 H_0 e^{j\vec{k}^{(r)} \cdot \vec{r}}, \end{aligned} \quad (12)$$

and the total field:

$$(\vec{E}_{tot}, \vec{H}_{tot}) = \begin{cases} 0, & z > 0 \\ (\vec{E}^{(i)} + \vec{E}^{(r)}, \vec{H}^{(i)} + \vec{H}^{(r)}), & z < 0. \end{cases} \quad (13)$$

The field in the presence of the holes, according to Bethe's approximation [11], can be computed by *adding* to the primary field (13) above, the field $\vec{E}^{(scatt.)}$, $\vec{H}^{(scatt.)}$ produced by elementary electric and magnetic sources radiating on a perfectly conducting (no hole) plane $z = 0$ and placed at $x = na_h$, $y = mb_h$, $m, n = -\infty, \dots, \infty$.

Specifically, the sources at $\vec{r} = \vec{r}_{nm} = na_h \hat{u}_x + mb_h \hat{u}_y$, have moment densities:

$$\begin{aligned} \vec{P}_{nm} &= \epsilon_0 \alpha_e \delta(\vec{r} - \vec{r}_{nm}) \vec{E}_z^{(tot)}(\vec{r}_{nm}) \hat{u}_z, \\ \vec{M}_{nm} &= \alpha_m \delta(\vec{r} - \vec{r}_{nm}) (\vec{I} - \hat{u}_z \hat{u}_z) \vec{H}^{(tot)}(\vec{r}_{nm}), \end{aligned} \quad (14)$$

where α_e and α_m are the hole electric and magnetic polarizabilities⁶.

It is seen from eq.s (13), (11) and (12) that:

⁶Since we are interested in computing the hole-produced field in $z < 0$, we use the so called *internal* hole polarizabilities in (14).

Figure 3: Equiverse magnetic Bethe-dipoles images.

$$\begin{aligned} E_z^{(tot)}(z=0) &= 2E_z^{(i)}(z=0) = -Z_0 H_0 \sin\theta e^{jk_0 x \sin\theta}, \\ (\bar{\vec{I}} - \hat{u}_z \hat{u}_z) \cdot \vec{H}^{(tot)}(z=0) &= (\bar{\vec{I}} - \hat{u}_z \hat{u}_z) \cdot 2\vec{H}^{(i)}(z=0) = 2H_0 \hat{u}_y e^{jk_0 x \sin\theta}. \end{aligned} \quad (15)$$

Furthermore, the fields radiated in $z < 0$ by the dipoles (14) sitting on the perfectly conducting plane $z = 0$ are the same as those radiated *in free space* by (14) and their images⁷. The images are equiverse and placed exactly at the same positions as the corresponding primary sources (see Fig. 3).

Thus the superposition of the primary and image sources is just *twice* (14), viz.:

$$\begin{aligned} \vec{P}_{tot} &= \sum_{m,n} (\vec{P}_{m,n} + \vec{P}_{m,n}^{(image)}) = \sum_{m,n} 2\vec{P}_{m,n} = \\ &= -4 c^{-1} H_0 \alpha_e \sin\theta \delta(z) \sum_{m,n} \delta(x - na_h) \delta(y - mb_h) e^{jk_0 na_h \sin\theta} \hat{u}_z, \end{aligned} \quad (16)$$

$$\begin{aligned} \vec{M}_{tot} &= \sum_{m,n} (\vec{M}_{m,n} + \vec{M}_{m,n}^{(image)}) = \sum_{m,n} 2\vec{M}_{m,n} = \\ &= 4H_0 \alpha_m \delta(z) \sum_{m,n} \delta(x - na_h) \delta(y - mb_h) e^{jk_0 na_h \sin\theta} \hat{u}_y. \end{aligned} \quad (17)$$

The fields produced by these sources can be quickly computed using the vector potential $\vec{A}$ and the magnetic Hertz potential $\vec{\Pi}$ which are related to $\vec{P}_{tot}$ and $\vec{M}_{tot}$ by:

⁷The field computed in this way fails to reproduce the *true* field for $z > 0$. If one were interested in computing the fields in $z > 0$, one should use the *external* polarizabilities in (14) before applying the image theorem. In this case, the computed field would not reproduce the true field for $z < 0$.

$$\begin{aligned}
(\nabla^2 + k_0^2)\vec{A} &= j\omega\mu_0 \vec{P}_{tot}, \\
(\nabla^2 + k_0^2)\vec{\Pi} &= -\vec{M}_{tot},
\end{aligned} \tag{18}$$

as follows:

$$\begin{aligned}
\vec{E}^{(scatt.)} &= -j\omega c^{-2} \nabla \times \nabla \times \vec{A} + j\omega\mu_0 \nabla \times \vec{\Pi}, \\
\vec{H}^{(scatt.)} &= \mu_0^{-1} \nabla \times \vec{A} + \nabla \times \nabla \times \vec{\Pi}.
\end{aligned} \tag{19}$$

In order to solve eq.s (18) it is first expedient to note that since $\vec{P}_{tot} = P\hat{u}_z$ and $\vec{M}_{tot} = M\hat{u}_y$ then $\vec{A} = A\hat{u}_z$ and $\vec{\Pi} = \Pi\hat{u}_y$. It is further convenient to use the (generalized) Fourier representation of the periodic delta functions:

$$\sum_n \delta(x - na_h) = \frac{1}{a_h} \sum_p e^{jq(2\pi/a_h)x}, \quad \sum_m \delta(y - mb_h) = \frac{1}{a_h} \sum_q e^{jq(2\pi/b_h)y}, \tag{20}$$

so as to recast the source terms into the following form:

$$\begin{aligned}
\vec{P}_{tot} &= -\frac{4H_0}{c} \frac{\alpha_e}{b_h a_h} \sin\theta \delta(z) \sum_{p,q} e^{2\pi j(px/a_h + qy/b_h)} e^{jk_0 x \sin\theta} \hat{u}_z, \\
\vec{M}_{tot} &= \frac{4H_0}{b_h a_h} \frac{\alpha_m}{b_h} \delta(z) \sum_{p,q} e^{2\pi j(px/a_h + qy/b_h)} e^{jk_0 x \sin\theta} \hat{u}_y,
\end{aligned} \tag{21}$$

and obtaining the following wave-equations:

$$\begin{aligned}
(\nabla^2 + k_0^2)A &= -j\omega\mu_0 \frac{4H_0\alpha_e}{c a_h b_h} \sin\theta \delta(z) \sum_{p,q} e^{2\pi j(px/a_h + qy/b_h)} e^{jk_0 x \sin\theta}, \\
(\nabla^2 + k_0^2)\Pi &= -\frac{4H_0\alpha_m}{a_h b_h} \delta(z) \sum_{p,q} e^{2\pi j(px/a_h + qy/b_h)} e^{jk_0 x \sin\theta}.
\end{aligned} \tag{22}$$

The forcing terms in (22) contain delta functions at $z = 0$, and thus the equations must be solved in the weak (Sobolev) sense.

The form of the equations suggests that the solutions have the same x, y dependance as the corresponding forcing terms, and thus can be written as:

$$\begin{aligned}
A &= \sum_{p,q} A_{p,q} e^{\pm j\gamma_{p,q}z} e^{j[k_0 x \sin\theta + 2\pi(px/a_h + qy/b_h)]}, \quad z \gtrless 0, \\
\Pi &= \sum_{p,q} \Pi_{p,q} e^{\pm j\eta_{p,q}z} e^{j[k_0 x \sin\theta + 2\pi(px/a_h + qy/b_h)]}, \quad z \gtrless 0
\end{aligned} \tag{23}$$

where $\gamma_{p,q}$ and $\eta_{p,q}$ are non-negative defined, and the choice of the sign in the z -dependent exponentials corresponds to the physical requirement that the waves

produced by the sources (21) propagate *away* from the plane where they lie ($z = 0$).

