(Non) singular Kantowski-Sachs Universe from quantum spherically reduced matter

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abstract

Using \(s\)-wave and large \(N\) approximation the one-loop effective action for 2d dilaton coupled scalars and spinors which are obtained by spherical reduction of 4d minimal matter is found. Quantum effective equations for reduced Einstein gravity are written. Their analytical solutions corresponding to 4d Kantowski-Sachs (KS) Universe are presented. For quantum-corrected Einstein gravity we get non-singular KS cosmology which represents 1) quantum-corrected KS cosmology which existed on classical level or 2) purely quantum solution which had no classical limit. The analogy with Nariai BH is briefly mentioned. For purely induced gravity (no Einstein term) we found general analytical solution but all KS cosmologies under discussion are singular. The corresponding equations of motion are reformulated as classical mechanics problem of motion of unit mass particle in some potential \(V\).

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1 Introduction

It is quite common belief that two-dimensional dilatonic gravity may be useful only as toy model for the study of realistic 4d gravity, especially in quantum regime (for a review of quantum gravity, see, for example, ref.[1]). However, it is quite well-known (for example, see ref.[2]) that spherical reduction of Einstein gravity leads to some specific dilatonic gravity (for its most general model, see ref.[3]). On the same time, spherical reduction of minimal 4d matter leads to 2d dilaton coupled matter.

The conformal anomaly for 2d conformally invariant, dilaton coupled scalar has been found in refs.[4, 5, 6, 7] and the correspondent anomaly induced effective action has been found in refs.[5, 6, 8, 9]. The same calculation for 2d and 4d dilaton coupled spinor has been presented recently in [10]. Using such anomaly induced effective action (i.e. working in s-wave and large \( N \) approximation) and adding it to reduced Einstein action one may study four-dimensional Kantowski-Sachs (KS) quantum cosmology [11] in consistent way as it was done in refs.[12] (for a discussion of 2d dilatonic quantum cosmology, see for example [13, 14, 15, 12]). (Note that using similar methods the inducing of wormholes in the early Universe has been recently investigated in ref. [16] confirming such inducing).

In the (mainly numerical) study of refs.[12] it was found that most of KS cosmologies under investigation are singular at the initial stage of the Universe evolution. The interesting question is : can we construct (non) singular KS quantum cosmologies using purely analytical methods?

In the present work we try to answer to this question. Using the analogy between KS cosmology and Schwarzschild BH (or its generalizations) after the interchange of time and radial coordinates (see [11, 17]) we found the particular solution of quantum equations of motion analytically. This solution represents non-singular KS cosmology (expanding Universe with always non-zero radius) which comes from Schwarzschild-de Sitter (or -anti de Sitter) BH after interchange of time and radial coordinates. For purely induced gravity (when cosmology is defined completely by quantum effects of matter) we present general analytical solution of quantum equations of motion. Unfortunately, in this case all found KS quantum cosmologies are singular. We also reformulate last problem as classical mechanics problem re-writing the system of equations as describing the motion of unit mass particle in some potential \( V \).
2 Anomaly induced effective action and non-singular KS cosmology

We will start from the action of Einstein gravity with $N$ minimal real scalars and $M$ Majorana fermions

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g(4)} \left( R^{(4)} - 2\Lambda \right) + \int d^4x \sqrt{-g(4)} \left( \frac{1}{2} \sum_{i=1}^{N} g^{\alpha\beta}_{(4)} \partial_\alpha \chi_i \partial_\beta \chi_i + \sum_{i=1}^{M} \bar{\psi}_i \gamma^\mu \nabla_\mu \psi_i \right)$$  \hspace{1cm} (1)

where $\chi_i$ and $\psi_i$ are real scalars and Majorana spinors, respectively. In order to apply large $N$ approach $N$ and $M$ are considered to be large, $N, M \gg 1$, $G$ and $\Lambda$ are gravitational and cosmological constants.

We now assume the spherically symmetric spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi} d\Omega^2$$  \hspace{1cm} (2)

where $\mu, \nu = 0, 1$, $g_{\mu\nu}$ and $\phi$ depend only from $x^0 = t$, and $d\Omega^2$ corresponds to two-dimensional sphere.

