Graviton and gauge boson propagators in $AdS_{d+1}$

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Abstract

We construct the gauge field and graviton propagators in Euclidean $AdS_{d+1}$ space-time by two different methods. In the first method the gauge invariant Maxwell or linearized Ricci operator is applied directly to bitensor ansatze for the propagators which reflect their gauge structure. This leads to a rapid determination of the physical part of the propagators in terms of elementary functions. The second method is a more traditional approach using covariant gauge fixing which leads to a solution for both physical and gauge parts of the propagators. The gauge invariant parts agree in both methods.

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1 Introduction

One important aspect of the AdS/CFT correspondence conjectured in [1, 2, 3] is the calculation of correlation functions in Type IIB supergravity on $AdS_5 \times S^5$ [4] in order to study the large $N$ strong coupling dynamics of $\mathcal{N} = 4$ superconformal Yang-Mills theory.

Bulk to bulk Green’s functions, describing propagation between two interior points of $AdS$, are required to compute 4-point functions [5, 6, 7, 8, 9, 10]. Scalar and spinor propagators for general mass have been discussed in the old literature [11] on $AdS$ field theory. The Feynman gauge propagator for massless gauge bosons, was obtained in [12]. The result involves transcendental functions which cancel for physical components of the gauge field. This motivated a recent simpler derivation [8] which used a convenient bitensor ansatz to obtain the physical part of the propagator as a simple algebraic function of the chordal distance variable. There are older covariant treatments of the graviton propagator [13, 14, 15] in de Sitter space (of constant positive curvature) which again involve transcendental functions, and there is recent work on the gauge boson and graviton propagators for $AdS$ in non–covariant gauges [6].

In the present paper, we shall construct the gauge boson and graviton propagator in $AdS_{d+1}$ using two different methods, both covariant under the isometry group $SO(d + 1, 1)$ of the Euclidean AdS space.

The first method is a modification of that of [8]. The key step is a pair of ansatze for the bitensor propagators which naturally separate gauge invariant parts from gauge artifacts. The latter do not contribute for conserved sources of the fields. The gauge invariant components can then be rather simply obtained by substituting the ansatz in the Maxwell or linearized Ricci equations. They are given by elementary functions. Gauge fixing is unnecessary because we work on the subspace of conserved sources. In the second method the propagators are constructed in the Landau gauge. This method is closely related to the work of [12, 13, 14, 15], with some new technical simplifications. In particular we find a simple and general way to solve the inhomogeneous equation that arises for the transverse traceless part of the propagator. It is shown explicitly that the gauge invariant parts of the propagators agree in both methods.

These results have immediate applications to 4-point functions in the $AdS/CFT$ correspondence. Only the gauge invariant part of the propagators is required, since they are coupled to conserved currents or stress tensors. The amplitude for gauge boson exchange between external scalar fields was presented in [8]. The calculation of graviton exchange for even $d$ and general integer scaling dimension $\Delta$ of external scalars is complete and will be presented shortly [16].

The paper is organized as follows. In Section 2 we introduce standard notations for Euclidean
AdS\(_{d+1}\) and apply the new method to obtain the gauge boson propagator. This method is then applied to the graviton propagator in Section 3. The more traditional Landau gauge methods are applied to the gauge and graviton propagators in Sections 4 and 5, respectively.

2 New method for the gauge propagator

We work on the Euclidean continuation of AdS\(_{d+1}\), viewed as the upper half space, \(z_0 > 0\), in \(\mathbb{R}^{d+1}\) with metric \(g_{\mu\nu}\) given by

\[
ds^2 = \sum_{\mu,\nu=0}^{d} g_{\mu\nu} dz_\mu dz_\nu = \frac{1}{z_0^2} (dz_0^2 + \sum_{i=1}^{d} dz_i^2).
\]

(2.1)

The AdS\(_{d+1}\) scale has been set to unity, and the metric above has constant negative curvature \(R = -d(d+1)\). It is well known that invariant functions and tensors on AdS\(_{d+1}\) are most simply expressed in terms of the chordal distance variable \(u\), defined by

\[
u \equiv \frac{(z - w)^2}{2z_0 w_0}
\]

(2.2)

where \((z - w)^2 = \delta_{\mu\nu}(z - w)_\mu(z - w)_\nu\) is the “flat Euclidean distance”.

The action of an abelian gauge field coupled to a conserved current source \(J^\mu\) in the AdS\(_{d+1}\) background is

\[
S_A = \int d^{d+1}z \sqrt{g} \left[ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_{\mu} J^\mu \right].
\]

(2.3)

We have not included a gauge fixing term, because we work directly on the restricted space of conserved currents and look for a solution of the Euler-Lagrange equation in the form

\[
A_\mu(z) = \int d^{d+1}w \sqrt{g} G_{\mu\nu'}(z, w) J^{\nu'}(w)
\]

(2.4)

with bitensor propagator \(G_{\mu\nu'}(z, w)\). We see that the propagator must satisfy the AdS-covariant equation \((\partial_\mu = \frac{\partial}{\partial z_\mu} \text{ and } \partial_{\mu'} = \frac{\partial}{\partial w_{\mu'}})\)

\[
D^\mu \partial_{\mu'} G_{\nu'\nu} = -g_{\nu'\nu} \delta(z, w) + \partial_{\nu'} \Lambda_\nu(z, w)
\]

(2.5)

Here, \(\Lambda_\nu\) is a vector function, which acts as a pure gauge term, and will cancel out when integrating (2.5) against any conserved current \(J^{\nu'}\).

Any bitensor can be expressed [12] as a sum of two linearly independent forms with scalar coefficients. Following [8] we choose two independent bitensors derived from the biscalalar variable \(u\), namely

\[
\partial_{\mu} \partial_{\nu'} u = -\frac{1}{z_0 w_0} \left[ \delta_{\mu\nu'} + \frac{1}{w_0} (z - w)_\mu \delta_{\nu'0} + \frac{1}{z_0} (w - z)_{\nu'} \delta_{\mu 0} - u \delta_{\mu 0} \delta_{\nu'0} \right]
\]

(2.6)
and \( \partial_\mu u \partial_\nu' u \) with
\[
\partial_\mu u = \frac{1}{z_0} \left[ (z - w)_\mu / w_0 - u \delta_\mu 0 \right] \\
\partial_\nu' u = \frac{1}{w_0} \left[ (w - z)_{\nu'} / z_0 - u \delta_{\nu'0} \right].
\]

The propagator can be represented as a superposition of the bitensors \( \partial_\mu \partial_\nu' u \) and \( \partial_\mu u \partial_\nu' u \) times scalar functions of \( u \). It is more convenient to use the equivalent form [8]
\[
G_{\mu\nu'}(z, w) = -(\partial_\mu \partial_\nu' u) F(u) + \partial_\mu \partial_\nu' S(u) \tag{2.8}
\]
in which \( F(u) \) describes the propagation of the physical components of \( A_\mu \) and \( S(u) \) is a gauge artifact. We see that \( S(u) \) drops out of the invariant equation (2.5) and the integral solution (2.4) and can be discarded. Thus we must determine only \( F(u) \) which we will find by substitution of (2.8) into (2.5).