Letting (23) into (22), for $z \neq 0$, one readily gets:

$$\gamma_{p,q} = \eta_{p,q} = +\sqrt{k_0^2 - \left\{ \left(\frac{2\pi q}{b_h} \right)^2 + \left[\left(\frac{2\pi p}{a_h} \right) + k_0 \sin\theta \right]^2 \right\}}. \quad (24)$$

Each term in (23) is thus recognized to represent a plane wave propagating in the (p, q) -*grating lobe* direction, with wave vector:

$$k_x = \frac{2\pi}{a_h} p + k_0 \sin\theta, \quad k_y = \frac{2\pi}{b_h} q, \quad k_z = \pm \left[k_0^2 - (k_x^2 + k_y^2) \right]^{1/2}, \quad z \gtrless 0. \quad (25)$$

In particular, the $p = q = 0$ term propagates, for $x < 0$, in the specular reflection direction ⁸, with wave-vector:

$$k_x = k_0 \sin\theta, \quad k_y = 0, \quad k_z = -k_0 \cos\theta. \quad (26)$$

In order to determine the coefficients $A_{p,q}$ and $\Pi_{p,q}$, we require that (23) be weak-solutions of (22), by integrating them across $(0-, 0+)$, viz.:

$$\begin{aligned} j\omega\mu_0 \int_{0-}^{0+} dz P_{tot} &= \int_{0-}^{0+} dz (\nabla^2 + k_0^2) A = \\ &= \int_{0-}^{0+} dz (\partial_{xx} + \partial_{yy} + k_0^2) A + |\partial_z A|_{z=0-}^{z=0+}, \end{aligned} \quad (27)$$

and the like for Π . The first term on the r.h.s. of (27) is zero, because the integrand is continuous and limited across $z = 0$. The second is non-zero, because $\partial_z A$ is *discontinuous* across $z = 0$, due to the *different* signs of the z -dependent exponential for $z > 0$ and $z < 0$ in (23). Hence, using (21) and (23) in (27) one gets:

$$A_{p,q} = 2 \frac{k_0 H_0}{ab \gamma_{p,q}} \sin\theta \mu_0 \alpha_e, \quad (28)$$

$$\Pi_{p,q} = -j \frac{2H_0}{ab \eta_{p,q}} \alpha_m. \quad (29)$$

The reflection coefficient of the perforated wall can now be defined with reference to the *specular* component ($p = q = 0$) of the scattered field⁹:

⁸Compare with $\vec{k}^{(r)}$ in (12).

⁹In the asymptotic case of our interest, $\theta \rightarrow \pi/2$, all higher-order grating lobe waves ($p, q \neq 0$) decay exponentially off the $z = 0$ plane, as seen from eq. (24).

$$\Gamma_H = \left. \frac{H_{0,0}^{(scatt)} + H^{(r)}}{H^{(i)}} \right|_{z=0} := - \frac{Z_{z>0} - Z_{z<0}}{Z_{z>0} + Z_{z<0}}, \quad (30)$$

where the last equality defines the reflection coefficient at $z = 0$ in terms of the *oblique* wave impedances¹⁰ $Z_{z<0}$ and $Z_{z>0}$ of the *media* filling the $z < 0$ and $z > 0$ half-spaces.

The scattered magnetic field can be easily computed using (19), (23) and (28), (29). One readily obtains¹¹:

$$H_{0,0}^{(scatt.)} = 2jk_0 \frac{\alpha_m + \alpha_e \sin^2 \theta}{a_h b_h \cos \theta} H_0 e^{j\vec{k}^{(r)} \cdot \vec{r}}, \quad (31)$$

and hence, substituting into (30):

$$\Gamma_H = 1 + 2jk_0 \frac{\alpha_m + \alpha_e \sin^2 \theta}{a_h b_h \cos \theta} \approx 1 - 2 \frac{Z_{z>0}}{Z_{z<0}} = 1 - 2 \frac{Z_{z>0}}{Z_0 \cos \theta}, \quad (32)$$

where the fraction on the r.h.s. of (30) has been expanded to lowest order in the ratio¹² $Z_{z>0}/Z_{z<0}$. Hence:

$$Z_{z>0} = -j \frac{k_0 Z_0}{a_h b_h} (\alpha_m + \alpha_e \sin^2 \theta). \quad (33)$$

We are thus led to conclude that, for *close-to-grazing* incident fields, a perforated perfectly conducting wall acts like the surface of a homogeneous medium with (oblique) wave impedance

$$Z_{z>0} = -j \frac{k_0 Z_0}{a_h b_h} (\alpha_m + \alpha_e). \quad (34)$$

Note that the spatial distribution of the holes appears in (34) only through the factor $(a_h b_h)^{-1}$ which represents the number of holes per unit area n_σ . Hence:

$$Z_{z>0} = -jk_0 Z_0 (\alpha_m + \alpha_e) n_\sigma, \quad (35)$$

which reproduces our heuristic ansatz (8).

As a matter of fact, the impedance (35) is usually *very small*, and thus provided the further condition:

$$\left| \left(\frac{Z_0}{Z_{z>0}} \right) k_0 \rho_S \right| \gg 1 \quad (36)$$

¹⁰We remind that the oblique wave impedance of a plane wave is defined as the ratio between the components of $\vec{E}$ and $\vec{H}$ transverse to the direction of propagation $\vec{k}$.

¹¹The scattered specular magnetic field has only a nonzero y -component.

¹²Note that in the limit of vanishingly small holes we get $Z_{z>0} = 0$ (perfect conductor), and hence we expect $|Z_{z>0}/Z_{z<0}| \ll 1$.

is satisfied (see *Appendix A*), where ρ_S is the (local) smallest radius of curvature of the surface S , then a Leontóvich boundary condition (3) with Z_{wall} given by (35), can be used even for a *non-planar*, perfectly conducting, perforated surface.

This provides a rigorous justification of our heuristic ansatz (8). In the following we shall denote the wall impedance of a perforated pipe wall in free space by $Z_{p.w.}^{(0)}$.

5 Possible Model Improvements.

It is conceivable and relatively straightforward to improve the model by, e.g.:

- i) including the effect of a non-zero wall thickness;
- ii) describing the effect of electromagnetic coupling among the holes;
- iii) taking into account the presence of a further (imperfectly conducting) tube surrounding the perforated beam pipe;

Note that all *potential* improvements of (35) should be gauged consistently against the omission of terms of higher order in $k \cdot (\text{hole size})$ in the standard (Bethe's) formulae for $\alpha_{e,m}$. These terms have been discussed in [19], [20] and [21].

In this section we shall briefly review points *i)* to *iii)* above.

Holes in a Thick Wall

A nonzero wall thickness is useful to reduce RF leakage through the pumping holes. A general formalism for computing electric and magnetic hole polarizabilities for thick walls has been set up in [22]-[24]. For circular holes one has, with good accuracy [25]:

$$\left(\alpha_e^{(i)} + \alpha_m^{(i)}\right) \approx 0.57 \left(\alpha_e^{(0)} + \alpha_m^{(0)}\right), \quad w/r_0 \leq 2 \quad (37)$$

$$\alpha_e^{(e)} = \frac{2}{3} r_0^3 \exp(-\xi_{TE} w/r_0), \quad (38)$$

$$\alpha_m^{(e)} = -\frac{4}{3} r_0^3 \exp(-\xi_{TM} w/r_0), \quad (39)$$

where the superfixes (i) , (e) , (0) identify the internal, external and thin-wall polarizabilities, r_0 is the hole radius and w the wall thickness, $\xi_{TE} = 2.405$ and $\xi_{TM} = 1.841$ are the damping constants of the dominant cut-off modes of a circular waveguide having the same radius as the holes.