The action (1) reduced according to (2) takes the form

$$S_{\text{red}} = \int d^2x \sqrt{-g} e^{-2\phi} \left[ -\frac{1}{16\pi G} \left\{ R + 2(\nabla \phi)^2 - 2\Lambda + 2e^{2\phi} \right\} + \frac{1}{2} \sum_{i=1}^{N} (\nabla \chi_i)^2 + \sum_{i=1}^{2M} \bar{\psi}_i \gamma^\mu \nabla_\mu \psi_i \right]$$  \hspace{1cm} (3)

Note that the fermion degrees of freedom after reduction are twice of original ones. In the spherical reduction $\gamma^\mu \nabla_\mu$ is replaced by $\gamma^\mu (\nabla_\mu + \partial_\mu)$ but the second term does not contribute to the action.

Working in large $N$ and $s$-wave approximation one can calculate the quantum correction to $S_{\text{red}}$ (effective action). Using 2d conformal anomaly for dilaton coupled scalar and dilaton coupled spinor, one can find the anomaly induced effective action. There is no consistent approach to calculate the conformally invariant part of this functional in closed form for scalars. However one can find this functional as some expansion of Schwinger–DeWitt type keeping only leading term. Then the effective action may be written in
the following form[6, 8, 10]

$$W = \frac{-1}{8\pi} \int d^2x \sqrt{-g} \left[ \frac{N + M}{12} R \frac{1}{\Delta} R - N \nabla^\lambda \phi \nabla_{\lambda} \phi \frac{1}{\Delta} R 
+ \left( N + \frac{2M}{3} \right) \phi R + 2N \ln \mu^2 \nabla^\lambda \phi \nabla_{\lambda} \phi \right].$$

(4)

Note that numerical coefficient in front of log term does not matter as it can be changed by rescaling of $\mu$. As it was shown in ref.[10] dilaton coupled spinors do not give contribution to this term, at least in leading order of SD expansion. The anomaly induced effective action for dilaton coupled spinor is found also in Appendix following ref.[10].

The equations of motion may be obtained by the variation of $\Gamma = S_{red} + W$ with respect to $g^{\pm\pm}, g^{\pm\mp}$ and $\phi$

$$0 = \frac{e^{-2\phi}}{4G} \left( 2 \partial_t \rho \partial_t \phi + (\partial_t \phi)^2 - \partial_t^2 \phi \right) - \frac{N + M}{12} \left( \partial_t^2 \rho - (\partial_t \rho)^2 \right) - \frac{N}{2} \left( \rho + \frac{1}{2} \right) (\partial_t \phi)^2 - \frac{N}{4} \ln \mu^2 (\partial_t \phi)^2 + t_0$$

(5)

$$0 = \frac{e^{-2\phi}}{8G} \left( 2 \partial_t^2 \phi - 4 (\partial_t \phi)^2 + 2 \Lambda e^{2\rho} - 2 e^{2\rho + 2\phi} \right) + \frac{N + M}{12} \partial_t^2 \rho + \frac{N}{4} (\partial_t \phi)^2 - \frac{N}{4} + \frac{2M}{3} \partial_t^2 \phi$$

(6)

$$0 = \frac{e^{-2\phi}}{4G} \left( - \partial_t^2 \phi + (\partial_t \phi)^2 + \partial_t^2 \rho - \Lambda e^{2\rho} \right) + \frac{N}{2} \partial_t (\rho \partial_t \phi) + \frac{N + \frac{M}{3}}{4} \partial_t^2 \rho + \frac{N}{2} \ln \mu^2 \partial_t^2 \phi.$$ 

(7)

Here we have chosen the conformal gauge $g_{\pm\mp} = -\frac{1}{2} e^{2\rho}, g_{\pm\mp} = 0$ ($x^{\pm} \equiv t \pm r$) and $t_0$ is a constant which is determined by the initial conditions. Combining (5) and (6) we get

$$0 = \frac{e^{-2\phi}}{4G} \left( - (\partial_t \phi)^2 + 2 \partial_t \rho \partial_t \phi + \Lambda e^{2\rho} - e^{2\rho + 2\phi} \right) + \frac{N + M}{12} (\partial_t \rho)^2$$

$$- \frac{N}{2} \rho (\partial_t \phi)^2 - \frac{N}{4} + \frac{2M}{3} \partial_t \rho \partial_t \phi - \frac{N}{4} \ln \mu^2 (\partial_t \phi)^2 + t_0.$$

(8)