For this we need certain properties of derivatives of \( u \), most of which were derived in [8] and which we list here. (Some of these will be needed only later.)

\[
\Box u = D^\mu \partial_\mu u = (d + 1)(1 + u) \tag{2.9}
\]
\[
D^\mu u \partial_\mu u = u(2 + u) \tag{2.10}
\]
\[
D_\mu \partial_\nu u = g_{\mu\nu}(1 + u) \tag{2.11}
\]
\[
(D^\mu u) \left( D_\mu \partial_\nu' u \right) = \partial_\nu' u \partial_\nu u \tag{2.12}
\]
\[
(D^\mu u) \left( \partial_\mu \partial_\nu' u \right) = (1 + u) \partial_\nu' u \tag{2.13}
\]
\[
(D^\mu \partial_\mu' u) \left( \partial_\mu \partial_\nu' u \right) = g_{\mu'\nu'} + \partial_\mu' u \partial_\nu' u \tag{2.14}
\]
\[
D_\mu \partial_\nu \partial_\nu' u = g_{\mu\nu'} \partial_\nu u \tag{2.15}
\]

With the help of (2.9-2.15), we readily find
\[
D^\mu \partial_\mu G_{\nu'\nu'} = \partial_\nu \partial_\nu' u \left[ -u(2 + u) F''(u) - d(1 + u) F'(u) \right] + \partial_\nu u \partial_\nu' u \left[ (1 + u) F''(u) + dF'(u) \right]. \tag{2.16}
\]

For distinct \( z \) and \( w \), the \( \delta \)-function term in (2.5) may be ignored. Using \( AdS \)-invariance, we also have \( \Lambda_\nu = \partial_\nu u \Lambda(u) \) in (2.5). Using (2.16) we find that (2.5) gives the set of two second order differential equations for the functions \( F(u) \) and \( \Lambda(u) \), namely
\[
u(2 + u) F'' + d(1 + u) F' = -\Lambda \tag{2.17}
\]
\[
(1 + u) F'' + dF' = \Lambda'. \tag{2.18}
\]
Now, (2.18) is readily integrated to give \((1 + u)F' + (d - 1)F = \Lambda\). (An integration constant is chosen to vanish so that \(F(u)\) vanishes on the boundary.) Eliminating \(\Lambda\) between (2.17) and (2.18) gives
\[
u(2 + u)F'' + (d + 1)(1 + u)F' + (d - 1)F = 0, \tag{2.19}
\]
which agrees with [8] and is just the invariant equation
\[(\Box - m^2)F(u) = 0 \tag{2.20}\]
for the propagator of a scalar field of mass \(m^2 = -(d - 1)\). The solution is the elementary function, normalized to include the \(\delta\) function in (2.5),
\[
F(u) = \frac{\Gamma \left(\frac{d-1}{2}\right)}{(4\pi)^{(d+1)/2}} \left[u(2 + u)\right]^{-(d-1)/2}. \tag{2.21}
\]
As a further check on the consistency of this approach, we note that \(\Lambda\) vanishes as \(1/u^{d+1}\) on the boundary. The current \(J_\nu'(w)\) is also expected to vanish, and there is then no surface term which could negate the statement following (2.5) that the gauge term cancels out.

There is an even faster variation of this method, a bit less clear pedagogically, which we now outline. The advantage is that we need not deal with the gauge function \(\Lambda\) nor the coupled equations (2.17-2.18). Instead we insert the ansatz (2.8), with \(S(u)\) dropped, in the integral (2.4) and apply the Maxwell operator inside the integral. After some calculation using (2.9-2.15) we find
\[
D^\mu \partial_\mu A_\nu(z) = - \int d^{d+1}w \sqrt{g} \left[\partial_\nu \partial_\nu' u \Box F - d\partial_\nu' u \partial_\nu F - \partial_\mu \partial_\nu' u D^\mu \partial_\nu F\right] J_\nu'(w) \tag{2.22}
\]
\[
= - \int d^{d+1}w \sqrt{g} \left[\partial_\nu \partial_\nu' u (\Box F + (d - 1)F) - \partial_\nu \partial_\nu' ((d - 1)F + (1 + u)F')\right].
\]
The manipulations of derivatives which lead from the first to second line are designed so that the last term is a total derivative with respect to \(w\). This can be partially integrated and dropped by current conservation. One then finds that
\[
D^\mu \partial_\mu A_\nu(z) = - \int d^{d+1}w \sqrt{g} \partial_\nu \partial_\nu' u (\Box F + (d - 1)F) J_\nu'(w) = -J_\nu(z) \tag{2.23}
\]
must hold for all choices of current source \(J_\nu(z)\). Therefore \(F(u)\) must satisfy (2.20) with \(-\delta(z, w)\) on the right side, and the solution with fastest decay on the boundary is (2.21) as before.

The method of this section is somewhat simpler than that of [8] in which a version of (2.5) with gauge-fixing term added to the wave operator and \(\Lambda_\nu = 0\) was solved. The immediate
extension of that method to the graviton propagator does not work readily because it leads to intricately coupled equations for 5 scalar functions. The new method is far simpler, as we will now see.

3 New method for the graviton propagator

The gravitational action in \(d+1\) dimensional Euclidean space with cosmological constant \(\Lambda\) and covariant matter action \((S_m)\) is

\[
S_g = \frac{1}{2\kappa^2} \int d^{d+1}z \sqrt{g} (R - \Lambda) + S_m. \tag{3.1}
\]

Henceforth we set \(\kappa = 1\). The first variation with respect to \(g^{\mu\nu}\) gives the Euler-Lagrange equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - \Lambda) = T_{\mu\nu}. \tag{3.2}
\]

We take \(\Lambda = -d(d-1)\) so that without sources \((T_{\mu\nu} = 0)\), we obtain Euclidean AdS space with \(R = -d(d+1)\) as the maximally symmetric solution of (3.2).

A matter source produces a fluctuation \(h_{\mu\nu} = \delta g_{\mu\nu}\) about the AdS metric \(g_{\mu\nu}\). To obtain the graviton propagator it is sufficient to consider (3.2) to linear order in \(h_{\mu\nu}\). It is simpler to work with the equivalent “Ricci form” of this equation, namely

\[
R_{\mu\nu} + d g_{\mu\nu} = \tilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} T^\sigma_{\sigma}. \tag{3.3}
\]

It is straightforward but tedious to use textbook results [17] on the linearized Ricci tensor and commute covariant derivatives to obtain the linearized equation

\[
-D^\sigma D_\sigma h_{\mu\nu} - D_\mu D_\nu h^\sigma_{\sigma} + D_\mu D^\sigma h_{\sigma\nu} + D_\nu D^\sigma h_{\mu\sigma} - 2(h_{\mu\nu} - g_{\mu\nu} h_{\sigma\sigma}) = 2\tilde{T}_{\mu\nu}. \tag{3.4}
\]

All covariant derivatives and contractions in (3.4) are those of the \(AdS\) metric \(g_{\mu\nu}\) of (2.1). At this point we could impose a gauge condition such as the deDonder gauge, \(D^\sigma h_{\sigma\mu} = \partial_\mu h_{\sigma\sigma}/2\), which simplifies (3.4), and an approach of this type is pursued in Sec 4. Here, however, we will generalize the method used for the gauge propagator in Section 2 and work more directly with (3.4) which is invariant under infinitesimal diffeomorphisms, \(\delta h_{\mu\nu} = D_\mu V_\nu + D_\nu V_\mu\). We represent the solution of (3.4) or, equivalently, the linearization of (3.2), as the integral

\[
h_{\mu\nu}(z) = \int d^{d+1}w \sqrt{g} G_{\mu\nu\mu'\nu'}(z, w) T^{\mu'\nu'}(w) \tag{3.5}
\]
This defines the propagator which must then satisfy

\[
- D^\sigma D_\sigma G_{\mu\nu;\mu'} - D_\mu D_{\nu} G_{\sigma}^{\ \mu;\nu'} + D_\mu D^\sigma G_{\sigma\nu;\mu'}
\]

\[
+ D_\nu D^\sigma G_{\mu\nu;\mu'} - 2(G_{\mu\nu;\mu'} - g_{\mu\nu} G_{\sigma}^{\ \mu;\nu'})
\]

\[
= \left( g_{\mu\nu} g_{\nu\nu'} + g_{\mu\nu} g_{\nu\nu'} - \frac{2}{d-1} g_{\mu\nu} g_{\nu\nu'} \right) \delta(z, w) + D_\nu' \Lambda_{\mu
u;\nu'} + D_{\nu'} \Lambda_{\mu\nu;\nu'}
\]

where \( \Lambda_{\mu\nu;\nu'}(z, w) \) represents a diffeomorphism (in the \( ' \) coordinates) whose contribution will vanish when (3.6) is integrated against the covariantly conserved stress tensor \( T^{\mu\nu'} \).