Coupling Among Holes

Electromagnetic coupling among the holes can be accounted by using in (35) the *effective* electric polarizabilities $\alpha'_{e,m}$ of each hole, *in the presence of the others*, viz.:

$$\alpha'_{e,m} = \frac{\alpha_{e,m}}{1 - C_{e,m}\alpha_{e,m}}, \quad (40)$$

where the coupling constants $C_{e,m}$ are given in *Appendix B*. For a relativistic beam, the induced *electric* dipoles are *normal* to the perforated wall, while the *magnetic* ones are *tangent* and, for circular holes, *parallel* to the magnetic field. Thus, eq.s (B5) to (B7), under the further assumption $a = b = s$ (isotropic hole array) specialize to:

$$C_e = C_\perp = s^{-3} \left[-\frac{12}{5\pi} + 16\pi K_0(2\pi), \right] \quad (41)$$

$$C_m = C_\parallel = s^{-3} \left[\frac{6}{5\pi} - 8\pi K_0(2\pi), \right] \quad (42)$$

where K_0 is a Bessel function of the 3rd kind. Equations (41), (42) imply a quasi static ($s \ll \lambda$) assumption, which could be removed in principle [26].

Perforated Beam Pipe in a Co-Axial Lossy Tube

The influence of an external imperfectly conducting tube, coaxial to the beam pipe (cold-bore) can be simply described by using *modified* polarizabilities in eq. (7) and (8). In *Appendix C* we show that for the special case of a circular liner of radius b surrounded by a coaxial tube of radius a the modified polarizabilities have the simple form:

$$\alpha'_{e,m} = \alpha_{e,m}^{(i)} + F \alpha_{e,m}^{(e)} \quad (43)$$

where¹³:

$$F = -\frac{\alpha_e^{(e)} + \alpha_m^{(e)}}{\alpha_e^{(i)} + \alpha_m^{(i)} + j \operatorname{sgn}(k) \hat{\delta}^* n_\sigma^{-1} (1 + b/a)}, \quad (44)$$

or, equivalently,

$$F = -\frac{(\alpha_e^{(e)} + \alpha_m^{(e)})(\alpha_e^{(i)} + \alpha_m^{(i)})^{-1}}{1 + \frac{(1 + b/a)Z_{cb}}{Z_{p.w.}^{(0)}}}. \quad (45)$$

In (43), (44) and (45) the superfixes $(e), (i)$ denote the external and internal polarizabilities, $\hat{\delta}$ is the *complex* EM penetration depth into the cold bore walls

¹³Equation (44) was first deduced by Gluckstern [27], [28], although apparently neglecting the *complex* character of $\hat{\delta}$. See *Appendix C*.

(both walls at $r = b+$ and $r = a$ assumed lossy, with finite conductivity σ_{cb}), and Z_{cb} the corresponding (complex) wall impedance:

$$\hat{\delta} = (|k_0|Z_0)^{-1} Z_{cb}^*, \quad Z_{cb} = [1 - j \operatorname{sgn}(\omega)] \left(\frac{|k_0|Z_0}{2\sigma_{cb}} \right)^{1/2}. \quad (46)$$

Hence, (7) and (8) become:

$$Z_{\parallel} = -j \frac{Z_0 k_0}{2\pi b} \left[(\alpha_e^{(i)} + \alpha_m^{(i)}) - \frac{(\alpha_e^{(e)} + \alpha_m^{(e)})^2 (\alpha_e^{(i)} + \alpha_m^{(i)})^{-1}}{1 + \frac{(1+b/a)Z_{cb}}{Z_{p.w.}^{(0)}}} \right] n_{\sigma}, \quad (47)$$

$$Z_{wall} = -j Z_0 k_0 n_{\sigma} \left[(\alpha_e^{(i)} + \alpha_m^{(i)}) - \frac{(\alpha_e^{(e)} + \alpha_m^{(e)})^2 (\alpha_e^{(i)} + \alpha_m^{(i)})^{-1}}{1 + \frac{(1+b/a)Z_{cb}}{Z_{p.w.}^{(0)}}} \right]. \quad (48)$$

In the limit of an infinitely thin liner's wall, where $\alpha_{e,m}^{(e)} = -\alpha_{e,m}^{(i)}$ equation (48) admits a simple and sound physical interpretation.

The complex propagation constant and characteristic impedance of the lossy coaxial cold bore waveguide, can be equivalently computed by assuming that the outer ($r = a$) cold-bore wall is a *perfect conductor*, (as assumed from the beginning in [29]) and placing an *equivalent* wall impedance:

$$Z_{eq.} = Z_{cb} \left(1 + \frac{b}{a} \right) \quad (49)$$

on the inner conductor ($r = b$) alone¹⁴. Then, it is readily seen that (48) is nothing but the parallel combination of $Z_{p.w.}^{(0)}$ given by eq. (8) and $Z_{eq.}$ (49),

$$Z_{wall} = \frac{Z_{p.w.}^{(0)} Z_{eq.}}{Z_{p.w.}^{(0)} + Z_{eq.}}. \quad (50)$$

The power lost in the coaxial region has been predicted and measured in [7], [8], and confirms the validity of the above analysis (see *Appendix C*.)

6 Numerical Results

In this section we shall apply our main results to the square geometry depicted in Fig. 4, where the holes are located in the rounded corners and the flat sides of length d are copper coated.

¹⁴This is a more or less obvious consequence of the r^{-1} dependance of the *TEM* magnetic field in the coaxial region

Figure 4: Square pipe with rounded corners. The radius of curvature is $(a - d)/2$ and the flat sides are copper coated, while the corners are uncoated.

We shall assume that the size of the holes is determined by the requirement of preventing excessive radiation loss through them, while their number is dictated by requiring an adequate pumping capacity. Typical numbers are shown in Table I. If the holes are confined to the rounded corners of the square-section liner depicted in Fig. 4, then

$$\begin{aligned} & (\text{no. of holes per unit length of liner}) = \\ & 4 (\text{no. of holes per unit length at each rounded corner}) \end{aligned} \quad (51)$$

It is readily seen that each rounded corner has a surface $(\pi/4)(a - d)$ per unit length of liner, thus, there are $4(\pi/4)(a - d) n_\sigma$ holes per unit length of liner. Furthermore, for a regular 2D lattice of circular holes with spacing s , one has ¹⁵

$$n_\sigma = s^{-2} \frac{N_\lambda}{4a} \left[\frac{\pi}{4} \left(1 - \frac{d}{a} \right) \right]^{-1}. \quad (52)$$

As a result, the coupling coefficients (41) and (42) become functions of d/a and so does the wall impedance.

Representative numerical results are displayed in Figs 5 and 6. In Fig. 5 we plot $R = \text{Re}(Z_{\text{wall}})$ and $X = \text{Im}(Z_{\text{wall}})$ vs. d/a , for $\omega/\omega_c = .1, .5, .9$, $\omega_c = (\pi c/a)$ being the (lowest) pipe cut-off frequency¹⁶. To draw these figures we used the parameters collected in Table I below, taking into account wall thickness via eq. (37), and the presence of the cold-bore via eq.s (43), (44).

Neglecting hole coupling, or including it via eq.s (40), (42) does not affect sensibly the curves in Fig. 5.

¹⁵Note that $s \geq 2r_0$, above which the holes merge.