This equation may be used to determine $t_0$ from the initial condition, it decouples from the rest two equations. Hence, Eq.(8) is not necessary in subsequent analysis.
It is often convenient to use the cosmological time $\tau$ instead of $t$, where the metric is given by

$$ds^2 = -d\tau^2 + e^{2\rho}d\tau^2 + e^{-2\phi}d\Omega^2. \quad (9)$$

Since we have $d\tau = e^\phi dt$, we obtain $\partial_t = e^\phi \partial_\tau$ and $\partial_t^2 = e^{2\phi} (\partial^2 + \partial_\tau \partial_\tau)$. Then Eqs. (6) and (7) may be rewritten as follows

$$0 = \left(\frac{e^{-2\phi}}{G} - N - \frac{2M}{3}\right) \partial^2_\tau \phi + \frac{N + M}{3} \partial^2_\tau \rho + \left(-\frac{2e^{-2\phi}}{G} + N\right) (\partial_\tau \phi)^2 \quad (10)$$

$$0 = \left(\frac{e^{-2\phi}}{G} - N - \frac{2M}{3}\right) \partial^2_\tau \rho + \left\{\frac{e^{-2\phi}}{G} + 2N \left(\rho + 1 + \ln \mu^2\right)\right\} \partial^2_\tau \phi$$

$$- \frac{e^{-2\phi}}{G} (\partial_\tau \phi)^2 + \left\{\frac{e^{-2\phi}}{G} + 2N \left(\rho + 1 + \ln \mu^2\right)\right\} \partial_\tau \rho \partial_\tau \phi$$

$$- \left(\frac{e^{-2\phi}}{G} - N - \frac{2M}{3}\right) (\partial_\tau \rho)^2 + \frac{e^{-2\phi}}{G} \Lambda. \quad (11)$$

We now consider a special solution for Eqs.(5), (6) and (7) corresponding to the (Wick-rotated) Nariai solution [18], where $\phi$ is a constant : $\phi = \phi_0$. Then the Eqs.(6) and (7) can be rewritten as follows:

$$0 = \frac{3}{(N + M)G} \left(\Lambda e^{-2\phi_0} - 1\right) e^{2\rho} + \partial^2_\tau \rho \quad (12)$$

$$0 = \frac{\Lambda e^{-2\phi_0}}{G} \left(-\frac{e^{-2\phi_0}}{G} + N + \frac{2}{3} M\right)^{-1} e^{2\rho} + \partial^2_\tau \rho. \quad (13)$$

Comparing (12) with (13), we obtain

$$e^{-2\phi_0} = \frac{(2N + M)G}{6} + \frac{1}{2\Lambda} \pm \frac{1}{2} \sqrt{\frac{(2N + M)^2G^2}{9} + \frac{1}{\Lambda^2} - \frac{(8N + 6M)G}{3\Lambda}} \quad (14)$$

The sign $\pm$ in (14) should be $+$ if we require the solution coincides with the classical one $e^{-2\phi_0} = \frac{1}{\Lambda}$ in the classical limit of $N, M \to 0$. On the other hand, in the solution with the $-$ sign, we have $e^{-2\phi_0} \sim \frac{(3N + 2M)G}{3} \to 0$ in
the classical limit. Therefore the second solution does not correspond to any classical solution but the solution is generated by the quantum effects.

The solution of (12) and (13) is given by

\[ e^{2\rho} = \begin{cases} 
\frac{2C}{R_0 \cos^2(t\sqrt{C})} & \text{when } R_0 > 0 \\
\frac{-2C}{R_0 \cosh^2(t\sqrt{C})} & \text{when } R_0 < 0 
\end{cases} \] (15)

Here \( C > 0 \) is a constant of the integration and \( R_0 \) is 2d scalar curvature, which is given by

\[ R_0 = 2e^{-2\phi_0} \partial_t^2 \rho = -\frac{3\Lambda}{(N + M)G} \left( \frac{(2N + M)G}{3} - \frac{1}{\Lambda} \right) \pm \sqrt{\left( \frac{(2N + M)^2G^2}{9} + \frac{1}{\Lambda^2} - \frac{(8N + 6M)G}{3\Lambda} \right)} \] (16)