Our strategy now is to solve (3.6) by expanding \( G_{\mu\nu;\mu'} \) and \( \Lambda_{\mu\nu;\nu'} \) in an appropriate basis of bitensors. One basis of fourth rank bitensors obtained by consideration of the geodesics between the points \( z \) and \( w \) was proposed by [13] and will be used in the next section. We start with a very similar basis of 5 independent fourth rank bitensors constructed from the basic forms \( \partial_\mu \partial_{\mu'} u, \partial_\mu u, \text{ and } \partial_{\mu'} u \) defined in (2.6-2.7)

\[
T^{(1)}_{\mu\nu;\mu'} = g_{\mu\nu} g_{\mu'\nu'}
\]

\[
T^{(2)}_{\mu\nu;\mu'} = \partial_\mu u \partial_{\nu'} u \partial_{\mu'} u \partial_{\nu} u
\]

\[
T^{(3)}_{\mu\nu;\mu'} = \partial_\mu \partial_{\mu'} u \partial_{\nu} \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \partial_{\nu} \partial_{\mu'} u
\]

\[
T^{(4)}_{\mu\nu;\mu'} = g_{\mu\nu} \partial_{\mu'} u \partial_{\nu'} u + g_{\mu\nu} \partial_{\mu} u \partial_{\nu} u
\]

\[
T^{(5)}_{\mu\nu;\mu'} = \partial_\mu \partial_{\mu'} u \partial_{\nu} \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \partial_{\nu} \partial_{\mu'} u + (\mu' \leftrightarrow \nu')
\]

and a general ansatz for the propagator as the superposition

\[
G_{\mu\nu;\mu'} = \sum_{i=1}^{5} T^{(i)}_{\mu\nu;\mu'} A_{(i)}(u)
\]

However, it is very advantageous to reorganize things so as to separate physical components of the propagator from gauge artifacts, and we therefore postulate the form

\[
G_{\mu\nu;\mu'} = (\partial_\mu \partial_{\mu'} u \partial_{\nu} \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \partial_{\nu} \partial_{\mu'} u) G(u) + g_{\mu\nu} g_{\mu'\nu'} H(u)
\]

\[
+ \partial_{\nu}[\partial_{\nu'} u \partial_{\mu'} u X(u)] + \partial_{\nu'}[\partial_{\nu} u \partial_{\mu} u X(u)]
\]

\[
+ \partial_{\nu}[\partial_{\nu'} u \partial_{\mu} u Y(u)] + \partial_{\nu'}[\partial_{\nu} u \partial_{\mu'} u Y(u)]
\]

\[
+ \partial_{\nu}[\partial_{\nu} u Z(u)] g_{\mu'\nu'} + \partial_{\nu'}[\partial_{\nu} u Z(u)] g_{\mu\nu}.
\]

where \( (\cdot) \) denotes symmetrization with strength 1. The terms involving \( X, Y \) and \( Z \) are gradients with respect to \( z \) or \( w \). The gradients with respect to \( z \) are diffeomorphisms which are annihilated in (3.6) and give an irrelevant modification of \( h_{\mu\nu} \) in (3.5). The gradients with respect to \( w \)
contribute to the left side of (3.6) as $D_{\mu'}\partial_{\nu'}(\cdots)$ and can be absorbed by changing $\Lambda_{\mu\nu;\mu'}$ on the right side. Thus we can restrict our attention to the physical $G(u)$ and $H(u)$ terms provided we are sure that the form (3.9) is actually equivalent to (3.8). It is not hard to convince oneself that this is the case by applying derivatives in (3.9) to obtain a set of five equations relating each $A_{(i)}(u)$ to a combination of $G, H$, and derivatives of $X, Y$ and $Z$. These expressions are simple enough that one can readily deduce that the $A_{(i)}$ are uniquely determined by $G, H, X, Y, Z$ and vice versa. There remains to write an analogous tensor decomposition of the diffeomorphism term on right side of (3.6) and we use the following

$$
\Lambda_{\mu\nu;\mu'} = g_{\mu\nu}\partial_{\nu'}uA(u) + \partial_{\mu}u\partial_{\nu'}uC(u) + (\partial_{\mu}u\partial_{\nu'}u + \partial_{\nu'}u\partial_{\mu}u)B(u)
$$

(3.10)

The stage is now set for the next part of our work which is to substitute the $G$ and $H$ terms of (3.9) and the decomposition (3.10) in (3.6), apply all derivatives using the relations (2.9-2.15), and express the result as a superposition of independent tensor terms. Actually 6 independent tensors appear because (3.6) does not have the symmetry under exchange of $z$ and $w$ that the propagator itself possesses. The full calculation is important but tedious, and we quote only the final result, which is

$$
- D^\sigma D_\sigma G_{\mu\nu;\mu'} + D_\mu D_\nu G_{\sigma;\mu'} + D_\mu D^\sigma G_{\sigma;\mu'} - D_\nu D^\sigma G_{\sigma;\mu'}
= 2(G_{\mu;\mu'} - g_{\mu\nu}G_{\sigma;\mu'}) - D_\mu\Lambda_{\mu\nu;\mu'} - \Lambda_{\mu\nu;\mu'}
= T^{(1)}[-u(2 + u)H'' - 2d(1 + u)H' + 2dH + 4G - 2(1 + u)G' - 2(1 + u)A]
+ T^{(2)}[-2G'' - 2C'] + T^{(3)}[-u(2 + u)G'' - (d - 1)(1 + u)G' + 2(d - 1)G - 2B]
+ g_{\mu\nu}\partial_{\mu'}u\partial_{\nu'}u[2(1 + u)G' + 4(d + 1)G - 2A']
+ g_{\mu\nu}\partial_{\mu'}u\partial_{\nu'}u[-2G'' - (d - 1)H'' - 4B - 2(1 + u)C]
+ T^{(5)}[(1 + u)G'' + (d - 1)G' - B' - C]
$$

(3.11)

For clarity, we have omitted the indices of the tensors $T^{(i)}$ defined in (3.7). We also assume that the points are separated so that the $\delta$-functions in (3.6) can be dropped.

The system of equations which determines the graviton propagator is obtained by setting the scalar coefficient of each independent tensor in (3.11) to 0. We focus first on the three equations

$$
C''(u) = -G''(u)
$$

(3.12)

$$
B(u) + C(u) = (1 + u)G'(u) + (d - 1)G(u)
$$

(3.13)

$$
-2B(u) = u(2 + u)G''(u) + (d - 1)(1 + u)G'(u) - 2(d - 1)G(u).
$$

(3.14)
The first two equations can be integrated immediately to give

\[ C(u) = -G'(u) \]  \hspace{2cm} (3.15)
\[ B(u) = (1 + u)G'(u) + (d - 1)G(u) \]

Integration constants are dropped so that all functions vanish on the boundary. When (3.15) is substituted in (3.14) we find an uncoupled differential equation for G(u), namely

\[ 0 = u(2 + u)G''(u) + (d + 1)(1 + u)G'(u) \]  \hspace{2cm} (3.16)

Comparing with (2.19-2.20) we see that this is exactly the equation for a massless scalar propagator. This result is consistent with the analysis of [6] in a non-covariant gauge. The correctly normalized solution can be obtained from older AdS literature [11] and is given by the hypergeometric function

\[ G(u) = \tilde{C}_d (2u^{-1})d F(d, \frac{d + 1}{2}; d + 1; -2u^{-1}) \]  \hspace{2cm} (3.17)

\[ \tilde{C}_d = \frac{\Gamma(d + 1)}{(4\pi)^{(d+1)/2d}} \]

An explicit solution in terms of elementary functions will be given below.

We must now study the three remaining tensor equations in (3.11). After eliminating \( B(u) \) and \( C(u) \) using (3.12) and (3.13) these become

\[ 0 = 2G'' + 2(1 + u)G' + 4(d - 1)G + (d - 1)H'' \]  \hspace{2cm} (3.18)
\[ 0 = -u(2 + u)H'' - 2d(1 + u)H' + 2dH + 4G - 2(1 + u)G' - 2(1 + u)A \]  \hspace{2cm} (3.19)
\[ 0 = (1 + u)G' + 2(d + 1)G - A' \]  \hspace{2cm} (3.20)

Since \( G(u) \) is already known from (3.16), this is an over-determined system, and we must both obtain a solution and check its consistency.