¹⁶Below the (lowest) cutoff frequency $(c/2a)$, coupling among the holes will be essentially static, consistent with (41) and (42). This can be loosely understood by noting that, in the relativistic regime, the beam field is deducible from a (transverse, scalar) potential depending on $z - ct$, and that below cut-off, no fields could propagate from one hole to another.

Figure 5: Thick liner in (left) free space and (right) coaxial cold bore: $R = \text{Re}(Z_{wall})$ and $X = \text{Im}(Z_{wall})$ vs. d/a , for $\omega/\omega_c = .1, .5, .9$, $\omega_c = (\pi c/a)$ being the (lowest) pipe cut-off frequency. We assume the geometry of *Fig. 4* with the holes confined to the rounded corners and the parameters of Table I

Figure 6: Thin liner in (left) free space and (right) coaxial cold bore: $R = \text{Re}(Z_{wall})$ and $X = \text{Im}(Z_{wall})$ vs. d/a , for $\omega/\omega_c = .1, .5, .9$, $\omega_c = (\pi c/a)$ being the (lowest) pipe cut-off frequency. We assume the geometry of *Fig. 4* with the holes confined to the rounded corners and the parameters of Table I

On the other hand, for a thin liner in free space at low frequencies ($\omega \ll \omega_c$) hole coupling can be relevant, as shown in Fig. 6, where we used Kurennoy's result:

$$\left| \frac{\text{Re } Z_{\text{wall}}}{\text{Im } Z_{\text{wall}}} \right| = \frac{\pi^2}{6} \frac{\alpha_e^2 + \alpha_m^2}{\alpha_e + \alpha_m} \left(\frac{\omega}{\omega_c} \right)^3 a^{-3} \quad (53)$$

and (8) together with eq.s (40), (42).

7 Parasitic Loss Due to Resistive Wall and Holes

The parasitic loss (energy lost by the beam per unit pipe-length), is given by [12]:

$$\frac{\Delta \mathcal{E}}{L} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |I(\omega)|^2 \Re Z_{\parallel}(\omega) d\omega. \quad (54)$$

where $I(\omega)$ is the beam current frequency spectrum, and $Z_{\parallel}(\omega)$ the longitudinal impedance.

In the following we shall refer to a stainless-steel, rounded-corner, square cross-section beam-pipe with copper-plated side walls, sketched in *Fig. 4*.

For gaussian bunches of r.m.s. length σ_z , and total charge Q , the current frequency spectrum is:

$$I(\omega) = Q e^{-\sigma_z^2 k^2 / 2}. \quad (55)$$

7.1 Ohmic Losses

The *pure* ohmic power losses $P_{Cu}^{(0)} + P_{ss}^{(0)}$ in the *unperforated* copper-plated and stainless steel (internal) beam-pipe surfaces, can be written [34]:

$$P_{Cu,ss}^{(0)} = N_b \nu_r \frac{\Delta \mathcal{E}}{L} = N_b \nu_r \frac{Q^2 c Z_0}{8\pi^3 a^2} W_{Cu,ss} \left(\frac{\sigma_z}{a} \right) G_{Cu,ss} \left(\frac{d}{a} \right), \quad (56)$$

where N_b is the number of bunches in the ring, ν_r the revolution frequency, a the (rounded) square side-length, Q the bunch charge, the functions $G_{Cu}(\cdot)$, $G_{ss}(\cdot)$ are defined in Appendix D, and the functions $W_{ss}(\cdot)$ and $W_{Cu}(\cdot)$ depend only on the bunch length and the (local, spectral) impedances $Z_{\text{wall}}^{(Cu)}$, $Z_{\text{wall}}^{(ss)}$ of the stainless-steel and copper-plated surfaces and are [34]:

$$W_{ss,Cu} \left(\frac{\sigma_z}{a} \right) = 2 \int_0^{+\infty} e^{-(\sigma_z^2/a^2)(y^2/\beta_0^2)} \Re \left\{ Y_0 Z_{\text{wall}}^{(ss,Cu)} \left(\frac{yc}{a} \right) dy \right\}. \quad (57)$$

where $y = \pi\omega/\omega_c$, $\omega_c = \pi/a$ being the cutoff frequency of the first waveguide mode in the (square) liner, and the wall impedances $Z_{\text{wall}}^{(ss,Cu)}$ refer to the unperforated (stainless steel, copper) chamber walls. In *Fig. 7* we plot $W_{ss}(\sigma_z/a)$, using the parameters in Table I. The corresponding values of $W_{Cu}(\sigma_z/a)$ are easily deduced since: $W_{ss}/W_{Cu} = \sqrt{\rho_{ss}/\rho_{Cu}} \approx 30$.

Figure 7: The function $W_{ss}(\sigma_z/a)$ for squared chamber geometry with unperforated walls. We assume the geometry of *Fig. 4* and the parameters in Table I.

ρ_{ss}	$5 \times 10^{-7} \Omega m$
ρ_{Cu}	$5.5 \times 10^{-10} \Omega m$
# of particles per bunch	10^{11}
# of bunches N_b	2835
revolution frequency ν_r	11.245 kHz
hole radius r_0	.75 mm
wall thickness	.75 mm
liner diameter a	3.48 cm
bunch length σ_z	7.5 cm

Table I: Model parameters.

7.2 Power Loss Through Holes

The power leaking through the holes, and dissipated in the stainless-steel inner and outer walls of the coaxial region can be written as $P_{Cu}^{(holes)} + P_{SS}^{(holes)}$, where :

$$P_{Cu}^{(holes)} = N_b \nu_r \frac{Q^2 c Z_0}{8\pi^3 a^2} W_{Cu}^{(holes)} \left(\frac{\sigma_z}{a} \right) G_{Cu} \left(\frac{d}{a} \right) \quad (58)$$

$$P_{ss}^{(holes)} = N_b \nu_r \frac{Q^2 c Z_0}{8\pi^3 a^2} W_{ss}^{(holes)} \left(\frac{\sigma_z}{a} \right) G_{SS} \left(\frac{d}{a} \right) \quad (59)$$

represent the contribution of the holes drilled on the stainless-steel (rounded-corners) and copper-plated (straight sides) portions of the liner's wall. In (58) and (59),

$$W_{ss,Cu}^{(holes)} \left(\frac{\sigma_z}{a} \right) = 2 \int_0^{+\infty} e^{-(\sigma_z^2/a^2)(y^2/\beta_0^2)} \Re \left\{ Y_0 Z_{wall(ss,Cu)}^{(holes)} \left(\frac{y^c}{a} \right) dy \right\}. \quad (60)$$

where $Z_{wall(ss,Cu)}^{(holes)}$ is the appropriate wall impedance (48), which depends on the number of holes per unit surface:

$$n_{\sigma}^{(Cu)} = \frac{N_{\lambda}^{(Cu)}}{4a} \left(\frac{d}{a} \right)^{-1}, \quad n_{\sigma}^{(ss)} = \frac{N_{\lambda}^{(ss)}}{4a} \left[\frac{\pi}{4} \left(1 - \frac{d}{a} \right) \right]^{-1} \quad (61)$$

$N_{\lambda}^{(Cu,ss)}$ being the number of holes per unit length on the copper-plated and stainless-steel portions of the liner's wall.

The ohmic power losses in the perforated pipe can be approximately written

$$P_{Cu,ss} = (1 - \alpha_{Cu,ss}) P_{Cu,ss}^{(0)} \quad (62)$$

where $P_{Cu,ss}^{(0)}$ are computed using (56) and (57), and $\alpha_{Cu,ss}$ are the hole-covered copper and steel surface fractions, respectively.