Note that 4d curvature \( R_4 = R_0 + 2e^{2\phi_0} \) becomes a constant. It should be also noted that the solution exists for the both cases: of positive \( \Lambda \) and negative \( \Lambda \). In Eq.(16), the + sign corresponds to the classical limit \((N, M \to 0)\). In the limit, we obtain \( R_0 \to 2\Lambda \) \((R_4 \to 4\Lambda)\). On the other hand, the – sign in Eq.(16) corresponds to the solution with – sign in (14) generated by the quantum effect. In the classical limit for the solution, the curvature \( R_0 \) in (16) diverges as \( R_0 \sim \frac{3}{2(N + M)G} \to +\infty \). Therefore from (15), we find that \( e^{2\rho} \) vanish: (Note that \( R_0 > 0 \) in the limit): \( e^{2\rho} = \frac{4(N + M)GC}{3 \cosh^2(t\sqrt{C})} \to 0 \). Therefore by using (2), we obtain the following metric near the classical limit:

\[ ds^2 = \frac{4(N + M)GC}{3 \cosh^2(t\sqrt{C})} (-dt^2 + dr^2) + \frac{(3N + 2M)G}{3} d\Omega^2 \] (17)

This is non-singular metric for fixed \( N, M \).

It should be interesting to consider the limit \( \Lambda \to 0 \), where there is no de Sitter or anti-de Sitter solution at the classical level. In the limit, we can have a finite solution:

\[ e^{-2\phi_0} \to \frac{(3N + 2M)G}{3} \], \[ e^{2\rho} \to \frac{(N + M)G}{3 \cosh^2(t\sqrt{C})} \left( R_0 \to \frac{6}{(N + M)G} \right) \] (18)
This tells that the Nariai space can be generated by the quantum effect even if $\Lambda = 0$.

The obtained solution (15) (and (17)) might appear to have a singularity when $\cos^2 \left( t\sqrt{C} \right) = 0$ (for $R_0 > 0$ case) but the singularity is apparent one. In fact the scalar curvature $R_0$ in (16) is always constant. If we change the conformal time coordinate $t$ by the cosmological time $\tau$ in (9), we find that the time $\cos^2 \left( t\sqrt{C} \right) = 0$ corresponds to infinite future or past.

In the following, we assume $R_0 > 0$ for simplicity. $R_0 < 0$ case can be easily obtained by changing the constant $C \rightarrow -C$ and analytically continuing solutions. We now change the time-coordinate by $\tau = \sqrt{\frac{2}{R_0}} \ln \left( \frac{1+\tan \left( \frac{t\sqrt{C}}{2} \right)}{1-\tan \left( \frac{t\sqrt{C}}{2} \right)} \right)$.

Then the time $\cos^2 \left( t\sqrt{C} \right) = 0$ ($t\sqrt{C} = \pm \frac{\pi}{2}$) corresponds to $\tau = \pm \infty$. Using the cosmological time $\tau$, we obtain the following metric

$$ds^2 = -d\tau^2 + \frac{2C}{R_0} \cosh^2 \left( \tau \sqrt{\frac{R_0}{2}} \right) dr^2 + e^{-2\phi_0} d\Omega^2.$$  \hspace{1cm} (19)

Here $e^{-2\phi_0}$ is given in (14). If we assume $r$ has the periodicity of $2\pi$, the metric describes non-singular Kantowski-Sachs Universe, whose topology is $S_1 \times S_2$. The radius of the $S_2$ is constant but the radius of $S_1$ has a minimum when $\tau = 0$ and increases exponentially with the absolute value of $\tau$.

Hence we found non-singular KS cosmology which exists on classical level and which also exists on quantum level (as quantum corrected KS cosmology). This metric may be considered as the one obtained from Schwarzschild-de Sitter (Nariai) BH (for positive cosmological constant) [18] and from Schwarzschild-anti-de Sitter BH (for negative cosmological constant). To make the correspondence one has to interchange time and radial coordinates assuming corresponding Wick-rotation. It is very interesting that the last case (of negative cosmological constant) may be relevant to AdS/CFT correspondence. We also found non-singular KS Universe which does not have the classical limit and which is completely induced by quantum effects (even in the case of zero cosmological constant). Hence we obtained expanding Universe with the radius which is never zero. This cosmology may be interesting in frames of inflationary Universe as it can describe some sub-stage of inflationary Universe where there is effective expansion only along one (or two) space coordinates.
3 Induced gravity and singular KS quantum cosmology

Let us discuss now the situation when we live in the regime where quantum (non-local) anomaly induced effective action gives major contribution to equations of motion. In other words, quantum cosmology is defined completely by quantum effects (effective gravity theory which at some point makes transition to classical gravity). As we will see in this case the equations of motion admit the analytical solutions which lead to singular KS cosmology.