For this purpose we introduce the function \( p(u) \) such that

\[ p''(u) = G(u) \hspace{1cm} \text{with} \hspace{1cm} p(\infty) = p'(\infty) = 0 \]  \hspace{2cm} (3.21)

In terms of this function, it is not hard to solve the equations of (3.18-3.20) explicitly. From (3.18) and (3.20), and the requirement that \( H(u) \) and \( A(u) \) vanish at \( u = \infty \), one obtains respectively

\[ -(d - 1)H(u) = 2p''(u) + 2(1 + u)p'(u) + 4(d - 2)p(u) \]  \hspace{2cm} (3.22)
\[ A(u) = (1 + u)p''(u) + (2d + 1)p'(u) \]  \hspace{2cm} (3.23)
One can then check to see that (3.19) is also satisfied, if we use (3.16) and its integrated versions expressed in terms of \( p(u) \), which are given by

\[
\begin{align*}
    u(2+u)p'''(u) + (d-1)(1+u)p''(u) - (d-1)p'(u) &= 0 \\
    u(2+u)p''(u) + (d-3)(1+u)p'(u) - 2(d-2)p(u) &= 0
\end{align*}
\]

A more useful expression for \( H(u) \) can be obtained by eliminating \( p(u) \) in (3.22) using (3.24).

This gives

\[
\begin{align*}
    -(d-1)H(u) &= 2(1+u)^2G(u) + 2(d-2)(1+u)p'(u) \\
    -(d-1)p'(u) &= 2\tilde{C}_d(2u^{-1})^{d-1}F(d-1, \frac{d+1}{2}; d+1; -2u^{-1})
\end{align*}
\]

Since \( G(u) \) and \( H(u) \) express the physical content of the propagator and occur in integrals over \( AdS_{d+1} \) in applications to the AdS/CFT correspondence, explicit expressions in terms of elementary functions are most useful. The first step to obtain them is to integrate (3.16) to obtain

\[
G'(u) \sim \frac{1}{u(2+u)^{\frac{d+1}{2}}}
\]

The next integral can be evaluated, but the general form of the result depends on whether \( d \) is even or odd. For \( d \) even we can rearrange the result given in [18] (see Sec. 2.263,(4)) to obtain

\[
G(u) = \frac{\Gamma(d/2)(-)^{d/2}}{4\pi^{d/2}} \left\{ (1+u) \sum_{k=1}^{d/2} \frac{\Gamma(k - \frac{1}{2})}{\pi^{k} \Gamma(k)} \frac{(-)^k}{[u(2+u)]^{k+\frac{1}{2}}} + 1 \right\}
\]

It can be verified directly that (3.28) is the integral of (3.27) which vanishes at \( u = \infty \). The normalization of (3.28) was chosen by requiring agreement with the hypergeometric form (3.17) as \( u \) approaches 0. One can then use (3.25) to obtain an analogous expression for \( H(u) \), namely

\[
H(u) = -\frac{\Gamma(d/2)(-)^{d/2}}{4\pi^{d/2}} \times \left\{ \frac{2(1+u)^d}{d-1} \sum_{k=1}^{d/2} \frac{\Gamma(k - \frac{1}{2})(-)^k}{\pi^{k} \Gamma(k)(2k-3)} \left[ \frac{2k-3}{[u(2+u)]^{k+\frac{1}{2}}} + \frac{2k-d-1}{[u(2+u)]^{k+\frac{3}{2}}} \right] + 2(1+u)^2 \right\}
\]

These expressions may be worked out quite explicitly for the special case of interest, \( d = 4 \),

\[
\begin{align*}
    G(u) &= -\frac{1}{8\pi^2} \left\{ \frac{(1+u)[2(1+u)^2 - 3]}{[u(2+u)]^{\frac{3}{2}}} - 2 \right\} \\
    H(u) &= \frac{1}{12\pi^2} \left\{ \frac{(1+u)[6(1+u)^4 - 9(1+u)^2 + 2]}{[u(2+u)]^{\frac{3}{2}}} - 6(1+u)^2 \right\}
\end{align*}
\]

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For $d = 2p + 1$, the hypergeometric functions entering into the definition of $G(u)$ and $H(u)$ are degenerate, and reduce to elementary functions involving logarithms. These expressions will not be given here.

As in the case of the photon propagator it is worth observing here that there is a more direct way of obtaining the gauge invariant functions $G(u)$ and $H(u)$ in which the compensating diffeomorphism (3.10) and the manipulation of a set of coupled equations for $G, H, A, B, C$ are entirely avoided. The procedure is to substitute the ansatz (3.9), with gauge terms dropped, in the integral (3.5). We then apply the Ricci operator (3.6) under the integral, using (3.11) (with diffeomorphism terms dropped). The various tensor terms are then manipulated so as to isolate total derivatives $\frac{\partial}{\partial w^\mu}$ which vanish by partial integration since $D_\mu T^{\mu \nu}(w) = 0$. There are 3 independent tensor structures in the remaining terms, which directly give (3.16) for $G(u)$, and (3.18) and (3.19) for $H(u)$ (the latter with the solution of (3.20) for $A(u)$ already in place).

Our final observation, from (3.17) and (3.25), is that $G(u)$ and $H(u)$ vanish on the boundary as $1/u^d$ and $1/u^{(d-2)}$ respectively. One can then look back to (3.12-3.14) and (3.23) and observe that the functions $A(u), B(u)$ and $C(u)$ of the compensating diffeomorphism in (3.10) vanish as $1/u^{(d-1)}, 1/u^d$ and $1/u^{(d+1)}$, respectively. There is then no boundary contribution in the integral of (3.6) with a conserved stress tensor provided that $T_{00}(w)$ vanishes on the boundary. This is a consistency check on the approach we have used. However, because the method is new, a further check is also desirable. For this reason we show in Section 5 that one obtains the same functions $G(u)$ and $H(u)$ from a more traditional approach.

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4 The photon propagator in the Landau gauge

In this second part of the paper we wish to obtain the photon and graviton propagators in $AdS_n$ ($n = d + 1$) by a more standard covariant gauge–fixing procedure. We will employ techniques developed in [12, 13, 14, 15], following quite closely [15] where a simplified derivation of the graviton propagator for 4–dimensional de Sitter space was given. In this and the following section we will adopt the conventions introduced in [12], which are followed throughout this earlier literature.
4.1 Notations

Let $\mu(z, w)$ be the geodesic distance between $z$ and $w$. Then the basic tensors are the unit tangents to the geodesics at $z$ and $w$:

$$n_\mu(z, w) = D_\mu \mu(z, w), \quad n_{\mu'}(z, w) = D_{\mu'} \mu(z, w)$$

(4.1)

together with the parallel transporter $g_{\mu \nu'}(z, w)$ and the metric tensor itself at $z$ and $w$. We summarize in Table 1 some useful geometric identities while Table 2 contains the dictionary that translates between the two different sets of conventions used in the first and second part of the paper. Note the the $AdS$ scale is always set to 1, i.e. the curvature scalar is $R = -n(n - 1)$.

\[

g_{\mu \nu'} n_{\nu'} = -n_\mu
\]

\[
D_\mu n_\nu = A \left( g_{\mu \nu} - n_\mu n_\nu \right)
\]

\[
D_\mu n_{\nu'} = C \left( g_{\mu \nu'} + n_\mu n_{\nu'} \right)
\]

\[
D_\mu g_{\nu \nu'} = -(A + C)(g_{\mu \nu'} n_{\nu'} + g_{\mu \nu'} n_{\nu'})
\]

where

\[
A(\mu) = \coth(\mu)
\]

\[
C(\mu) = -\frac{1}{\sinh(\mu)}
\]

with

\[
A^2 - C^2 = 1
\]

\[
\frac{dA}{d\mu} = -C^2
\]

\[
\frac{dC}{d\mu} = -AC
\]

Table 1

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\[ n = d + 1 \]
\[ u = \cosh(\mu) - 1 = 2(x - 1) \]
\[ n_\mu = \frac{\partial_\mu u}{\sqrt{u(u + 2)}} \]
\[ g_{\mu\nu'} = \partial_\mu \partial_{\nu'} u + \frac{\partial_\mu u \partial_{\nu'} u}{u + 2} \]

Table 2

4.2 Gauge choice and field equations for the photon propagator

Let us write the gauge potential as
\[ A_\mu = A_\mu^\perp + D_\mu \Lambda, \quad (4.2) \]
where
\[ D^\mu A_\mu^\perp = 0. \quad (4.3) \]
A pleasant generic feature of Anti de Sitter space (as opposed to de Sitter) is the absence of normalizable zero modes. It follows that the decomposition (4.2) is unique, as long as one is careful in demanding proper (fast enough) falloff at the boundary of all the quantities involved. One has
\[ A_\mu^\perp = P_\mu^{(1)} \nu' A_{\nu'}, \quad (4.4) \]
where
\[ P_\mu^{(1)} \nu' \equiv g_{\mu' \nu}' \delta(z, w) - D_\mu D^{-1}_\nu' \quad (4.5) \]
is the projector onto transverse vectors.