In the following we shall focus on the special cases where the pumping holes are either confined to the copper-plated strips (case-I, $N_{\lambda}^{(ss)} = \alpha_{ss} = P_{ss}^{(holes)} = 0$) or to the stainless-steel rounded corners (case-II, $N_{\lambda}^{(Cu)} = \alpha_{Cu} = P_{Cu}^{(holes)} = 0$).

In *Fig. 8* we plot the functions $W_{ss,Cu}^{(holes)}(\sigma_z/a)$ for different values of d/a , for case-I and case-II respectively¹⁷, as functions of the ratio σ_z/a , assuming $N_{\lambda} = 2660 \text{ holes/m}$ and circular holes of 1.5 mm diameter (fixed pumping capacity).

As a result, we get the values summarized in Tables II and III below:

In the limiting case where $d/a = 0$ (all-steel, circular chamber) we get: $P_{SS}^{(0)} \approx 2.06 \text{ W/m}$, $P_{SS}^{(holes)} \approx 14.4 \text{ mW/m}$ whereas in the other limiting case $d/a = 1$ (all-copper square chamber) we obtain¹⁸ $P_{Cu}^{(0)} \approx 68 \text{ mW/m}$, $P_{Cu}^{(holes)} \approx 10 \text{ mW/m}$.

¹⁷Note that, for case-I and case-II, d/a must be, respectively, larger than $\pi^{-1}(S_h N_{\lambda}/4a)$ and smaller than $1 - (4/\pi)^{-2}(S_h N_{\lambda}/4a)$, to prevent adjacent holes from merging, S_h being the hole surface.

¹⁸Note that the quoted ohmic loss value is smaller than measured by a factor 2. This discrepancy is probably due to the roughness of the internal chamber wall [35].

Figure 8: (Top) $W_{Cu}^{holes}(\sigma_z/a)$ versus σ_z/a , holes confined to the the copper-plated strips (case I). (Bottom) $W_{ss}^{holes}(\sigma_z/a)$ versus σ_z/a , holes confined to the stainless-steel rounded corners (case II). In both cases we assume the geometry of *Fig. 4* with $d/a = .5, .7, .9$ and all other parameters as in Table I.

	$d/a = .5$	$d/a = .7$
P_{Cu}	54 mW/m	63 mW/m
P_{SS}	326 mW/m	72 mW/m
$P_{Cu}^{(holes)}$	30 mW/m	19 mW/m
P_{total}	410 mW/m	154 mW/m

Table II: Parasitic losses for holes on copper strips (case I).

	$d/a = .5$	$d/a = .7$
P_{Cu}	58 mW/m	66 mW/m
P_{SS}	298 mW/m	62 mW/m
$P_{SS}^{(holes)}$	7.2 mW/m	3.3 mW/m
P_{total}	363 mW/m	131 mW/m

Table III: Parasitic losses for holes on stainless-steel rounded corners (case II).

8 Conclusions

While relying on several simplifying assumptions (*perturbative* computation of beam coupling impedances and *approximate* boundary condition), the proposed approach includes in a remarkably simple fashion several non obvious features which are not as easily included in the standard approach.

We do believe that the combined use of reciprocity formulae (equations (1) and (2)) and impedance boundary conditions like (35) provide a powerful tool to obtain analytical estimates of the beam coupling impedances in realistic, complex and heterogeneous geometries.

As hints for future work we mention *i)* the possibility of deriving variational formulae for beam coupling impedances, *ii)* a statistical characterization of the beam coupling impedances for randomly placed holes, *iii)* the extension to hopefully more accurate higher order impedance boundary conditions, as discussed in [18],[30]-[33], and *iv)* finally the study of pipe wall roughness.

Acknowledgements

The Author wishes to thank the SL/AP group for hospitality and help during her stay at CERN, in particular Francesco Ruggiero for stimulating discussions.

References

- [1] The LHC Study Group, CERN/AC/95-05 (LHC), 1995.

- [2] S.S. Kurennoy, CERN SL 91-29 (AP), 1991; Part. Acc., **39**, 1, 1992; Proc. 3rd EPAC, Berlin, 1992.
- [3] S.S. Kurennoy, IHEP 92-84; SSCL-636, 1993; Proc. EPAC, London, 1994; Part. Acc., **50**, 167, 1995.
- [4] R.L. Gluckstern, CERN SL 92-05 (AP); CERN SL 92-18 (AP); Phys. Rev. **A46**, 1106, 1992.
- [5] S.S. Kurennoy and R.L. Gluckstern, Proc. PAC95, Dallas, USA, 1995..
- [6] S. De Santis et al., Proc. EPAC-96, Sitges, Spain, 1996; Phys. Rev. **E54**, 800, 1996.
- [7] F. Caspers, E. Jensen, F. Ruggiero, CERN/PS 92-24 (RF-AR); CERN/SL 92-15 (DI); LHC Note 186, 1992.
- [8] F. Ruggiero, Part. Acc. **50**, 83, 1995.
- [9] M.A. Leontovich, *Issledovaniya po Raspostraneniyu Radiovoln*., USSR Acad. Press, 1948.
- [10] S. Petracca, Part. Acc., **50**, 211, 1995.
- [11] H. Bethe, Phys. Rev. **66**, 163-182, 1944.
- [12] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, Wiley, 1993, Sect. 2.4.
- [13] G.V. Stupakov and S.S. Kurennoy, Phys. Rev. **E49**, 794, 1994.
- [14] S.B. Cohn, IRE Proc. **39**, 1416, 1952.
- [15] F. de Meulenaere and J. van Bladel, IEEE Trans., **AP-25**, 198-205, 1977.
- [16] R. de Smedt and J. van Bladel, IEEE Trans., **AP-28**, 703-707, 1980.
- [17] S. Kurennoy, Proc. EPAC 96, Sitges, Spain, 1996, p. 871.
- [18] T.B.A. Senior, IEEE Trans. **AP-29**, 826, 1981.
- [19] J. Van Bladel, Radio Sci. **14**, 319-331, 1979.
- [20] Cheng et al., Proc. PAC95, Dallas, USA, 1995.
- [21] A.V. Fedotov and R.L. Gluckstern, Phys. Rev. E **56**, 3583-3601, 1997.
- [22] N. McDonald, IEEE Trans., **MTT-20**, 689-695, 1972.
- [23] R.L. Gluckstern and J.A. Diamond, IEEE Trans. **MTT-39**, 274, 1991.
- [24] W.H. Cheng, et al. Proc. PAC95, Dallas (USA), 1995.
- [25] B. Radak and R.L. Gluckstern, Univ. of Maryland Rept. UMDP 93-112, 1993.
- [26] W.H. Eggiman, IRE Trans. **MTT-9**, 408, 1961.
- [27] R.L. Gluckstern, CERN SL 92-06 (AP), 1992.

- [28] R.L. Gluckstern, CERN SL 92-31 (AP), 1992.
- [29] R.L. Gluckstern and B.W. Zotter, CERN SL 96-56 (AP), 1996.
- [30] L.N. Trefethen and L. Halpern, Math. Comp., **47**, 421, 1986.
- [31] J.L. Volakis and T.B.A. Senior, Proc. IEEE, **77**, 5-11, 1989.
- [32] M. Idemen, Proc. Phys. Soc. Japan, **59**, 71, 1990.
- [33] D.J. Hoppe and Y. Rahmat-Samii, J. Electr. Waves Appl., **8**, 1303, 1994.
- [34] S. Petracca, CERN SL 95-109 (AP), 1995.
- [35] F. Caspers, M. Morvillo and F. Ruggiero, CERN LHC Project Report 115 (1997) and Proc. PAC97, Vancouver (Canada), 1997.
- [36] T.B.A. Senior, Rev. Sci. Instr., **B8**, 419, 1960.
- [37] S.M. Rytov, J. Exp. Th. Phys. USSR, **10**, 120, 1940.
- [38] V.A. Fock, J. Phys. USSR, **10**, 13, 1946.
- [39] E.L. Feinberg, Suppl. Nuovo Cimento **11**, 60, 1959.
- [40] J. Brown and W. Jackson, Proc. IEE, **B102**, 37-42, 1955.
- [41] H.S. Bennett, J. Appl. Phys., **24**, 785-810, 1953.
- [42] R.E. Collin, *Field Theory of Guided Waves*, IEEE Press, 1991.
- [43] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals Series and Products*, Academic Press, 1980.
- [44] R.F. Harrington, *Time Harmonic Electromagnetic Fields*, McGraw-Hill, 1966.