We consider purely induced gravity, i.e. \(N, M \to \infty\) case. Then the Einstein action can be dropped away. For this case, the field equations have the form

\[
0 = -\left(N + \frac{2M}{3}\right)\partial^2_\tau \phi + \frac{N + M}{3} \partial^2_\tau \rho + N(\partial_\tau \phi)^2
\]

\[
-\left(N + \frac{2M}{3}\right) \partial_\tau \rho \partial_\tau \phi + \frac{N + M}{3} (\partial_\tau \rho)^2
\]

\[
0 = \left(N + \frac{2M}{3}\right) \partial^2_\tau \rho + 2N(\rho + a) \partial^2_\tau \phi
\]

\[
+ 2N(\rho + 1 + a) \partial_\tau \rho \partial_\tau \phi + \left(N + \frac{2M}{3}\right)(\partial_\tau \rho)^2.
\]  

(20)

(21)

where \(a = \ln \mu^2\).

Equation (21) admits the following integral of motion

\[
I_1 = e^\rho \left[(N + \frac{2M}{3})\rho' + 2N(\rho + a)\phi'\right]
\]  

(22)

Here \(\prime = \frac{d}{d\tau}\). For the case \(\phi = \text{const} = \phi_0\), we have the following solution of equations (20-21)

\[
r(\tau) = \text{const} = e^{-\phi_0}, \quad f(\tau) = e^\rho = (f'_0 \tau + f_0).
\]  

(23)

For the metric (9), scalar curvature has the following form

\[
R = 2 \left(e^{2\phi} + 3 \phi'^2 - 2 \phi' \rho' + \rho'^2 - 2 \phi'' + \rho''\right).
\]  

(24)
Then for the solution (23) we have $R = 2e^{2\phi_0} = \text{const}$. For the case $\rho = \text{const} = \rho_0$, we have the solution of equations (20-21)

$$\rho_0 = -a, \quad r(\tau) = e^{-\phi} = (c_1 \tau + c_2)^{1+\frac{2M}{3N}}, \quad c_1, c_2 = \text{const}.$$  \hspace{1cm} (25)

$$c_2 = \exp \left( \frac{-3N \phi_0}{2M + 3N} \right), \quad c_1 = c_2 \frac{-3N \phi'_0}{(2M + 3N)}.$$  \hspace{1cm} (26)

Here $\phi_0$ and $\phi'_0$ are the values of $\phi$ and $\phi'$ at $\tau = 0$, respectively. For the solution (25) we have the following scalar curvature

$$R = \frac{2 \left( c_1^2 (4M^2 + 8MN + 3N^2) + \frac{3N^2 M}{(c_2 + c_1 \tau)^{1+\frac{4M}{3N}}} \right)}{3N^2 (c_2 + c_1 \tau)^2}.$$  \hspace{1cm} (27)

The solution (25) has a singularity at $\tau = -\frac{c_2}{c_1}$.

If $\rho \neq \text{const}$, then we obtain the following special solution

$$f(\tau) = e^\rho = f_0' \tau + f_0, \quad r(\tau) = e^{-\phi} = c_3 \left[ a + \ln(f_0' \tau + f_0) \right]^{1+\frac{4M}{3N}}.$$  \hspace{1cm} (28)

Here $f_0$ and $f_0'$ are the values of $f = e^\rho$ and $f'$ at $\tau = 0$ respectively. For the solution (28) we have

$$R = \frac{2 \left( f_0'^2 \left( \frac{4M^2 + 8MN + 3N^2}{N^2 (f_0' \tau + f_0)^2} + \frac{3e^{\phi_0} (a + \rho_0)^{2+\frac{4M}{3N}}}{(a + \ln(f_0' \tau + f_0))^{1+\frac{4M}{3N}}} \right) \right)}{3 (a + \ln(f_0' \tau + f_0))^2}.$$  \hspace{1cm} (29)

The solution (28) has a singularity when $\tau = -\frac{f_0}{f_0'} - e^{-a} f_0$.