Let us now impose the Landau gauge
\[ D^\mu A_\mu = 0 \quad (4.6) \]
which implies \( \Lambda \equiv 0 \). With this gauge choice the Maxwell equations read:
\[ -(g_{\mu\nu} \Box - R_{\mu\nu}) A^\nu = j_\mu \quad (4.7) \]
where the current is understood to be conserved, \( D_\mu j^\mu = 0 \). The photon propagator \( G_{\mu\nu'} \) in the Landau gauge satisfies
\[ -(\Box + (n - 1)) G_{\mu\nu'} = P_{\mu\nu'}^{(1)} \quad (4.8) \]
where we have used \( R_{\mu\nu} = -(n - 1)g_{\mu\nu} \) for \( AdS_n \).
4.3 Ansatz for the propagator using bitensors

We will now closely follow the manipulations in Section III of [12], where the spin 1 massive propagator was derived. The crucial difference between the discussion in [12] and our case is the presence of the projector on the r.h.s. of (4.8). We thus expect, and we are going to confirm, that the naive $m^2 \to 0$ limit of the result in [12] does not give the correct Landau gauge massless propagator. It gives instead a pure gauge part that can be added to the massless propagator.

The most general ansatz for $G_{\mu'\nu'}$ consistent with the symmetries of $AdS_n$ is

$$G_{\mu'\nu'} = \alpha(\mu) g_{\mu'\nu'} + \beta(\mu) n_{\mu' n_{\nu'}}.$$  \hfill (4.9)

Let

$$\gamma = \alpha - \beta.$$  \hfill (4.10)

Then using the identities in Table 1 (see [12]) one finds that the gauge condition $D^\mu G_{\mu'\nu'} = 0$ gives:

$$-(n - 1) C \alpha = \gamma' + (n - 1) A \gamma,$$  \hfill (4.11)

We can apply the operator $- (\Box + (n - 1))$ onto (4.9) to obtain the l.h.s. of (4.8) [12]. We can extract an equation for $\gamma$ by contracting the resulting bitensor with $n_{\mu} n_{\nu'}$. The r.h.s. of (4.8) for $z \neq w$ is, after an integration by parts, $D_\mu D_{\nu'} \Box^{-1}$. The contraction with $n_{\mu} n_{\nu'}$ gives

$$n_{\mu} P^{(1)}_{\mu \nu'} n_{\nu'} = \frac{d^2}{d\mu^2} \Box^{-1}.$$  \hfill (4.12)

Then we obtain the equation

$$\left( \frac{d^2}{d\mu^2} + (n + 1) A(\mu) \frac{d}{d\mu} + 2(n - 1) \right) \gamma = -\frac{d^2}{d\mu^2} \Delta_0,$$ \hfill (4.13)

where $\Delta_0$ is the massless scalar propagator$^1$

$$\Box \Delta_0 (\mu(z, w)) = -\delta(z, w).$$  \hfill (4.14)

4.4 Solving for $\gamma$

Equation (4.13) can be rewritten in terms of the $(n + 2)$–dimensional Laplace operator:

$$\Box_{n+2} + 2(n - 1) \gamma = -\frac{d^2}{d\mu^2} \Delta_0.$$ \hfill (4.15)

$^1$$\Delta_0$ is the same as $G(u)$ in equations (3.16,3.17).
Introducing the variable \( x \equiv \cosh^2(\mu/2) = (u + 2)/2 \), the Laplace operator in \( D \)-dimensional \( \text{AdS} \) is

\[
\square_D = -x(1-x) \frac{d^2}{dx^2} - \frac{D}{2}(1-2x) \frac{d}{dx} \quad (4.16)
\]

and (4.13) reads, for \( x \neq 1 \):

\[
\left( -x(1-x) \frac{d^2}{dx^2} - \frac{n+2}{2} (1-2x) \frac{d}{dx} + 2(n-1) \right) \gamma = \frac{n-1}{2} (2x - 1) \frac{d}{dx} \Delta_0 \quad (4.17)
\]

where we have used (4.14) to simplify the r.h.s\(^2\). Observe that the r.h.s. is an eigenvector of zero eigenvalue of the differential operator on the l.h.s., up to a delta function term at \( x = 1 \). This has a simple physical interpretation: for \( z \neq w \), the projector on the r.h.s of (4.8) is a pure gradient (see(4.5)), and thus a zero mode of the Maxwell operator.

To find a more explicit form of the r.h.s. we use the following observation. For any \( f \),

\[
\frac{d}{dx} \left( \square_D f \right) = \left( \square_{D+2} + D \right) \frac{df}{dx}. \quad (4.18)
\]

It follows that, for \( x \neq 1 \),

\[
\left( \square_{n+2} + n \right) \frac{d}{dx} \Delta_0 = 0. \quad (4.19)
\]

It is now easy to solve for \( \frac{d}{dx} \Delta_0 \) by noting that \( \left( \square_{n+2} + n \right) \) is the hypergeometric operator with parameters \( a = n, b = 1, c = n/2 + 1 \). The solution with fast falloff at the boundary \( (x \to \infty) \) is, since \( a > b \) [18]

\[
(-x)^{-a} F(a, a + 1 - c; a + 1 - b; x^{-1}) = (-x)^{-n/2}(1-x)^{-n/2}. \quad (4.20)
\]

Normalizing the short distance \( (x \to 1) \) singularity as in (4.14) we have

\[
\frac{d}{dx} \Delta_0 = -\Gamma \left( \frac{n}{2} + 1 \right) 2^{n-1} n \pi^{3/2} 2^{n+1} \pi^{3/2} \frac{1}{\left[ x(x-1) \right]^{3/2}}. \quad (4.21)
\]

We now want to solve the inhomogeneous equation (4.17). The l.h.s. is \( \left( \square_{n+2} + 2(n-1) \right) \gamma \). By using the relation (4.18) twice we find a massless operator acting on \( \int \int \gamma \):

\[
\square_{n-2} \left( \int \int \gamma \right) = \int \int \left( \frac{n-1}{2} (2x - 1) \frac{d}{dx} \Delta_0 \right), \quad (4.22)
\]

or

\[
\left( -x(1-x) \frac{d}{dx} - \frac{n-2}{2} (1-2x) \right) \int \gamma = \frac{(n-1)\Gamma \left( \frac{n}{2} - 2 \right) \left[ -n + 4, 1; -\frac{n}{2} + 3; x \right]}{2^{n+1} \pi^{3/2} \left[ x(x-1) \right]^{3/2}}. \quad (4.23)
\]

\(^2\)Our \( x \) is called \( z \) in [12]–[15].
which is a first order differential equation in $\gamma$ and can be solved by quadrature.

The two homogeneous solutions of (4.17) are readily obtained by recognizing that the l.h.s. is the hypergeometric operator with parameters $a = 2$, $b = n - 1$, $c = n/2 + 1$. We then have for the general solution of (4.17):

$$\gamma = \gamma_{\text{part}} + a \gamma_1 + b \gamma_2,$$

where

$$\gamma_{\text{part}} = \frac{(n - 1)\Gamma\left(\frac{n}{2} - 2\right)}{2^{n+1}\pi^{n/2}} d\left[\frac{\int_0^x dx' \ F(-n + 4; 1; -\frac{n}{2} + 3; x')}{[x(x - 1)]^{n/2 - 1}}\right]$$

(4.25)

$$\gamma_1 = \frac{2x - 1}{[x(x - 1)]^{n/2}}$$

(4.26)

$$\gamma_2 = F(2, n - 1; \frac{n}{2} + 1, 1 - x),$$

(4.27)

For odd dimensions $n \geq 5$ the hypergeometric function in (4.25) is a polynomial of degree $n - 4$. For example, for $n = 5$

$$\gamma_{\text{part}}^{n=5} = \frac{1}{32\pi^2} \frac{2x - 1}{(x(x - 1))^{3/2}}.$$  (4.28)

Notice that we have chosen a basis of homogeneous solutions in which $\gamma_1$ is singular at short distance ($\sim 1/(x - 1)^{n/2}$ as $x \to 1$) and has fast falloff at the boundary ($\sim 1/x^{n-1}$ for $x \to \infty$), while $\gamma_2$ is smooth for $x \to 1$ and falls like $1/x^2$ at infinity.