Appendix A - Impedance Boundary Conditions

In this section we summarize a number of less obvious issues about impedance boundary conditions. No explicit derivations are provided, but pertinent references are given.

Impedance boundary conditions were introduced and extensively studied by the Russian School, [36] - [39], and are usually credited to Leontovich [9]. They relate the tangential electric and magnetic fields on the exterior boundary ∂V^- of a given domain V , thus allowing to solve an electromagnetic boundary value problem by solving Maxwell equations in the exterior (interior) domain only. In the simplest form, they are:

$$\left| (\bar{\vec{I}} - \hat{u}_n \hat{u}_n) \cdot \vec{E} - Z_{wall} \hat{u}_n \times \vec{H} \right|_{\partial V^-} = 0, \quad (A1)$$

where Z_{wall} is the (local) characteristic impedance of the medium in V , and the fields are computed at ∂V^- .

These conditions can be applied at the surface ∂V of a *homogeneous*, isotropic body with refractive index n and smallest curvature radius or dimension R provided that [39], [36]:

$$n \gg 1, \quad \text{Im}(n) kR \gg 1. \quad (A2)$$

For an *open* surface ∂V (limiting a medium of infinite extent) for which *no* inward normal does intersect the surface more than once, the second condition in (A2) can be relaxed into a milder one:

$$|n| kR \gg 1. \quad (A3)$$

For a *flat* open surface, the first condition in (A2) is sufficient. These equations admit simple physical interpretations.

For a plane open interface, the first of (A2) implies via Snell's law that the transmitted field is a plane wave, and (A1) follows from the continuity of the field components tangent to ∂V .

For non-planar open interfaces, eq. (A3) implies that the surface is planar at the level of the leading Fresnel zone¹⁹, which makes (A1) asymptotically valid in $(|n| kR)^{-1}$.

For a *compact* body, or an open but *re-entrant* surface, the second equation in (A2) essentially ensures that the field penetration is much smaller than R , so that the waves are *not* transmitted beyond the body.

For *non-homogeneous* bodies, it can be shown that [37]:

$$\begin{aligned} & \left| (\bar{I} - \hat{u}_n \hat{u}_n) \cdot \vec{E} - Z_{wall} \hat{u}_n \times \vec{H} \right|_{\partial V} = \\ & = \mathcal{O} \left(\frac{1}{kZ_0} \frac{\partial Z_{wall}}{\partial n} \right) + \mathcal{O} \left(\frac{1}{kZ_0} |\nabla_t Z_{wall}| \right)^2. \end{aligned}$$

Equations (A1) therefore do apply *locally*, provided Z_{wall} is uniform over scales of the order of the wavelength in V . Higher order boundary conditions have been introduced by several Authors [18],[30]-[33], For *locally* plane *stratified* media, simple transmission line formulae are sufficiently accurate for all practical purposes.

Appendix B - Effective Polarizability in a Plane Regular Hole Array

In this Appendix, for the Reader's ease we summarize the approach developed by Brown [40], Bennett [41] and Collin [42] to compute the effective (electrical

¹⁹We recall that the first Fresnel zone around some point is a circle such that the distance of the observer from the center is $\lambda/2$ less than the distance from the border.

or magnetical) polarizability α' of a single hole (possibly noncircular) in a plane regular array. The induced dipole moment f is related to the pertinent field component by:

$$\vec{f} = \alpha(\vec{F}_0 + \vec{F}_{int}), \quad (B1)$$

where α , is the polarizability of a single hole, $\vec{F}_0$, is the incident field, and $\vec{F}_{int}$ is the *interaction* field acting on the each and any hole due to the presence of all other holes. These latter are due to the very existence of the induced dipoles, and can thus be written:

$$\vec{F}_{int} = C\vec{f}, \quad (B2)$$

where C depends only on the dipole orientation and the array geometry, *not* on the type of field (electric or magnetic).

According to eq.s (B1), (B2), the effective polarizability is given by:

$$\alpha' = \frac{\alpha}{1 - \alpha C}, \quad (B3)$$

The interaction constant depends on the direction of the dipoles. It is convenient (superposition) to solve for the simplest cases where each induced dipole f is parallel to one of the co-ordinate axes. For the canonical problem sketched in Fig. 2, showing a plane regular array of y -directed dipoles placed at $\vec{r}_{nm} = (na_h)\vec{u}_x + (mb_h)\vec{u}_y$. the interaction constant will be denoted as C_y .

The general solution of this problem, which implies *no* restriction about the ratio between the dipole spacing and the wavelength has been obtained by Collin ([42], prob.s 12.7 and 12.8). Here we shall confine to the simple case where a quasi-static approximation can be invoked (hole spacing $\ll$ wavelength), which is appropriate for our present purposes.

The field at $\vec{r} = 0$ (electrostatic or magnetostatic) due to all other dipoles at $\vec{r}_{nm} \neq 0$ can be derived from a scalar potential, viz.:

$$\phi = \vec{u}_y \cdot \sum_{r_{nm} \neq 0} \frac{\vec{r}_{nm}}{4\pi r^3}, \quad (B4)$$

and the corresponding field is $F_y = -\partial\phi/\partial y|_{\vec{r}=0}$. Hence, from (B4),

$$\begin{aligned} C_y &= F_y = \frac{1}{4\pi} \sum_{(n,m) \neq (0,0)}^{-\infty, \infty} \frac{2(mb_h)^2 - (na_h)^2}{[(mb_h)^2 + (na_h)^2]^{5/2}} = \\ &= \frac{1}{\pi} \sum_{m=1}^{\infty} (mb_h)^{-3} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{2(mb_h)^2 - (na_h)^2}{[(mb_h)^2 + (na_h)^2]^{5/2}} = \\ &= \frac{6}{5\pi b_h^3} - \frac{2}{\pi b} \sum_{m,n=1}^{\infty} \left(\frac{2m\pi}{b_h}\right)^2 K_0\left(\frac{2mn\pi a_h}{b_h}\right) \sim \end{aligned}$$

$$= \frac{6}{5\pi b_h^3} - \frac{8\pi}{b_h^3} K_0\left(\frac{2\pi a_h}{b_h}\right), \quad (B5)$$

where the Poisson summation formula has been used²⁰.

More or less obviously, if the induced dipoles were directed along the x -direction, one should interchange a_h and b_h in eq. (B5). Hence:

$$C_x \approx \frac{6}{5\pi a_h^3} - \frac{8\pi}{a_h^3} K_0\left(\frac{2\pi b_h}{a_h}\right). \quad (B6)$$

Finally, for z -directed dipoles the interaction constant can be written as:

$$\begin{aligned} C_z &= \frac{1}{4\pi} \sum_{(n,m) \neq (0,0)}^{-\infty, \infty} \frac{(mb_h)^2 + (na_h)^2}{[(mb_h)^2 + (na_h)^2]^{5/2}} = \\ &= -\frac{1}{4\pi} \sum_{(n,m) \neq (0,0)}^{-\infty, \infty} \left\{ \frac{2(mb_h)^2 - (na_h)^2}{[(mb_h)^2 + (na_h)^2]^{5/2}} + \frac{2(na_h)^2 - (mb_h)^2}{[(na_h)^2 + (mb_h)^2]^{5/2}} \right\} = \\ &= -(C_y + C_x). \end{aligned} \quad (B7)$$

where the last equality is obtained by comparison with eq.s (B5), (B6).