Let us consider the case when $I_1 = 0$, $\rho \neq \text{const}$, then we have the following special solution

$$\frac{d\tilde{\rho}}{d\tau} = \pm \frac{6N}{2M + 3N} \tilde{\rho} e^{-\tilde{\rho}} \sqrt{ \frac{c_1}{\tilde{\rho} \left[ 1 + \frac{6(M+N)}{(2M+3N)\tilde{\rho}} \right]}}, \quad \tilde{\rho} = \rho + a,$$  \hspace{1cm} (30)

$$r(\tau) = e^{-\phi} = c_2 |\tilde{\rho}|^{\frac{2M+3N}{6N}}, \quad c_1, c_2 = \text{const}.$$  \hspace{1cm} (31)

For the solution (30-31), we have the following scalar curvature

$$R = \frac{e^{2\phi} + \frac{3c_1 (2M + 3N)}{e^{2|\tilde{\rho}| \left( (2M + 3N)^2 + 6(M + N) \tilde{\rho} \right)^2}} \times \left\{ (2M + N)(2M + 3N)^2 + 6(2M^2 + MN(2N + 1) + N^2(3N - 1)) \tilde{\rho} \right\}}{1}.$$  \hspace{1cm} (32)

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The solution (30-31) is singular when \( \tilde{\rho} = 0 \) or \( \tilde{\rho} = \frac{(2M+3N)^2}{6(M+N)} \).

We now consider more general cases. First we should note that the equation (20) and (21) admit one more integral besides \( I_1 \) in (22)

\[
I_2 = e^{2\rho} \left[ \frac{N + M}{12} (\rho')^2 - \frac{N}{2} \rho (\phi')^2 - \frac{N + \frac{2}{3} M}{2} \rho \phi' - \frac{N}{2} a (\phi')^2 \right] \tag{33}
\]

Since Eq.(22) can be solved with respect to \( \phi' \)
\[
\phi' = \frac{1}{2N(\rho + a)} \left[ - \left( N + \frac{2}{3} M \right) \rho' + I_1 e^{-\rho} \right] \tag{34}
\]
we can delete \( \phi \) by substituting (34) into (33) and we obtain
\[
0 = \frac{1}{2} (\rho')^2 + V(\rho) , \quad V(\rho) = -\frac{6I_2}{N + M} \rho + a - \alpha e^{-2\rho} . \tag{35}
\]

Here \( \alpha \equiv -\frac{I_2^2}{8N^2} \) and \( \beta \equiv -\frac{3(N+\frac{2}{3}M)^2}{2N(N+M)} \). Note that \( \beta \) is negative and \( \alpha \) is positive (negative) when \( I_2 \) is negative (positive).

Since the 4d scalar curvature is given in (24), Eq.(35) tells that there would be a curvature singularity when \( \rho + a \to \beta \pm 0 \). In fact, when \( \rho + a \sim \beta \pm 0 \), we obtain from (35) \( (\rho')^2 \sim \frac{A}{\rho + a - \beta} \), \( A \equiv \frac{12I_2(\beta + \alpha)e^{2(a-\beta)}}{N+M} \). Therefore we find
\[
\rho + a = \beta + \left( \frac{3A}{2} (\tau - \tau_0) \right)^{\frac{2}{3}} . \tag{36}
\]
Here \( \tau_0 \) is a constant of the integration and \( \rho + a = \beta \) when \( \tau = \tau_0 \). By substituting (36) into (24), we find the behavior of the scalar curvature \( R \)
\[
R \sim \frac{4N}{27 \left( N + \frac{2}{3} M \right)} \left( \frac{3A}{2} \right)^{\frac{2}{3}} (\tau - \tau_0)^{-\frac{4}{3}} + \frac{4M^2}{243 \left( N + \frac{2}{3} M \right)^2} \left( \frac{3A}{2} \right)^{\frac{2}{3}} (\tau - \tau_0)^{-\frac{2}{3}} . \tag{37}
\]
Therefore there is always a singularity when \( \rho + a \sim \beta \pm 0 \) except \( \alpha = \beta \) case, when \( A \) vanishes (we should note that \( A \) is finite when \( I_2 = 0 \)).

In the case \( \alpha = \beta \), (35) can be explicitly solved to give
\[
e^\rho = \pm \frac{12I_2}{N + M} (\tau - \tau_0) . \tag{38}
\]
Here $\tau_0$ is a constant of the integration. (38) tells that there is a singularity when $\tau = \tau_0$. In case of the expanding universe (+ sign in (38)), (34) tells $\phi' = 0$, i.e., $\phi$ is a constant as in the Nariai space [18] (note that there is a singularity even in this case, which is different from the Nariai space). On the other hand, in case of the shrinking universe ($-$ sign in (38)), from (34), we obtain

$$\phi' = -\frac{(N + M)I_1}{2NI_2(\tau_0 - \tau)}\left\{a + \ln \left(\frac{12I_2(\tau - \tau_0)}{N + M}\right)\right\}. \quad (39)$$

Eq.(39) tells that there is a curvature singularity when $\tau - \tau_0 = -\frac{N + M}{12I_2}e^{-a}$ besides $\tau = \tau_0$.