The photon propagator to have the same short distance singularity as in flat space. This requires $\gamma \sim 1/(x - 1)^{n/2 - 1}$ and hence $a \equiv 0$. Second, we choose the boundary condition of fastest possible falloff as $x \to \infty$, which fixes $b \equiv (-1)^{n/2} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{2^n n(n-1)}$. With this choice, $\gamma \sim \log x/x^{n-1}$ as $x \to \infty$. The final answer for the Landau gauge propagator (4.9) can be obtained by solving (4.10) and (4.11) for $\alpha$ and $\beta$:

$$\alpha(x) = \frac{2}{n - 1} x(x - 1)\gamma'(x) + (2x - 1)\gamma(x)$$

(4.29)

$$\beta(x) = \frac{2}{n - 1} x(x - 1)\gamma'(x) + 2(x - 1)\gamma(x).$$

(4.30)

4.5 Comparison to the new form of the propagator

We now wish to make contact between this standard Landau gauge propagator and the new form of the propagator described in Section 2. To compare, we go back to the variable $u = 2x - 2$
and write our Landau gauge propagator in the same tensor basis used in (2.8):

\[ G_{\mu\nu}^{\text{Landau}} = -(\partial_{\mu}\partial_{\nu'}u)\tilde{F}(u) + \partial_{\mu}\partial_{\nu'}\tilde{S}(u) . \]  

(4.31)

Using the conversion formulas of Table 2, it is easy to show:

\[-\tilde{F}(u) + \tilde{S}'(u) = -\alpha \]

(4.32)

\[ \tilde{S}''(u) = \frac{\alpha}{u + 2} + \frac{\beta}{u(u + 2)} . \]

(4.33)

One can show that if \( \gamma \) satisfies (4.17) then \( \tilde{F} \) obeys (2.19). This is enough to prove that \( \tilde{F} = F \), since the solution of (2.19) is uniquely specified (up to a factor) by the boundary condition of fastest falloff at the boundary, and the normalization is fixed in both cases by matching to the flat space limit. As expected, the “Landau gauge” \( \tilde{F} \) coincides with the “universal” \( F \) found in Section 2.

The functional form of \( \tilde{S} \) is instead specific to our Landau gauge choice. As a final consistency check, one can show that the “gauge” piece \( \partial_{\mu}\partial_{\nu'}\tilde{S}(u) \) gives no contribution when integrated with conserved currents. Indeed, \( \tilde{S}(u) \sim 1/u^{n-3} \) as \( u \to \infty \) and \( \tilde{S}(u) \sim 1/u^{n/2-2} \) as \( u \to 0 \), which is enough to ensure the vanishing of the boundary terms in the integration by parts. It is interesting to observe that, for \( u \neq 0 \), the homogeneous solutions \( \gamma_1 \) and \( \gamma_2 \) do not contribute to the physical term \( \tilde{F} \). One may then be tempted to treat \( \gamma_1 \) and \( \gamma_2 \) as pure gauge artifacts and conclude that the constants \( a \) and \( b \) in (4.24) can be arbitrarily specified. This is however not quite correct. Changing \( a \) and \( b \) from the determined values would make \( \tilde{S} \) too singular at the origin or at infinity, so that \( \partial_{\mu}\partial_{\nu'}\tilde{S}(u) \) could not be “gauged away” from physical amplitudes.

5 The graviton propagator in the Landau gauge

We now wish to obtain the graviton propagator in the Landau gauge. Our analysis will parallel in many aspects the discussion of the photon in the previous section, although some new features will emerge.

5.1 Gauge choice and field equations for the graviton propagator

The linear fluctuations of the metric \( h_{\mu\nu} \) can always be decomposed as:

\[ h_{\mu\nu} = h_{\mu\nu}^{\perp} + D_{\mu}A_{\nu}^{\perp} + D_{\nu}A_{\mu}^{\perp} + (D_{\mu}D_{\nu} - \frac{1}{n}g_{\mu\nu}\Box)B + \frac{1}{n}g_{\mu\nu}h \]  

(5.1)
where
\[ D^\mu A_{\mu}^\perp = D^\mu h_{\mu}^\perp = g^{\mu\nu}h_{\mu\nu}^\perp = 0 \] (5.2)
and \( h = h_{\mu}^\mu \). The tensor decomposition (5.1) can be uniquely inverted:
\[ B = \frac{n}{n-1} \left( \Box + \frac{R}{n-1} \right)^{-1} \left( D^\mu \not D \nu - \frac{1}{n} g^{\mu\nu} \not D \right) h_{\mu\nu} \] (5.3)
\[ A_{\mu}^\perp = -Q_{\mu}^\nu D^\nu h_{\mu\nu} \] (5.4)
\[ h_{\mu\nu}^\perp = P_{\mu\nu}^{(2)\mu'\nu'}(z, w) h_{\mu'\nu'} \] (5.5)
\[ P_{\mu\nu}^{(2)\mu'\nu'}(z, w) = g^{(\mu'\nu')}(z, w) - \frac{1}{n} g_{\mu\nu} g^{\mu'\nu'} \delta(z, w) - 2 D_{(\mu} D^{\mu'} Q_{\nu')} \] (5.6)
\[ -(\Box + R/n) Q_{\mu'\nu'} = P_{\mu}^{(1)\mu'} , \] (5.7)
where \( (\cdot) \) indicates symmetrization with strength 1. \( Q_{\mu'\nu'} \) obeys the equation

where the right–hand side is the transverse spin 1 projector (4.5). Note that this is not the same as (4.8), the equation obeyed by the massless spin 1 propagator in the Landau gauge, the difference being in the value of the “mass”. It is understood the all the Green’s functions in (5.3–5.7) obey the boundary conditions appropriate for Anti de Sitter space, namely fastest possible falloff at the boundary. Notice that in contrast to the de Sitter case discussed in [15] we do not have a contribution to the r.h.s. of (5.7) from zero modes.

The covariant gauge choice which yields the simplest results is the “Landau gauge”:
\[ D^\mu h_{\mu\nu} = \frac{1}{n} D^\nu h \] (5.8)
With this gauge choice, \( B = A_{\mu}^\perp = 0 \) and the graviton propagator can be written as:
\[ G_{\mu\nu}(z, w) = g_{\mu\nu} g_{\mu'\nu'} T(\mu(z, w)) + G_{\mu\nu}(z, w) \] (5.9)
with \( G_{\mu\nu}(z, w) \) transverse and traceless:
\[ D^\mu G_{\mu\nu} = D^\mu G_{\mu\nu} = 0 \] (5.10)
\[ g^{\mu\nu} G_{\mu\nu} = g^{\mu'\nu'} G_{\mu'\nu'} = 0 \] (5.11)

Projection of the linearized Einstein equations onto the “pure trace” and “transverse symmetric traceless” subspaces gives:
\[ \left( \Box + \frac{R}{n-1} \right) T = \delta(z, w) \] (5.12)
\[ \left( -\Box + \frac{2R}{n(n-1)} \right) G_{\mu\nu}(z, w) = P_{\mu\nu}(z, w) \] (5.13)
where the spin 2 projector has been defined in (5.6). We now proceed to solve (5.12) and (5.13).