Appendix C - Derivation of Equation (47).

In the following we shall sketch the deduction of eq. (48), starting from first principles, following a route different (though equivalent) from the one in [27], and point out all approximations involved.

We work in the spectral domain ($z - ct \longrightarrow k$), so that an $\exp(-j\omega t)$, $\omega = \beta ck$ time-dependance is understood, different from the one usually adopted in applied electromagnetics. We assume that the spectral content of the primary field is such that *only* the fundamental (TEM) mode propagates in the coaxial region. The only non zero field components will be denoted by E (radial electric field) and H

²⁰The Poisson summation formula in its simplest form states that:

$$\sum_m^{-\infty, \infty} f_m(\alpha) = \alpha^{-1} \sum_m^{-\infty, \infty} F(2m\pi/\alpha),$$

where $F(\cdot)$ is the Fourier transform of f , the equality should be intended in the L^2 spirit (i.e., almost for every α , but on a set of zero Lebesgue measure), and the theorem is a mere consequence of Parseval spectral completeness theorem. In order to prove the formula, we only need the following result [43]:

$$\int_{-\infty}^{\infty} e^{jkz} \frac{2z^2 - r^2}{(z^2 + r^2)^{5/2}} dz = -2k^2 K_0(|k|r).$$

together with the exponentially fast decay to zero of the Bessel function K_0 with increasing argument.

(azimuthal magnetic field). We shall assume the holes as located at $z = L, 2L, \dots$ so that $n_\sigma = (2\pi bL)^{-1}$.

In the presence of an external co-axial tube, the elementary sources radiating *into* the liner will be:

$$\begin{cases} \vec{P}_i = \hat{r} \epsilon_0 \left[\alpha_e^{(i)} E_0(i) + \alpha_e^{(e)} E_{cb}(i) \right], \\ \vec{M}_i = \hat{\phi} \left[\alpha_m^{(i)} H_0(i) + \alpha_m^{(e)} H_{cb}(i) \right], \quad i = -\infty, \dots, \infty \end{cases} \quad (C1)$$

where E_0, H_0 are the primary (forward-propagating) fields:

$$\begin{cases} E_0 = \frac{Q}{2\pi r} e^{jkz}, \\ H_0 = Y_0 E_0, \end{cases} \quad (C2)$$

E_{cb}, H_{cb} are the fields in the coaxial region between the liner and the cold-bore, the argument i refers to the position the i -th hole ($r = b, z = iL$), and the superfixes $(e), (i)$ denote the external and internal hole polarizabilities.

We shall show that:

$$E_{cb}(i) = F E_0(i), \quad H_{cb}(i) = F H_0(i), \quad (C3)$$

so that (C1) can be equally written:

$$\begin{cases} \vec{P}_i = \hat{r} \epsilon_0 \left[\alpha_e^{(i)} + \alpha_e^{(e)} F \right] E_0(i), \\ \vec{M}_i = \hat{\phi} \left[\alpha_m^{(i)} + \alpha_m^{(e)} F \right] H_0(i) \end{cases} \quad (C4)$$

which involves *only* the primary (unperturbed) fields, and corresponds to *pretending* that the liner is located *in vacuo*, and using the *modified* polarizabilities:

$$\alpha_{e,m}^{(i)} + F \alpha_{e,m}^{(e)} \quad (C5)$$

for the liner's holes, to account for the effect of the coaxial region, in computing the perforated wall impedance (8), viz.:

$$Z_{wall} = -jZ_0 k_0 \left[\left(\alpha_e^{(i)} + \alpha_e^{(i)} \right) + F \left(\alpha_e^{(e)} + \alpha_m^{(e)} \right) \right] n_\sigma \quad (C6)$$

In order to establish (C3), we proceed as follows.

The TEM fields radiated into the coaxial region by a *single* hole at $z = z_q = lL$, i.e., by the Bethe dipoles:

$$\begin{cases} \vec{P}_l = \hat{r} \epsilon_0 \left[\alpha_e^{(e)} E_0(l) + \alpha_e^{(i)} E_{cb}(l) \right] \delta(z - lL), \\ \vec{M}_l = \hat{\phi} \left[\alpha_m^{(e)} H_0(l) + \alpha_m^{(i)} H_{cb}(l) \right] \delta(z - lL) \end{cases} \quad (C7)$$

can be written:

$$\begin{cases} E_{cb}^{(l)}(z) = \frac{C_l^\pm}{r} e^{\pm jk_g(z-lL)}, & z \gtrless lL, \\ H_{cb}^{(l)}(z) = \pm \frac{Y_0 C_l^\pm}{r} e^{\pm jk_g(z-lL)}, & z \gtrless lL \end{cases} \quad (C8)$$

where k_g is the TEM propagation constant in the coaxial region, and the coefficients $C_l^\pm$ are readily evaluated by resorting to Lorentz (reciprocity) theorem in the form [42]:

$$\begin{aligned} \iint_{\partial V} (\vec{E} \times \vec{\mathcal{H}}^\pm - \vec{\mathcal{E}}^\pm \times \vec{H}) \cdot \vec{n} d\Sigma &= \\ = j\omega\mu_0 \vec{\mathcal{H}}^\pm(lL) \cdot \vec{M}_l - j\omega \vec{\mathcal{E}}^\pm(lL) \cdot \vec{P}_l \end{aligned} \quad (C9)$$

where V is a slice of the coaxial region limited by the planes $z = lL \pm \delta$, δ as small as one wishes, and $(\vec{\mathcal{E}}^\pm, \vec{\mathcal{H}}^\pm)$ is a source-free forward or backward TEM mode. Hence:

$$\begin{cases} C_l^{(+)} = \frac{jk}{4\pi b \log(a/b)} \left[(\alpha_m^{(e)} + \alpha_e^{(e)})E_0(l) + \right. \\ \left. + (\alpha_m^{(i)} + \alpha_e^{(i)})E_{cb}^+(l) - (\alpha_m^{(i)} - \alpha_e^{(i)})E_{cb}^-(l) \right], \\ C_l^{(-)} = \frac{-jk}{4\pi b \log(a/b)} \left[(\alpha_m^{(e)} - \alpha_e^{(e)})E_0(l) + \right. \\ \left. + (\alpha_m^{(i)} - \alpha_e^{(i)})E_{cb}^+(l) - (\alpha_m^{(i)} + \alpha_e^{(i)})E_{cb}^-(l) \right]. \end{cases} \quad (C10)$$

Here the fields $E_{cb}^{(\pm)}$ denote the forward and backward part of E_{cb} , respectively.

Following [27], we shall make the assumption that the backward fields are negligible w.r.t. the forward ones ²¹.