Eq.(35) might tell that there would be a kind of singularity (not always curvature singularity) when $\rho + a \to \alpha \pm 0$. We now investigate the behavior near $\rho + a \to \alpha \pm 0$. Then Eq.(35) has the form of: $(\rho')^2 \sim B(\rho + a - \alpha)$ ($B \equiv \frac{12I_2e^{2(a - \alpha)}}{(N + M)(\alpha - \beta)}$). $B$ should be positive (negative) if $\rho + a \to \alpha + 0$ ($\rho + a \to \alpha - 0$) since $(\rho')^2 \geq 0$. Then we obtain

$$\rho + a \sim \alpha + B \frac{4}{\tau - \tau_\alpha}^2. \quad (40)$$

Here $\tau_\alpha$ is a constant of the integration and $\rho + a = \alpha$ when $\tau = \tau_\alpha$. Eq.(40) tells that $\rho$ is ‘reflected’ (i.e., $\rho'$ changes its sign) at $\tau = \tau_\alpha$ smoothly without curvature singularity.

Eq. (34) also tells that there might be a singularity when $\rho + a = 0$. When $\rho + a \sim 0$, the behaviour of $\rho'$ is given from (35) by

$$\rho' \sim \pm \frac{I_1e^a}{N + \frac{2}{3}M} \left\{1 - \frac{1}{2} \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)(\rho + a)\right\}. \quad (41)$$

Substituting (41) into (35), we find that there is no singularity at $\rho + a = 0$ if + sign in (41) is chosen. This means that the singularity does not appear $\rho' > 0$ if $I_1 > 0$ or $\rho' < 0$ if $I_1 < 0$ but the singularity would appear in other cases since (35) tells

$$\phi - \phi_0 \sim \frac{I_1(N + \frac{2}{3}M)}{N} \ln |\tau - \tau_\phi| \cdot \quad (42)$$

Here $\phi_0$ and $\tau_\phi$ are constants of integration and $\rho + a = 0$ when $\tau = \tau_\phi$. 

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Substituting (42) into (24), we find $R$ has a singularity when $\tau = \tau_\phi$:

$$R \sim 2e^{-2\phi_0} |\tau - \tau_\phi| \frac{I_1(N + \frac{2}{3}M)}{N} + \frac{2I_1 \left(N + \frac{2}{3}M\right)}{N} \left\{ \frac{3I_1 \left(N + \frac{2}{3}M\right)}{N} + 2 \right\} \frac{1}{(\tau - \tau_\phi)^2} .$$

(43)

If $I_1 > 0$, the second term diverges when $\tau \sim \tau_\phi$. On the other hand, if $I_1 < 0$, the first term diverges. Therefore there is a singularity if $I \neq 0$.

When $I_1 = 0$, $\alpha$ also vanishes. When $\rho + a \sim 0$, the behavior of $\rho$ is given by (40) by putting $\alpha = I_1 = 0$. Then the behavior of $\phi$ is given by using (34),

$$\phi \sim -\left(N + \frac{2}{3}\right) \ln |\tau - \tau_\phi| + \phi_0 .$$

(44)

Here $\phi_0$ is a constant of the integration and $\tau_\phi = \tau_\alpha$. From (24), we find that there is a curvature singularity when $\tau = \tau_\phi$:

$$R \sim 2e^{-2\phi_0} |\tau - \tau_\phi|^{-2(N + \frac{2}{3}M)} .$$

(45)

Eq.(35) can be compared with the system of one particle with unit mass and in the potential $V(\rho)$ in the classical mechanical system when the total energy vanishes. Since the “kinetic energy” $\frac{1}{2} (\rho')^2$ is positive, $\rho$ can have its value in the region where $V(\rho)$ is negative. Therefore the following cases can be allowed:

1) $0 > \beta > \alpha$ (I$_2$ > 0). In this case, the region with $\rho < \alpha$ and the region with $\rho > \beta$ are allowed. The region $\rho > \beta$ would correspond to the expanding universe but there is always a curvature singularity of (37) at $\rho + a = \beta$ ($\tau = \tau_\beta$).

2) $\alpha = \beta$ (I$_2$ > 0). In this case, from the solution (38), we find that there is a singularity when $\tau = \tau_0$ coming from (38). In case of the expanding universe (+ sign in (38)), (34) tells $\phi' = 0$, i.e., $\phi$ is a constant as in Nariai space [18]. On the other hand, in case of the shrinking universe (− sign in (38)), from (39), we find there is a curvature singularity when $\tau - \tau_0 = -\frac{N+M}{12I_2} e^{-a}$ besides $\tau = \tau_0$.