5.2 Solving for $T$

Using $R = -n(n-1)$ we see that (5.12) defines a scalar propagator of $m^2 = n$. Introducing the variable $x \equiv \cosh^2(\mu/2) = (u + 2)/2$ we recognize (5.12) (see (4.16)) as the hypergeometric equation of parameters $a = n$, $b = -1$, $c = n/2$. The fastest falloff solution with short distance singularity normalized as in (5.12) is [11] [18]:

$$T(x) = -\frac{1}{(n-2)(n-1)} \frac{\Gamma(n)\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma(n+2)\pi^{2n/2}} \frac{1}{x^n} F(n, n/2 + 1; n + 2; 1/x).$$

(5.14)

Observe that using (4.18) the operator $(\Box_n - n)$ on $T$ changes to $\Box_{n+2}$ on $\frac{d}{dx} T$, so that $T(x)$ could also be obtained by quadrature. $T(x)$ is actually an algebraic function for odd $n$. For example, for $n = 5$:

$$T^{n=5}(x) = \frac{1}{768\pi^2} \frac{128 x^4 - 256 x^3 + 144 x^2 - 16 x - 1}{(x - 1)^{3/2} x^3} - \frac{2 x - 1}{12\pi^2}.$$  

(5.15)

5.3 Ansatz for $G^{(2)}_{\mu\nu\mu'\nu'}$ in terms of bitensors

We want to write the most general ansatz for $G^{(2)}_{\mu\nu\mu'\nu'}$ consistent with AdS symmetry. The 5 independent fourth rank bitensors which are symmetric under $\mu \leftrightarrow \nu$, $\mu' \leftrightarrow \nu'$ and $(\mu, \nu) \leftrightarrow (\mu', \nu')$ are taken to be:

$$
\begin{align*}
\mathcal{O}^{(1)}_{\mu\nu\mu'\nu'} &= g_{\mu\nu}g_{\mu'\nu'} \\
\mathcal{O}^{(2)}_{\mu\nu\mu'\nu'} &= n_{\mu}n_{\nu}n_{\mu'}n_{\nu'} \\
\mathcal{O}^{(3)}_{\mu\nu\mu'\nu'} &= g_{\mu\nu}g_{\mu'\nu'} + g_{\mu'\nu}g_{\mu\nu'} \\
\mathcal{O}^{(4)}_{\mu\nu\mu'\nu'} &= g_{\mu\nu}n_{\mu'}n_{\nu'} + g_{\mu'\nu}n_{\mu}n_{\nu} \\
\mathcal{O}^{(5)}_{\mu\nu\mu'\nu'} &= g_{\mu\nu}n_{\nu}n_{\mu'} + g_{\mu'\nu}n_{\mu}n_{\nu'} + g_{\mu\nu'}n_{\mu}n_{\nu} + g_{\mu\nu}n_{\mu'}n_{\nu'}.
\end{align*}
$$

(5.16)

This basis is linearly related that of (3.7), as shown in Table 3. We then have

$$G^{(2)}_{\mu\nu\mu'\nu'} = \sum_{i=1}^{5} G_i(\mu) \mathcal{O}^{(i)}_{\mu\nu\mu'\nu'}. \quad (5.17)$$

The tracelessness conditions (5.11) imply:

$$
\begin{align*}
G_2 &= 4G_5 - nG_4 \\
G_1 &= \frac{1}{n}(2G_3 + G_4).
\end{align*}
$$

(5.18) \quad (5.19)
Define $f$ and $g$ as
\[
    f \equiv G_5 - G_3 \quad \text{(5.20)}
\]
\[
    g \equiv (n-1)G_4 - 2G_3 \quad \text{(5.21)}
\]

Then the transversality conditions (5.10) imply:
\[
    f'(\mu) + nA(\mu)f(\mu) + \frac{n}{2}C(\mu)g(\mu) - \left(\frac{n(n-1)}{2} - 1\right)C(\mu)G_4(\mu) = 0 \quad \text{(5.22)}
\]
\[
    g'(\mu) + nA(\mu)g(\mu) + 2nC(\mu)f(\mu) = 0 \quad \text{(5.23)}
\]

It is easy to see that the five functions $G_i$ can be determined immediately from the single function $g$. To find $g$, we need to use the equation of motion (5.13). Operating $(-\Box - 2)$ onto the ansatz (5.17) (see formulas (A4) in [14]) we get the l.h.s. of (5.13). Contracting the resulting bitensor with $n_\mu n_\nu n_\mu' n_\nu'$ we can extract an expression containing only $g$. Equ.(5.13) then implies:
\[
    -\left(\frac{n-1}{n}\right)\left(\frac{d^2}{dx^2} + (\frac{n}{2} + 2)(1 - 2x)\frac{d}{dx} - 2(n+1)\right)g = n_\mu n_\nu n_\mu' n_\nu' P^{(2)}_{\mu\nu'\mu'\nu'} .
\]

\section*{5.4 Finding the r.h.s.}

We now need a more explicit form for the spin 2 projector (5.6), which appears in (5.24). Define $\Delta_0$ as in (4.14) and
\[
    (\Box - n) \Delta_1(z, w) = -\delta(z, w) .
\]

Further, as in the case of the photon propagator (see (4.9)), let us write the ansatz for $Q_{\mu'\nu'}$ in (5.7) as
\[
    Q_{\mu'\nu'} = \tilde{\alpha}(\mu)g_{\mu'\nu'} + \tilde{\beta}(\mu)n_{\nu'}n_\mu .
\]

Going through the same steps as in Section 4.3, we define
\[
    \tilde{\gamma} = \tilde{\alpha} - \tilde{\beta} .
\]

and we arrive at
\[
    \left(\frac{d^2}{d\mu^2} + (n+1)A(\mu)\frac{d}{d\mu}\right)\tilde{\gamma} = -\frac{d^2}{d\mu^2}\Delta_0 .
\]

Using (4.14) we can rewrite (5.28) as
\[
    \left(\frac{d^2}{d\mu^2} + (n+1)A(\mu)\frac{d}{d\mu}\right)\left[\tilde{\gamma} - \frac{n-1}{2}\Delta_0\right] - 0 .
\]
We require that \( \tilde{\gamma} \) obeys the usual boundary condition of fastest falloff at infinity and matches at short distance the flat space singularity. Since the homogeneous solutions of (5.29) are either too singular at short distance or fall off too slowly at infinity, we choose the particular solution:

\[
\tilde{\gamma} = \frac{n - 1}{2} \Delta_0 .
\]  
(5.30)

We are now in the position to evaluate the r.h.s. of the basic equation (5.24). Let \( z \neq w \).

From the expansion (5.26) of \( Q \) we find

\[
n^{\mu} n^{\nu} n_{\mu} n_{\nu} \left[ -2 D_{\mu} D^{\nu} Q_{\mu}^{\nu} \right] = 2 \frac{d^2}{d\mu^2} \gamma
\]  
(5.31)

To evaluate the contribution of the last term in \( P^{(2)} \) in (5.6), note that

\[
\left( \square + \frac{R}{n - 1} \right)^{-1} \left( \square - 1 \right) = \frac{n - 1}{R} (\Delta_0 - \Delta_1)
\]  
(5.32)

Then we find

\[
n^{\mu} n^{\nu} (D_{\mu} D_{\nu} - \frac{1}{n} g_{\mu\nu} \square) = \frac{d^2}{d\mu^2} - \frac{1}{n} \square
\]

Now recalling (5.30), using the equations obeyed by \( \Delta_0 \) and \( \Delta_1 \) and changing to the variable \( x, \mu = \arccosh(2x - 1) \), we can rewrite (5.33) as

\[
n^{\mu} n^{\nu} n_{\mu} n_{\nu} P^{(2)}_{\mu\nu\mu'\nu'} = S_0(x) + S_1(x),
\]  
(5.34)

where \( S_0(x) \) and \( S_1(x) \) are defined as

\[
S_0(x) = \frac{1}{8} \frac{n(n + 1)(2x - 1)}{x(x - 1)} \frac{d}{dx} \Delta_0(x)
\]  
(5.35)

\[
S_1(x) = \frac{1}{4} \frac{(n + 1)(4(n - 1)x(x - 1) + n)}{n} \frac{d^2}{dx^2} \Delta_1(x).
\]  
(5.36)

The expression for \( \frac{d}{dx} \Delta_0 \) was given in (4.21). To find an explicit functional form for \( \frac{d^2}{dx^2} \Delta_1 \), we use (4.18) twice to get, for \( x \neq 1 \):

\[
(\square_{n+4} + n + 2) \frac{d^2}{dx^2} \Delta_1(x) = 0
\]  
(5.37)

which is the same as (4.19) with \( n \rightarrow n + 2 \). The fast falloff solution normalized as in (5.25) is

\[
\frac{d^2}{dx^2} \Delta_1(x) = \Gamma \left( \frac{n}{2} + 1 \right) \frac{1}{2^n \pi^{\frac{n}{2}}} \frac{1}{(x(x - 1))^{\frac{n}{2} + 1}}.
\]  
(5.38)
We thus get
\begin{align}
S_0(x) &= -\frac{(n+1) \Gamma \left( \frac{n}{2} + 1 \right)}{2^{n+2} \pi^{\frac{n}{2}}} \frac{2x - 1}{x(x - 1)^{\frac{n}{2} + 1}} \quad \text{(5.39)}
S_1(x) &= \frac{(n+1) \Gamma \left( \frac{n}{2} + 1 \right)}{2^{n+2} \pi^{\frac{n}{2}}} \frac{4(n - 1) x(x - 1) + n}{x(x - 1)^{\frac{n}{2} + 1}}. \quad \text{(5.40)}
\end{align}

Observe that \(S_0\) and \(S_1\) have opposite symmetry properties under “antipodal reflection” \(x \to 1 - x\).