Thus we shall focus on the first of (C10), neglect the last term on its r.h.s., and drop the suffix '+'. Using the first of (C10) and (C8), the (forward propagating) field at $r = b$, $z = z_p = pL$ due to the source at $z = z_l = lL$ is obtained:

$$\begin{aligned} E_{cb}^{(l)}(p) &= \frac{jk}{4\pi b^2 \log(a/b)} \left[(\alpha_m^{(e)} + \alpha_e^{(e)})E_0(l) + \right. \\ &\quad \left. + (\alpha_m^{(i)} + \alpha_e^{(i)})E_{cb}(l) \right] e^{jk_g(z_p - z_l)}. \end{aligned} \quad (C11)$$

The total field at $z = pL$ due to *all* holes is obtained by summing (C11) over all $l < p$:

$$E_{cb}^{(tot.)}(p) = \frac{jk}{4\pi b^2 \log(a/b)} \sum_{l < p} \left[(\alpha_m^{(e)} + \alpha_e^{(e)})E_0(l) + \right.$$

²¹The TEM waves produced by the holes accumulate coherently in the forward direction, since they travel at the same speed as the primary field of an (assumed) ultrarelativistic particle in the liner. The hole spacing is assumed to be such that no backward phasing may occur within the spectrum of the primary field.

$$+ (\alpha_m^{(i)} + \alpha_e^{(i)}) E_{cb}(l) \Big] e^{jk_g(z_p - z_l)}. \quad (C12)$$

At this point we *postulate* eq. (C3), viz.:

$$E_{cb}(p) = F E_0(p), \quad (C13)$$

and take F as independent of p , because the (infinite) structure is invariant under the group of z -translations by multiples of L . Letting (C13) into (C12) we get:

$$\begin{aligned} F E_0(p) &= \frac{jk}{4\pi b^2 \log(a/b)} \sum_{l < p} \left[(\alpha_m^{(e)} + \alpha_e^{(e)}) E_0(l) + \right. \\ &\quad \left. + (\alpha_m^{(i)} + \alpha_e^{(i)}) F E_0(l) \right] e^{jk_g(z_p - z_l)}. \end{aligned} \quad (C14)$$

Then we use (C2) to express $E_0(l)$ in terms of $E_0(p)$, whence:

$$\begin{aligned} F E_0(p) &= \frac{jk}{4\pi b^2 \log(a/b)} \sum_{l < p} \left[(\alpha_m^{(e)} + \alpha_e^{(e)}) + \right. \\ &\quad \left. + (\alpha_m^{(i)} + \alpha_e^{(i)}) F \right] E_0(p) e^{j(k_g - k)(z_p - z_l)}. \end{aligned} \quad (C15)$$

The geometric series in (C15) is readily summed:

$$\sum_{l < p} e^{j(k_g - k)(p - l)L} = \frac{1}{1 - e^{j(k_g - k)L}} \approx \frac{-1}{j(k_g - k)L} \quad (C16)$$

where the last approximation is justified in view of the expected smallness of the exponent. Thus, from (C15),

$$F = \frac{jk(\alpha_m^{(e)} + \alpha_e^{(e)})}{jk(\alpha_m^{(i)} + \alpha_e^{(i)}) - 4\pi j(k_g - k) L b^2 \log(a/b)}. \quad (C17)$$

To evaluate the difference $k_g - k$ in (C17), we note that the free-space wavenumber k is equal to the loss-free TEM propagation constant in the coaxial region [42]. Hence²²:

$$k_g - k \approx j \frac{Z_{cb}}{Z_0} \frac{a + b}{2 ab \log(a/b)}, \quad (C18)$$

²²Equation (C16) follows immediately using the following 1st-order formula for the (longitudinal) wavenumber in perturbed waveguides whose walls are described by an impedance boundary condition [44]:

$$k_{pert.} - k_{unpert.} = j \frac{\int_{\partial S} Z_{cb} |\vec{H}_0|^2 d\ell}{2 \operatorname{Re} \int_S \vec{E}_0^* \times \vec{H}_0 \cdot \hat{n} dS},$$

where S is the cross-section, ∂S its contour, (E_0, H_0) the unperturbed mode, and Z_{cb} is the complex (cold-bore) wall characteristic impedance.

Substitution of (C18) into (C17) gives after a little algebra (*no* further approximations involved):

$$F = - \frac{(\alpha_e^{(e)} + \alpha_m^{(e)})(\alpha_e^{(i)} + \alpha_m^{(i)})^{-1}}{1 + \frac{(1+b/a)Z_{cb}}{Z_{p.w.}^{(0)}}} \quad (C19)$$

where $Z_{p.w.}^{(0)}$ is the wall impedance of the perforated wall in free space, given by eq. (8). Equation (C19) can also be written:

$$F = - \frac{\alpha_e^{(e)} + \alpha_m^{(e)}}{\alpha_e^{(i)} + \alpha_m^{(i)} + j \operatorname{sgn}(k) \hat{\delta}^* n_\sigma^{-1} (1+b/a)}, \quad (C20)$$

where $\hat{\delta}$ is the complex EM penetration depth into the lossy coax-walls, viz.:

$$\hat{\delta} = (|k_0|Z_0)^{-1} Z_{wall}^*, \quad Z_{wall} = [1 - j \operatorname{sgn}(\omega)] \left(\frac{|k_0|Z_0}{2\sigma_{cb}} \right)^{1/2}. \quad (C21)$$

Note that in [27], [28] the *complex* character of (C16) is apparently ignored.

The parasitic losses in the co-axial region have been computed in [7] in terms of the co-axial transmission line current I_{cb} . In view of the relationship between this latter and the magnetic field in the co-axial region, using eq.s (C2), (C3), (C13) one gets (for a pointlike bunch $I(z, t) = Q\beta c\delta(z - \beta ct)$):

$$\left| \frac{I_{cb}(z, \omega)}{I(z, \omega)} \right| = |F|. \quad (C20)$$

Remarkably, the result obtained in [7] using a different approach is recovered, provided

$$|\alpha_e^{(i)} + \alpha_m^{(i)}| \ll |n_\sigma^{-1} \hat{\delta} (1+b/a)|, \quad (C21)$$

which holds true for the case discussed in [7].

Appendix D - The Functions $G_{Cu,SS}(d/a)$.

The general formula (1) can be applied to estimate the real part of the longitudinal impedance of a possible LHC liner, sketched in *Fig. 4*, using the exact solution E_0 for the field produced by a relativistic unit-charge particle traveling along the axis of a perfectly conducting square-section pipe obtained in [34], yielding:

$$\Re [Z_{\parallel}(\omega)] = \frac{\epsilon_0}{\beta_0 c Q^2} Y_0 \oint_{\partial S} Z_{wall} |E_{0n}(\vec{r}, 0)|^2 d\ell. \quad (D1)$$

For the geometry of *Fig. 4*, the pipe wall cross-section contour ∂S , can be written as $\partial S_{SS} + \partial S_{Cu}$, the first term representing the stainless-steel rounded

Figure 9: The functions $G_{Cu, SS}(d/a)$ for the geometry of *Fig. 4*.

corners, the second one the copper plated flat sides. Thus, eq. (D1) can be further written:

$$\Re [Z_{||}(\omega)] = \frac{Z_{wall}^{(ss)}}{4\pi^2 a} G_{SS}(d/a) + \frac{Z_{wall}^{(Cu)}}{4\pi^2 a} G_{Cu}(d/a), \quad (D2)$$

where $Z_{wall}^{(ss)}$, $Z_{wall}^{(Cu)}$ are the wall impedances for the unperforated stainless steel and copper plated) chamber walls, respectively, and

$$G_{Cu, SS} = \oint_{\partial S_{Cu, SS}} |\mathcal{E}_n(\vec{r}, 0)|^2 \frac{d\ell}{a}, \quad (D3)$$

with

$$\vec{\mathcal{E}}(\vec{r}, 0) = (2\pi\epsilon_0 a) \vec{E}_0(\vec{r}, 0). \quad (D4)$$

The functions $G_{Cu, SS}$ are shown in *Fig. 9* below.