3) $0 > \alpha > \beta$ (I$_2$ > 0). The region $\rho + a < \beta$ and the region $\rho + a > \alpha$ are allowed. In the latter case, the shrinking universe turns to expand at $\rho + a = \alpha$ ($\tau = \tau_\alpha$) but there is always a curvature singularity coming
from the singularity as explained around Eq.(41) at $\rho + a = 0$ ($\tau = \tau_\phi$) when the universe is shrinking ($\tau_\phi < \tau_\alpha$) if $I_1 > 0$ or when the universe is expanding ($\tau_\phi > \tau_\alpha$) if $I_1 < 0$.

4) $0 = \alpha > \beta$ ($I_1 = 0$). When $I_2 < 0$, the region $\beta < \rho + a < 0$ can be allowed. On the other hand, when $I_2 > 0$, the region $\rho + a < \beta$ and the region $\rho + a > \alpha$ are allowed. In the case of $\rho + a > \alpha$, however, we find from (44) that there is a curvature singularity when $\rho + a \sim 0$ ($\tau \sim \tau_\alpha = \tau_\phi$).

5) $\alpha > 0 > \beta$ ($I_2 < 0$). Only the region $\beta < \rho < \alpha$ can be allowed. There is no any solution describing the expanding universe in this case.

As it follows from above analysis in purely induced gravity case when expanding Universe is constructed due to matter quantum effects one always gets the curvature singularity like in the case discussed in ref.[12]. Nevertheless, it is remarkable that equations of motion in this case admit analytical solutions.

In summary, using s-wave and large $N$ approximation we studied gravitational equations of motion with quantum corrections. The analytical solutions representing (non) singular KS cosmology are found. In this derivation, for non-singular KS Universe the analogy with Nariai BH (after interchange of time and radial coordinates) is used. In the same way, starting from more complicated multiple horizon BHs with constant curvature one can find other families of non-singular quantum cosmologies.

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A Anomaly induced effective action for dilaton coupled spinor

In this appendix, we present the conformal anomaly for 2d dilaton coupled spinors (it was derived in [10]). We start from 2d dilaton coupled spinor Lagrangian:

\[ L = \sqrt{-g} f(\phi) \bar{\psi} \gamma^\mu \partial_\mu \psi \]  \hspace{1cm} (46)

where \( \psi \) is 2d Majorana spinor, \( f(\phi) \) is an arbitrary function and \( \phi \) is dilaton.

Let us make now the following classical transformation of background field \( g_{\mu\nu} \):

\[ g_{\mu\nu} \rightarrow f^{-2}(\phi) \tilde{g}_{\mu\nu} . \]  \hspace{1cm} (47)

Then it is easy to see that \( \gamma^\mu(x) \rightarrow f(\phi) \tilde{\gamma}^\mu(x) \) and in terms of new classical metric we obtain usual, non-coupled with dilaton (minimal) Lagrangian for 2d spinor:

\[ L = \sqrt{-\tilde{g}} \bar{\psi} \tilde{\gamma}^\mu \partial_\mu \psi . \]  \hspace{1cm} (48)

Then we get the following conformal anomaly for dilaton coupled Majorana spinor (46):

\[
\sqrt{-g} T = \frac{\sqrt{-g}}{24\pi} \left[ \frac{1}{2} R - \Delta \ln f \right] \\
= \frac{\sqrt{-g}}{24\pi} \left[ \frac{1}{2} R - \frac{f'}{f} \Delta f - \frac{(f'' f - f'^2)}{f^2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] .
\]  \hspace{1cm} (49)

Integrating the anomaly, we find the anomaly induced effective action \( W \) in the covariant, non-local form [10]:

\[
W = -\frac{1}{2} \int d^2 x \sqrt{-g} \left\{ \frac{1}{96\pi} \frac{1}{\Delta} R - \frac{1}{24\pi} R \ln f(\phi) \right\}
\]  \hspace{1cm} (50)

That gives anomaly induced effective action for dilaton coupled spinors. It is interesting that adding this \( W \) to classical part of CGHS model[19] we get RST model[20] where last term in \( W \) has been introduced by hands in ref.[20]. Dilaton coupled quantum spinor gives the natural realization of RST model.
References