### 5.5 Solving for \(g\)

If we imagine to turn off the source \(S_1\), the equation (5.24) is the same as the corresponding photon equation (4.17) with \(n \to n + 2\) and a different normalization of the source. The functional form of a particular solution \(g^0_{\text{part}}\) corresponding to the source \(S_0\) can thus be read from (4.25). Accounting for the different normalization we find:

\[g^0_{\text{part}}(x) = \frac{n(n + 1) \Gamma \left( \frac{n}{2} - 1 \right)}{(n - 1) 2^{n+2} \pi^{\frac{n}{2}}} \frac{d}{dx} \left[ \int_0^x \frac{dx'}{x(x - 1)^{\frac{n}{2} + 1}} \right] \quad \text{(5.41)}\]

For odd \(n\), the hypergeometric function in (5.41) is actually a polynomial of degree \(n - 2\).

As in the case of the photon, for \(x \neq 1\), \(S_0\) is an eigenfunction of the differential operator on the l.h.s. of (5.24) with zero eigenvalue. It turns out that the source \(S_1\) is also an eigenfunction of the same operator with eigenvalue \(-\frac{(n-2)(n-1)}{n}\). Thus the particular solution \(g^1_{\text{part}}\) is

\[g^1_{\text{part}} = -\frac{n}{(n-2)(n-1)} S_1. \quad \text{(5.42)}\]

The general solution of (5.24) is then given by (compare with (4.25–4.27)):

\begin{align}
g &= g^1_{\text{part}} + g^0_{\text{part}} + \hat{a} g_1 + \hat{b} g_2 \quad \text{(5.43)}
g_1 &= \frac{2x - 1}{x(x - 1)^{\frac{n}{2} + 1}} \quad \text{(5.44)}
g_2 &= F(2, n + 1; \frac{n}{2} + 2, 1 - x). \quad \text{(5.45)}
\end{align}

Notice that \(g_1\) is singular at short distance \(\sim 1/(x - 1)^{n/2+1}\) as \(x \to 1\) and falls like \(1/x^{n+1}\) as \(x \to \infty\), while \(g_2\) is smooth at \(x = 1\) and falls like \(1/x^2\) at infinity. The constant \(\hat{a}\) is determined by demanding that the graviton propagator has the same short distance singularity as in flat
space, which requires \( g \sim (x - 1)^{n/2 - 1} \). This fixes \( \tilde{a} \equiv \frac{n(n+1)F(\frac{n}{2}+1)}{(n-2)(n-1)^{2n+2} \pi^2} \). The requirement of fastest possible falloff at the boundary determines \( \tilde{b} \equiv (-1)^{\frac{n-1}{2}} \frac{F(\frac{n+3}{2}) n}{(n+2)(n-1)^2 \pi^2} \). With this choice, \( g \sim \log x/x^{n+1} \) as \( x \to \infty \).

For example, in five dimensions we have

\[
g^{n=5} = -\frac{15}{1024 \pi^2} \frac{8 x^3 - 4 x^2 - 6 x + 5}{x^2 (x - 1)^2} + \frac{15}{56 \pi^2} F(2, 6; \frac{9}{2}; 1 - x). \tag{5.46}
\]

The final answer for the transverse traceless part of the graviton propagator is given by (5.17), where the functions \( G_i \) are obtained by solving (5.18–5.23) in terms of \( g \):

\[
G_1(x) = \frac{1}{n(n-2)} \left( \frac{4(x - 1)^2 x^2}{n+1} g''(x) + 4x(x-1)(2x-1) g'(x) + (4nx(x-1) + n - 2) g(x) \right) \tag{5.47}
\]

\[
G_2(x) = -4(x - 1)^2 \left( \frac{x^2}{n(n+1)} g''(x) + \frac{2(n+2)x}{n(n+1)} g'(x) + g(x) \right) \tag{5.48}
\]

\[
G_3(x) = -\frac{1}{2(n-2)} \left( \frac{4(n-1)x^2(x-1)^2}{n(n+1)} g''(x) + \frac{4(n-1)x(x-1)(2x-1)}{n} g'(x) + (4(n-1)x(x-1) + n - 2) g(x) \right) \tag{5.49}
\]

\[
G_4(x) = -\frac{4x(x-1)}{n(n-2)} \left( \frac{x(x-1)}{n+1} g''(x) + (2x-1) g'(x) + n g(x) \right) \tag{5.50}
\]

\[
G_5(x) = -\frac{x-1}{n-2} \left( \frac{2(n-1)x^2(x-1)}{n(n+1)} g''(x) + \frac{x(4(n-1)x - 3n + 4)}{n} g'(x) + (2(n-1)x - n + 2) g(x) \right) \tag{5.51}
\]

The full graviton propagator in the Landau gauge is given by (5.9), so that its components in the tensor basis (5.16) are simply \( (T + G_1, G_2, G_3, G_4, G_5) \), where the trace term \( T \) was determined in (5.14).

### 5.6 Comparison to the new form of the propagator

To make contact with the discussion in Section 3, we go back to the variable \( u = 2x - 2 \) and write the Landau gauge graviton propagator in a tensor basis analogous to (3.9):

\[
C_{\mu\nu\rho\sigma}^{\text{Landau}} = (\partial_\mu \partial_\nu u \partial_\rho \partial_\sigma u + \partial_\mu \partial_\rho u \partial_\nu \partial_\sigma u) G(u) + g_{\mu\nu} g_{\rho\sigma} \bar{H}(u) \tag{5.52}
\]
Using the conversion formulas between the two tensor basis (3.7) and (5.16) given in Table 3, one can show:

\[
\begin{align*}
\tilde{Y}'(u) &= \frac{1}{4} G_2(u) + 2u^2 G_3(u) + 4u G_5(u) \quad (5.53) \\
\tilde{X}'(u) &= -\frac{G_5(u)}{u(u + 2)} - 2\tilde{Y}(u) \quad (5.54) \\
\tilde{Z}'(u) &= \frac{G_4(u)}{u(u + 2)} - 2\tilde{X}(u) - 2(1 + u)\tilde{Y}(u) \quad (5.55) \\
\tilde{G}(u) &= G_3(u) - 2\tilde{X}(u) \quad (5.56) \\
\tilde{H}(u) &= T(u) + G_1(u) - 2(1 + u)\tilde{Z}(u). \quad (5.57)
\end{align*}
\]

As in the case of the photon, one expects the “Landau gauge” \(\tilde{H}\) and \(\tilde{G}\) to be exactly the “universal” \(H\) and \(G\) found in Section 3. We directly checked this for \(n = 5\) working with the explicit expressions for \(G_i\) and \(T\). It is worth commenting on some interesting features of this check. First, the homogeneous solutions \(g_1\) and \(g_2\) do not contribute at all to \(\tilde{H}\) and \(\tilde{G}\), at least for \(u \neq 0\). They however do contribute to \(\tilde{X}, \tilde{Y}, \tilde{Z}\). The role of the homogeneous solutions is again to ensure the proper falloff and short distance singularity of the “gauge terms”, so that they can be removed when integrating the propagator with conserved currents. Second, the contribution of \(g_{\text{part}}^1\) to the physical functions \(\tilde{H}, \tilde{G}\) is precisely removed by adding the trace term \(T\). Therefore there is no effective propagation of the scalar of \(m^2 = n\).
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References


C.P. Burgess and C. A. Lutken, ‘Propagators and Effective Potentials in Anti-de Sitter


